# NOVEL SOLUTION METHODOLOGY FOR STOCHASTIC LQ PROBLEMS WITH BOUNDED CONTROL

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**Abstract**

In the paper a novel methodology is proposed to solve problems of stochastic optimal control with bounded in magnitude control law. Namely, the developed strategy allows to fond an exact analytical solution to the modified Hamilton-Jacobi-Bellman equation. Together with the hybrid solution method, the proposed strategy makes possible to build a solution to a whole class of stochastic optimal control problems with bounded in magnitude control force.

**Key words**

Dynamic programming, stochastic optimal control, bounded control

## 1 Introduction

The dynamic programming approach (Bellman (1957)) provides a methodology to deal with problems of stochastic optimal control (Dreyfus (1965), Fleming and Rishel (1975)). The basic idea of the method is to convert a stochastic optimal control problem to a problem of finding a solution for the deterministic partial differential equation, so called Hamilton-Jacobi-Bellman (HJB) equation, within the whole state-space. Unlike its trivial appearance, this problem is far from being simple, and a very limited number of exact (and approximate) analytical solutions available today is a direct proof of it.

There is a number of difficulties that arises when dealing with the HJB equation, especially if one considers a stochastic system. First of all, the asymptotic behavior of the Bellman function is unknown, which does not allow one to use classical numerical methods to solve it numerically. It must be mentioned that in certain problems, related to the probability of system's state, such boundary conditions are known from the problem statement. Secondly, if the introduced control force is bounded in magnitude, as it is considered here, the corresponding HJB equation becomes nonlinear with a signum type nonlinearity. Finding an analytical solution to a nonlinear, degenerate, multidimensional equation of parabolic type becomes almost an impossible task.

Moreover, problems of stochastic optimal control of dynamic systems may form a whole class of problems, which stays apart from other optimal control problems and has its own mathematical as well as engineering issues. It happens because the governing equation of motion of any dynamic system, written in a state-space form would not contain noise in all of the equations. This leads to the fact that the corresponding HJB equation becomes degenerate equation of parabolic type. Mathematical challenge here is the theorem of existence and uniqueness of a solution in the classical sense for a degenerate parabolic equation, which is yet to be proven. Although, certain promises has already been brought by Lio and Ley (2006), who has recently proven existence and uniqueness of a solution to such an equation in the viscosity sense.

From practical (engineering) point of view, a Linear Quadratic (LQ) problem is the only problem, which has an exact explicit analytical solution, which therefore can be implemented in practical applications. On the other hand, assumption of an unbounded in magnitude control force, made in LQ problem, seems unreasonable and simply unfeasible in a variety of practical applications. Thus, consideration of a bounded in magnitude control force appears to be well justified. Until recently, the only technique available for constructing an approximate solution for such problems was a perturbation approach, where either noise intensity or control bound were assumed as a small parameter. Another possibility is to use asymptotic techniques, such as the stochastic averaging for instance, to handle problems of stochastic control approximately (for instance Ying and Zhu (2006)).

## 2 Hybrid Solution Method for solving a HJB equation

Recently in Bratus et.al. (2006) the hybrid solution method has been proposed for finding a solution to HJB equation, with a bounded (in magnitude) control force. The method consists of two steps, which are demonstrated on the following example. Consider a SDOF system, subjected to external white noise Gaussian excitation. The governing equation of motion may be written as

$$\begin{cases} \ddot{x} = h(t, x, \dot{x}) + \nu + \sigma(x)\dot{w}, \quad t < s \le T, \\ x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \end{cases} \tag{1}$$

where $w = w(s)$ is a Wiener process (its derivative should be understood formally), $\sigma^2$ is a white noise intensity and the control $\nu = \nu(s)$ is an adapted random process satisfying $|\nu| \le R$, for a fixed constant $R > 0$. In the paper by Bratus et.al. (2006) a special case of equation (1) has been considered, namely $h(t, x, \dot{x}) = -2\alpha\dot{x} - \beta^2 x$ and $\sigma - const$, e.g. the system is autonomous. Let's rewrite equation (1) in the state-space form, introduce the Bellman function and taking into account that the control goal was to minimize the following cost function (mean system response energy if $a = b = 1$)

$$J_{x_1,x_2,t}(\nu) = \mathbb{E}\Big\{\frac{a}{2}[\beta^2 x_1^2(T) + x_2^2(T)] + \\ + \int_t^T \frac{b}{2}[\beta^2 x_1^2(s) + x_2^2(s)]ds\Big\} \tag{2}$$

one may arrive to the following HJB equation

$$\begin{aligned} &\partial_t u + Lu + \inf_{|\nu|\le R}\{\nu\partial_2 u\} + p = 0, \\ &Lu := x_2\partial_1 u - (2\alpha x_2 + \beta^2 x_1)\partial_2 u + \frac{\sigma^2}{2}\partial_2^2 u, \\ &p(x_1, x_2) := \frac{b}{2}(\beta^2 x_1^2 + x_2^2) \end{aligned} \tag{3}$$

where $\partial_t$, $\partial_1$ and $\partial_2$ denote the partial derivative with respect to $t$, $x_1$ and $x_2$. Notice that the optimal control is defined as

$$\nu = -R\, sign(\partial_2 u), \quad \inf_{|\nu|\le R}\{\nu\partial_2 u\} = -R|\partial_2 u|, \tag{4}$$

and the terminal condition

$$u(x_1, x_2, T) = \frac{a}{2}(\beta^2 x_1^2 + x_2^2). \tag{5}$$

The developed hybrid solution method offered two-step procedure of finding a solution to the corresponding HJB equation:

• Assume the existence of the domain $D$ of the state-space plane, where the sign of the derivative $\partial_2 u$ stays unchanged for any values of $x_1, x_2$ and $t > 0$. In other words, this domain does not contain switching lines $\partial_2 u = 0$, transition of which leads to the sign changes to the opposite. Introducing $z(x_1, x_2, t) = \text{sign}[\partial_2 u]$ one arrives from (3) to the *modified* HJB equation

$$\partial_t u + Lu - Rz\partial_2 u + p = 0 \tag{6}$$

Temporally disregarding the second derivative in (6), due to similarity of deterministic and stochastic problems, one arrives to the hyperbolic equation, exact solution to which can be found by the method of characteristics, and therefore solution to the modified HJB equation (6). Then one can find the boundaries of the domain $D$, where the derived solution is valid. This domain has been called the *outer* domain.

• Having an exact analytical solution in the outer domain, the solution within the *inner*, complementary to the outer domain, may be found numerically (for details see Bratus et.al. (2006))

There are several issues, which must be emphasized here. First, the optimal control law within the outer domain for the system (1) is found as $\nu = -Rsign(x_2)$, which is proved to be an optimal control law for the system's steady-state response with Lagrange's cost function ($a = 0$) (Iourtchenko D.V. (2000)). Secondly, results obtained by other authors using different approaches, for instance Crespo L.G. and Sun J.Q., (2002), Park J. et.al., (2005) completely confirm and agree with the results, obtained by the hybrid solution method. Finally, it has been proven in Bratus et.al. (2006) that the solution, obtained by this approach provides an asymptotic behavior for the Bellman function, i.e. when $|x_1|, |x_2| \longrightarrow \infty$. This allows one to use derived analytical solution as a boundary condition, thereby solving the HJB equation in the whole state-space domain.

## 3 A quadratic function approach

The major point of the hybrid solution method is an ability to derive an analytical solution to the modified HJB equation. Deriving a solution to the modified HJB equation by the method of characteristics becomes extremely cumbersome or even impossible if the considered system either non-autonomous $h(t, x_1, x_2)$, has parametric broad band excitation $\sigma(x_1, x_2)$, or its equation of motion has higher degree. The later happens when the system is subjected to a narrow band noise, which appears as a result of filtering the Gaussian white noise process. Therefore, it is proposed to look for a solution to the modified equation in the quadratic form, similar to the classical LQR or LQG problems:

$$u(t,\overline{x}) = \sum_{i,j=0}^{N} f_{ij}(t)x_i x_j, \ \ x_0 = 1. \qquad (7)$$

Such substitution allows one to solve stochastic optimal control problem in more general formulation, like described by equation (1). Such a generalization does not change in principle the proof, described above from the paper by Bratus et.al. (2006), but allows and significantly simplifies a way of finding a solution, by reducing the problem to a Cauchy problem for a set of ordinary differential equations with respect to unknown functions $f_{ij}(t)$. Let's show the implementation of the proposed technique in three examples.

## 4 Dynamic system under periodic and white noise external excitations

Consider a dynamic system, subjected to external white noise and periodic excitations. The governing equation of motion in a state-space form may be written as:

$$\begin{cases} \dot{x}_1 = x_2, \quad 0 < t \le T, \\ \dot{x}_2 = -\beta^2 x_1 + \nu + \sigma\xi(t) + \lambda sin(\omega t), \\ x_1(0) = x_{10}, \ \ x_2(0) = x_{20}, \ \ |\nu(t)| \le R \end{cases} \qquad (8)$$

where description of all the parameters is kept as above, $\lambda$ - excitation amplitude, $\omega$ - excitation frequency. Assuming that the aim of the control is to minimize system's mean response energy (2). Then, keeping the notation from previous paragraph, the corresponding HJB equation has a from:

$$\frac{\partial u}{\partial t} + Lu + \lambda sin(\omega t)\frac{\partial u}{\partial x_2} - Rz\frac{\partial u}{\partial x_2} + p = 0; \quad (9)$$

Two cases, resonant ($\beta = \omega$) and non-resonant ($\beta \neq \omega$) should be considered separately. A solution to the *modified* equation (3) is sought for in the form:

$$\begin{aligned} u(t,x_1,x_2) &= \sum_{i,j=0}^{2} f_{ij}(t)x_i x_j = \\ & f_{00}(t) + f_{10}(t)x_1 + f_{20}(t)x_2 + f_{12}(t)x_1x_2 + \\ & + f_{11}(t)x_1^2 + f_{22}(t)x_2^2. \end{aligned} \qquad (10)$$

This brings one to the following set of equations:

$$\begin{cases} \dot{f}_{11}(t) = \beta^2 f_{22}(t), \ \dot{f}_{22}(t) = -\frac{1}{2}, \ \dot{f}_{10}(t) = -\beta^2 f_{20}, \\ \dot{f}_{20}(t) = 2[Rz - \lambda sin(\omega t)]f_{22}(t) - f_{10}(t), \\ \dot{f}_{00}(t) = [Rz - \lambda sin(\omega t)]f_{20}(t) - \sigma^2 f_{22}(t), \\ f_{11}(t=T) = a\beta^2/2, \\ f_{22}(t=T) = a/2, \ \ others \ \ f_{i,j}(t=T) = 0 \end{cases} \qquad (11)$$

### 4.1 Case of nonresonant excitation

Solution to set (11) in the nonresonant case and ($b = 1$, $a = 0$) was found by the hybrid solution method in Iourtchenko D.V. (2007) and may be written as:

$$\begin{aligned} & f_{12}(\tau) = 0, \quad \tau = T - t, \\ & f_{11}(\tau) = \frac{\beta^2}{2}\tau, \ \ f_{22}(\tau) = \frac{1}{2}\tau, \\ & f_{10}(\tau) = \frac{Rz}{\beta^2}(\beta^2\tau - \beta sin(\beta\tau)) - \frac{\lambda\beta}{2}\left[\frac{4\beta\omega cos\phi_1}{(\beta^2-\omega^2)^2} - \right. \\ & \left. - \frac{cos\phi_2}{(\beta-\omega)^2} + \frac{cos\phi_3}{(\beta+\omega)^2} + \frac{2\beta\tau sin\phi_1}{(\beta+\omega)^2}\right], \\ & f_{20}(\tau) = \frac{Rz}{\beta^2}(cos(\beta\tau) - 1) + \lambda\left[\frac{(\beta^2+\omega^2)sin\phi_1}{(\beta^2-\omega^2)^2} - \right. \\ & \left. - \frac{sin\phi_2}{2(\beta-\omega)^2} - \frac{sin\phi_3}{2(\beta+\omega)^2} + \frac{\tau\omega(\omega^2-\beta^2)cos\phi_1}{(\beta^2-\omega^2)^2}\right], \\ & \phi_1 = \omega(T-\tau), \ \phi_2 = \omega T - \beta\tau, \ \phi_3 = \omega T + \beta\tau. \end{aligned} \qquad (12)$$

It has been shown that for the steady-state response (when $T$ and $\tau$ are very large) the outer domain is defined by the inequality:

$$|x_2| \ge \frac{\lambda\omega}{|\beta^2 - \omega^2|} \qquad (13)$$

Obviously, in the case of $\omega >> \beta$ or $\omega << \beta$ the outer domain size will be proportional to a small parameter (the excitation amplitude $\lambda$ may be taken equal to unity, since the original equation of motion may be scaled to it). Thus, the outer domain, where the bang-bang control law ($\nu = -Rsign[x_2]$) is optimal, would expand. This clearly indicates the resembles in control policy between the autonomous ($|x_2| > 0$) (Iourtchenko (2000)) and non-autonomous systems in the nonresonant mode. To find a solution to the HJB equation in the whole state-space domain for any values of parameters, one need to solve the corresponding HJB equation numerically.

### 4.2 Case of resonant excitation

Solution to the case when ($\beta = \omega$) and ($b = 1, a = 0$) may be written as:

$$\begin{aligned} & f_{11}(\tau) = \frac{\beta^2}{2}\tau, \ \ f_{22}(\tau) = \frac{1}{2}\tau, \ \ f_{12}(\tau) = 0, \\ & f_{10}(\tau) = \frac{Rz}{\beta}(\beta^2\tau - sin(\beta\tau)) - \\ & - \frac{\lambda}{8\beta}\left[cos\phi_1(1 - 2\beta^2\tau^2) - \cos\phi_3 - 2\beta\tau sin\phi_1\right], \\ & f_{20}(\tau) = \frac{Rz}{\beta^2}(cos(\beta\tau) - 1) + \\ & + \frac{\lambda}{8\beta^2}\left[sin\phi_1(1 + 2\beta^2\tau^2) + +2\tau\beta cos\phi_1 - sin\phi_3\right]. \end{aligned} \qquad (14)$$

The outer domain is defined as:

$$|x_2| \geq \frac{\lambda\tau}{\omega} \tag{15}$$

It is clear that the inner domain, where the analytical solution (14) is not valid, expands away for large values of $\tau$ (and $T$ correspondingly), although it remains finite, since the $T$ is given in advance. Therefore the domain of the state-space where the bang-bang control law ($\nu = -Rsign[x_2]$) is valid reduces. This is exactly what was expected since such a control law cannot minimize the response energy of the system (8) at the resonance. Indeed, if one considers system (8) without white noise excitation, then the system at the resonance becomes unstable, when $\nu = -Rsign[x_2]$. Therefore, at the resonance the optimal control law is the one, defined by formula (4).

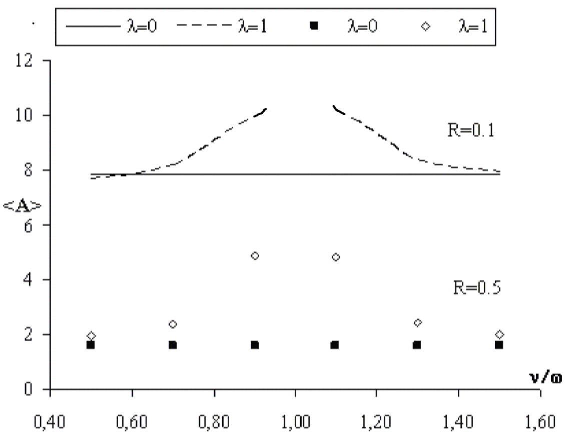


Figure 1. Mean response amplitude of the controlled system.

Figure1. demonstrates influence of the control magnitude on the system mean response amplitude behavior for different values of excitation frequency.

## Conclusions

The paper proposes a novel approach for finding an analytical solution to the modified HJB equation within the outer state-space domain. This solution then may be used as a boundary condition to solve the corresponding HJB equation within the entire state-space domain numerically, thereby finding an optimal control policy. The proposed strategy allowed in the first time to find a solution to problems of stochastic optimal control of dynamic systems subjected to combined external periodic and white noise excitations, as well as other previously unsolved problems. It turns out that the developed methodology allows to construct a solution to the whole class of stochastic optimal control problems with bounded in magnitude control force (Iourtchenko (2008)), characterized by the following equation of motion $\ddot{x} = f(x, \dot{x}, t) + \nu + \sigma(x, \dot{x})\xi(t)$, where $f$ and $\sigma$ are linear functions of its state variables.


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