LYAPUNOV FUNCTIONS IN HAUSDORFF DIMENSION ESTIMATES OF COCYCLE ATTRACTORS

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Abstract
We consider global attractors or invariant sets of cocycles which are generated by nonautonomous ordinary differential equations. Using the singular values of the linearized flow and adapted Lyapunov functions we give upper Hausdorff dimension estimates for a class of global cocycle attractors or invariant sets.

Key Words
Cocycle, nonautonomous system, global attractor, invariant set, Hausdorff dimension.

1 Introduction

General upper estimates of the Hausdorff dimension of attractors of dynamical systems have been derived for the first time in [Douady, Oesterlé, 1980]. Later these results were generalized by other authors (see [Smith, 1986; Temam, 1988]). For the first time Lyapunov functions have been introduced into the estimates of Hausdorff dimension in [Boichenko, Leonov, 1992]. The investigation of nonautonomous differential equations leads to the theory of cocycles and their attractors ([Wakeman, 1975; Bebutov, 1941; Kloeden, Schmalfuss, 1997; Chepyzhov, Vishik, 1994]). In a certain way one can consider random dynamical systems and the associated random attractors. Elements of the Douady-Oesterlé theory of upper Hausdorff dimension estimates for random attractors were developed in [Crauel, Flandoli, 1998]. The concept of kernel sections for nonautonomous dynamical systems has been developed in [Chepyzhov, Vishik, 1994].

In this paper we state two theorems about upper Hausdorff dimension estimates of cocycle attractors or invariant sets which include Lyapunov functions. These results can be looked as generalization of the estimates for attractors or invariant sets of autonomous systems (Boichenko, Leonov, 1992]; [Boichenko, Leonov and Reitmann, 2005) to cocycle attractors.

The paper is organized as follows. In Section 2 we recall the concept of cocycles and their global B-pullback attractors. In Section 3 we briefly introduce the basic tools of Hausdorff dimension and singular values of a linear operator. Using these termini we can formulate an upper Hausdorff dimension estimate of cocycle attractors or invariant sets. In Section 4 we consider attractors of cocycles which are generated by nonautonomous differential equations in $\mathbb{R}^n$. Using the partial trace of the Jacobi matrix of the linearization along orbits and the derivative of a Lyapunov function along the orbits we give a realization of the general theorem of Section 3. In Section 5 we investigate the well-known Rössler system ([Rössler, 1976]) with time-dependent coefficients and derive an upper estimate for the Hausdorff dimension of the associated cocycle invariant set.

2 Basic tools for cocycle theory

Let (Θ, ρ_θ) be a compact complete metric space.

A *base flow* $(\{\sigma^t\}_{t\in\mathbb{R}}, \Theta)$ is defined by a continuous mapping $\sigma: \mathbb{R} \times \Theta \to \Theta$, $(t, \theta) \longmapsto \sigma^t(\theta)$ satisfying

[1] G. A. Leonov is partly supported by CONACYT 000000000078890 and by Ministry of Education and Science of the Russian Federation.
[2] V. Reitmann is supported by the DAAD (German Academic Exchange Service).

1) $\sigma^0(\cdot) = \mathrm{id}_\Theta$,
2) $\sigma^{t+s}(\cdot) = \sigma^t(\cdot) \circ \sigma^s(\cdot)$ for each $t, s \in \mathbb{R}$. (1)

A *cocycle over the base flow* $(\{\sigma^t\}_{t\in\mathbb{R}}, \Theta)$ is defined by the pair $\left(\{\varphi^t(\theta,\cdot)\}_{\substack{t\in\mathbb{R},\\ \theta\in\Theta}}, \mathbb{R}^n\right)$ where

$$\begin{aligned} &1)\varphi^t(\theta,\cdot): \mathbb{R}^n \to \mathbb{R}^n, \quad \forall t \in \mathbb{R}, \ \forall \theta \in \Theta, \\ &2)\varphi^0(\theta,\cdot) = \mathrm{id}_{\mathbb{R}^n}, \quad \forall \theta \in \Theta, \\ &3)\varphi^{t+s}(\theta,\cdot) = \varphi^t\big(\sigma^s(\theta), \varphi^s(\theta,\cdot)\big), \\ &\qquad \forall t, s \in \mathbb{R}, \forall \theta \in \Theta. \end{aligned} \tag{2}$$

In the sequel we shortly denote a cocycle $\left(\{\varphi^t(\theta,\cdot)\}_{\substack{t\in\mathbb{R},\\ \theta\in\Theta}}, \mathbb{R}^n\right)$ over the base flow $(\{\sigma^t\}_{t\in\mathbb{R}}, \Theta)$ by (φ, σ).

If $\theta \in \Theta \mapsto Z(\theta) \subset \mathbb{R}^n$ is a map, we call $\hat{Z} = \{Z(\theta)\}_{\theta\in\Theta}$ a *nonautonomous set.*

The nonautonomous set $\{Z(\theta)\}_{\theta\in\Theta}$ is said to be *compact* if all sets $Z(\theta) \subset \mathbb{R}^n$, $\theta \in \Theta$ are compact, and *invariant* for the cocycle (φ, σ) if

$$\varphi^t\big(\theta, Z(\theta)\big) = Z\big(\sigma^t(\theta)\big) \tag{3}$$

for all $t \in \mathbb{R}$ and $\theta \in \Theta$.

The set $\hat{Z} = \{Z(\theta)\}_{\theta\in\Theta}$ is said to be *globally B-pullback attracting* for the cocycle (φ, σ) if

$$\lim_{t\to\infty} \mathrm{dist}\big(\varphi^t(\sigma^{-t}(\theta), B), Z(\theta)\big) = 0 \tag{4}$$

for any $\theta \in \Theta$ and any bounded $B \subset \mathbb{R}^n$.

A nonautonomous set $\hat{A} = \{A(\theta)\}_{\theta\in\Theta}$ is called *global B-pullback attractor* for the cocycle (φ, σ) if the set $\hat{A}$ is compact, invariant and globally B-pulback attracting for the cocycle.

In order to get the existence of a global B-pullback attractor for the cocycle the following property is useful.

The set $\{Z(\theta)\}_{\theta\in\Theta}$ is said to be *B-forward absorbing* for the cocycle (φ, σ) if for each $\theta \in \Theta$ and each bounded set $B \subset \mathbb{R}^n$ there exists a time $T = T(\theta, B)$ such that

$$\varphi^t(\theta, B) \subset Z\big(\sigma^t(\theta)\big) \tag{5}$$

for all $t \geq T(\theta, B)$.

If the cocycle has a globally B-forward absorbing compact set $\hat{Z}$ there exists by the Kloeden-Schmalfuss theorem ([Kloeden, Schmalfuss, 1997]) a unique global B-pullback attractor $\hat{A} = \{A(\theta)\}_{\theta\in\Theta}$ for the cocycle (φ, σ) which is given by

$$A(\theta) = \bigcap_{\substack{t\geq 0\\ t\in\mathbb{R}}} \overline{\bigcup_{\substack{\tau\geq t\\ \tau\in\mathbb{R}}} \varphi^\tau\big(\sigma^{-\tau}(\theta), Z(\theta)\big)}, \quad \forall \theta \in \Theta. \tag{6}$$

3 Upper Hausdorff dimension estimates for cocycles

Let (M, ρ) be a metric space and $Z \subset M$ be an arbitrary subset of M. Assume that $d \geq 0$ and $\varepsilon > 0$ are arbitrary numbers. Let us cover Z by at least countable many balls B_{r_i} of radii $r_i \leq \varepsilon$ and define

$$\mu_H(Z, d, \varepsilon) := \inf\left\{\sum_i {r_i}^d \,\middle|\, r_i \leq \varepsilon, Z \subset \bigcup_i B_{r_i}\right\}, \tag{7}$$

where the infimum is taken over all such countable ε-covers of Z under the convention that $\inf \emptyset = \infty$.

It is obvious that for fixed Z and d the function $\mu_H(Z, d, \varepsilon)$ does not decrease with decreasing ε. Thus there exists the limit (which may be infinite)

$$\mu_H(Z, d) := \lim_{\varepsilon\to 0+0} \mu_H(Z, d, \varepsilon) = \sup_{\varepsilon>0} \mu_H(Z, d, \varepsilon). \tag{8}$$

It is known (e.g. [Boichenko, Leonov and Reitmann, 2005]) that there exists an unique $d_{cr} = d_{cr}(Z) \in [0, \infty]$ such that

$$\mu_H(Z, d) = \begin{cases} 0, & \text{for any } d > d_{cr,} \\ \infty, & \text{for any } d < d_{cr.} \end{cases} \tag{9}$$

The value

$$\dim_H Z := d_{cr}(Z) \tag{10}$$

is called the *Hausdorff dimension* of Z.

For an $n \times n$ matrix L the *singular values* are the nonnegative square roots of the eigenvalues of $L^T L$. The singular values of the matrix L are denoted by $\alpha_i(L)$ and are arranged in a non-decreasing order

$$\alpha_1(L) \geq \alpha_2(L) \geq \cdots \geq \alpha_n(L). \tag{11}$$

For any $k \in \{0,1,\ldots,n\}$ we put

$$\omega_k(L) := \begin{cases} \alpha_1(L)\alpha_2(L) \ldots \alpha_k(L), & \text{for } k > 0, \\ 1, & \text{for } k = 0. \end{cases} \tag{12}$$

Suppose $d \in [0, n]$ is an arbitrary number. It can be represented as $d = d_0 + s$, where $d_0 \in \{0,1,\ldots,n-1\}$ and $s \in (0,1]$. Now we put

$$\omega_d(L) := \begin{cases} \omega_{d_0}(L)^{1-s}\omega_{d_0}(L)^{1+s}, & \text{for } d \in (0, n], \\ 1, & \text{for } d = 0 \end{cases} \tag{13}$$

and we call $\omega_d(L)$ the *singular value function* of L of order d.

Suppose that (φ, σ) is a cocycle for which the maps $\varphi^t(\theta,\cdot): \mathbb{R}^n \to \mathbb{R}^n$ are smooth enough for all $t \in \mathbb{R}$ and $\theta \in \Theta$.

Let us make the following assumptions:

(A1) The nonautonomous set $\hat{Z} = \{Z(\theta)\}_{\theta\in\Theta}$ is a compact invariant set for the cocycle (φ, σ).

(A2) For each $\theta \in \Theta$ and $t > 0$ let

$$\partial_2\varphi^t(\theta,\cdot): \mathbb{R}^n \to \mathbb{R}^n \quad (14)$$

be the differential of $\varphi^t(\theta,\cdot)$ with respect to the second argument, i.e. u, which has the following properties:

a) For each $\varepsilon > 0$ and $t > 0$ the function

$$\eta_\varepsilon(t,\theta) := \sup_{\substack{v,u\in Z(\theta)\\ 0<\|v-u\|\le\varepsilon}} \frac{\|\varphi^t(\theta,v)-\varphi^t(\theta,u)-\partial_2\varphi^t(\theta,u)(v-u)\|}{\|v-u\|} \quad (15)$$

is bounded on Θ and converges to zero as $\varepsilon \to 0$ for each fixed $t > 0$.

b) For each $t > 0$

$$\sup_{\theta\in\Theta}\sup_{u\in Z(\theta)} \|\partial_2\varphi^t(\theta,u)\|_{\mathrm{op}} < \infty \quad (16)$$

where $\|L\|_{\mathrm{op}}$ denotes the operator norm of an $n \times n$ matrix L.

Now we can state the main result of our paper.

Theorem 1 *Suppose that the assumptions* **(A1)** *and* **(A2)** *are satisfied and the following conditions hold:*

1) There exists a compact set $\tilde{K} \subset \mathbb{R}^n$ such that

$$\overline{\bigcup_{\theta\in\Theta} Z(\theta)} \subset \tilde{K}. \quad (17)$$

2) There exist a continuous function $\kappa: \Theta \times \mathbb{R}^n \to \mathbb{R}_{>0}$, a time $\tau > 0$ and a number $d \in (0, n]$ such that

$$\sup_{(\theta,u)\in\Theta\times\tilde{K}} \frac{\kappa\big(\sigma^\tau(\theta), \varphi^\tau(\theta,u)\big)}{\kappa(\theta,u)} \omega_d\big(\partial_2\varphi^\tau(\theta,u)\big) < 1. \quad (18)$$

Then

$$\dim_H Z(\theta) \le d \quad \textit{for each } \theta \in \Theta. \quad (19)$$

The proof of Theorem 1 and the next Theorem 2 will be presented in [Leonov, Reitmann and Slepukhin, to appear]. A short announcement of these results has already been given in [Leonov, Reitmann and Slepukhin, 2010].

4 Cocycles generated by differential equations

Let us consider the nonautonomous ODE

$$\dot{u} = f(t,u), \quad (20)$$

where $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k-smooth $(k \ge 1)$ vector field. With respect to the vector field (20) we introduce the *hull* of f given by

$$\mathcal{H}(f) = \overline{\{f(\cdot+t,\cdot), t \in \mathbb{R}\}}, \quad (21)$$

where the closure is taken in the compact-open topology. One can show that $\mathcal{H}(f)$ is metrizable with a metric ρ. As a result we get the complete metric space $(\mathcal{H}(f), \rho)$ on which a base flow called *Bebutov flow* ([Bebutov, 1941]) is given by the shift map

$$\sigma^t(\tilde{f}) = \tilde{f}(\cdot+t,\cdot) \quad (22)$$

for $\tilde{f} \in \mathcal{H}(f)$.

We assume that $\mathcal{H}(f)$ is compact. A sufficient condition for this is the almost-periodicity of $f(t,u)$ with respect to t.

Suppose now that we have on $\Theta = \mathcal{H}(f)$ the "evaluation map" given by

$$(\theta,u) \in \Theta \times \mathbb{R}^n \longmapsto \theta(0,u). \quad (23)$$

In particular we get for $\theta = f \in \mathcal{H}(f)$

$$\hat{f}(t,u) = f(0,u). \quad (24)$$

It follows that

$$\hat{f}(\sigma^t(f),u) = f(t,u) \quad (25)$$

for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

Using this map we can associate to (20) the family of vector fields

$$\dot{u} = \hat{f}(\sigma^t(\theta),u), \quad (26)$$

where $\theta \in \mathcal{H}(f)$ is arbitrary. As special case the given system (20) is included into (26).

Under the following additional assumptions on (20) one can show for system (26) the existence of a cocycle over the base flow $\big(\{\sigma^t\}_{t\in\mathbb{R}}, \mathcal{H}(f)\big)$ (cf. [Wakeman, 1975]).

(A3) The map $(t,u) \in \mathbb{R} \times \mathbb{R}^n \longmapsto f(t,u)$ is continuous and satisfies a local Lipschitz condition with respect to u.

(A4) There exist locally integrable functions $p, q: \mathbb{R} \to \mathbb{R}$ such that

$$\|f(t,u)\| \le p(t)\|u\|^2 + q(t) \quad (27)$$

for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$.

For a point $(\theta_0, u_0) \in \Theta \times \mathbb{R}^n$ we denote by $w(t, u_0)$ the solution of the variational equation. along the orbit of the cocycle through (θ_0, u_0), i.e., the equation

$$\dot{w} = \partial_2 \hat{f}\big(\sigma^t(\theta_0), \varphi^t(\theta_0, u_0)\big)w \qquad (28)$$

with the initial condition $w(0, w_0) = w_0 \in \mathbb{R}^n$.

Then we have

$$\partial_2 \varphi^t(\theta_0, u_0)w_0 = w(t, w_0) \quad \text{for } t \geq 0, \qquad (29)$$

i.e. $\partial_2 \varphi^t(\theta_0, u_0)w_0$ is a solution of the variational equation (28).

Let

$$\lambda_1(\theta, u) \geq \lambda_2(\theta, u) \geq \cdots \geq \lambda_n(\theta, u) \qquad (30)$$

be the eigenvalues of the matrix

$$\tfrac{1}{2}\big[\partial_2 \hat{f}(\theta, u) + \partial_2 \hat{f}(\theta, u)^T\big]. \qquad (31)$$

Theorem 2 *Suppose that there exist a continuous function $V: \Theta \times \mathbb{R}^n \to \mathbb{R}$ for which the derivative*

$$\tfrac{d}{dt} V\big(\sigma^t(\theta), \varphi^t(\theta, u_0)\big) \qquad (32)$$

exists along the given trajectory. Suppose further that there are a number $d \in (0, n]$ written as $d = d_0 + s$ with $d_0 \in \{0, 1, \dots, n-1\}$ and $s \in (0, 1]$ and a time $\tau > 0$ such that

$$\varphi^\tau\big(\theta, Z(\theta)\big) = Z\big(\sigma^\tau(\theta)\big) \quad \textit{for all } \theta \in \Theta, \qquad (33)$$

the condition (17) is satisfied and

$$\begin{aligned} \int_0^\tau [\lambda_1\big(\sigma^t(\theta), \varphi^t(\theta, u_0)\big) + \cdots + \\ + \lambda_{d_0}\big(\sigma^t(\theta), \varphi^t(\theta, u_0)\big) + \\ + s\lambda_{d_0+1}\big(\sigma^t(\theta), \varphi^t(\theta, u_0)\big) + \\ + \tfrac{d}{dt} V\big(\sigma^t(\theta), \varphi^t(\theta, u_0)\big)]dt < 0 \end{aligned} \qquad (34)$$

for all $\theta \in \Theta$ and $u_0 \in \widetilde{K}$.

Then

$$\dim_H Z(\theta) \leq d \textit{ for all } \theta \in \Theta. \qquad (35)$$

5 Upper Hausdorff dimension estimate for invariant set of nonautonomous Rössler System

We consider the nonautonomous Rössler system ([Rössler, 1976])

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x, \\ \dot{z} = -b(t)z + a(t)(y - y^2), \end{cases} \qquad (36)$$

where the parameters are functions $a, b: \mathbb{R} \to \mathbb{R}_{>0}$ which we write as

$$\begin{aligned} a(t) = a_0 + a_1(t), \\ b(t) = b_0 + b_1(t). \end{aligned} \qquad (37)$$

Here a_0 and b_0 are positive constants; $a_1(\cdot)$ and $b_1(\cdot)$ are smooth functions satisfying the inequalities

$$\begin{aligned} |a_1(t)| \leq \varepsilon a_0, \\ |b_1(t)| \leq \varepsilon b_0 \quad \text{for all } t \in \mathbb{R}, \end{aligned} \qquad (38)$$

where $\varepsilon \in (0,1)$ is a small parameter. Assume also, that there is an $l > 0$ such that

$$\big|\dot{b}(t)\big| \leq \varepsilon l \quad \text{for all } t \in \mathbb{R} \qquad (39)$$

and the hull $\mathcal{H}(f)$ with f as right-hand side of (36) is compact in compact-open topology. A sufficient condition for this is the almost periodicity of a and b.

It follows that system (36) is a special type of system (20) for which the assumptions of Wakeman's theorem are satisfied. Thus (36) generates a cocycle $\left(\{\varphi^t(\theta, \cdot)\}_{\substack{t \in \mathbb{R} \\ \theta \in \mathcal{H}(f)}}, \mathbb{R}^n\right)$ over the base flow $\big(\{\sigma^t\}_{t \in \mathbb{R}}, \mathcal{H}(f)\big)$. We assume that for this cocycle there exist a compact set $\hat{Z} = \{Z(\theta)\}_{\theta \in \mathcal{H}(f)}$, which satisfies (17) with a compact $\widetilde{K}$, and a time $\tau > 0$ such that

$$\varphi^\tau\big(\theta, Z(\theta)\big) = Z\big(\sigma^\tau(\theta)\big) \quad \textit{for all } \theta \in \mathcal{H}(f). \qquad (40)$$

Instead of (36) we consider the family of systems

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x, \\ \dot{z} = -b_\theta(t)z + a_\theta(t)(y - y^2), \end{cases} \qquad (41)$$

where for brevity we have written

$$\begin{aligned} a_\theta(t) \equiv \hat{a}\big(\sigma^t(\theta)\big) \\ \text{and } b_\theta(t) \equiv \hat{b}\big(\sigma^t(\theta)\big). \end{aligned} \qquad (42)$$

Our aim is to estimate from above the Hausdorff dimension of $\hat{Z}$ with the help of

Theorem 2. To do so we have to check the inequality

$$\lambda_{1,\theta}(t,x,y,z)+\lambda_{2,\theta}(t,x,y,z)+ +s\lambda_{3,\theta}(t,x,y,z)+\frac{d}{dt}V_\theta(t,x,y,z)<0, \quad (43)$$

for all $t\in[0,\tau]$, $(x,y,z)\in\widetilde{K}$ and $\theta\in\mathcal{H}(f)$,

in which

$$\lambda_{k,\theta}(t,x,y,z)\equiv\lambda_k\big(\sigma^t(\theta),\varphi^t(\theta,x,y,z)\big), \quad k=1,2,3 \quad (44)$$

are the eigenvalues of the symmetrized Jacobian matrix for the right-hand side of (41) ordered with respect to their size as

$$\lambda_{1,\theta}\ge\lambda_{2,\theta}\ge\lambda_{3,\theta} \quad (45)$$

and

$$V_\theta(t,x,y,z)\equiv V\big(\sigma^t(\theta),\varphi^t(\theta,x,y,z)\big) \quad (46)$$

is a Lyapunov function.

It is easy to see that

$$\lambda_{1,\theta}=\frac{1}{2}\Big(-b_\theta(t)+\sqrt{b_\theta^2(t)+1+a_\theta^2(t)(1-2y)^2}\Big),$$
$$\lambda_{2,\theta}=0, \quad (47)$$
$$\lambda_{3,\theta}=\frac{1}{2}\Big(-b_\theta(t)-\sqrt{b_\theta^2(t)+1+a_\theta^2(t)(1-2y)^2}\Big).$$

Let us choose the Lyapunov function V as

$$V(\sigma^t(\theta),x,z):=\frac{1}{2}(1-s)\xi(z-b_\theta(t)x), \quad (48)$$

where ξ is a varying parameter. A direct calculation shows that

$$\dot V_\theta=\frac{1}{2}(1-s)\xi\Big(\big(a_\theta(t)+b_\theta(t)\big)y-\dot b_\theta(t)x-a_\theta(t)y^2\Big). \quad (49)$$

It follows that the inequality (43) is satisfied if

$$-b_\theta(t)(1+s)+(1-s)h_\theta(t,x,y;\xi)<0$$

for all $t\in[0,\tau]$, $\theta\in\mathcal{H}(f)$ and $(x,y)\in\mathrm{Pr}_{x,y}\widetilde{K}$, (50)

where

$$h_\theta(t,x,y;\xi)=\sqrt{b_\theta^2(t)+1+a_\theta^2(t)(1-2y)^2}+ +\xi\Big(\big(a_\theta(t)+b_\theta(t)\big)y-\dot b_\theta(t)x-a_\theta(t)y^2\Big) \quad (51)$$

and $\mathrm{Pr}_{x,y}\widetilde{K}$ is the projection of $\widetilde{K}$ on the subspace of x and y.

Let us estimate $h(t,x,y;\xi)$ from above. We can write this expression as

$$h_\theta(t,x,y;\xi)= =-\Big(\eta\sqrt{b_\theta^2(t)+1+a_\theta^2(t)(1-2y)^2}-\frac{1}{2\eta}\Big)^2+ +\eta^2\big(b_\theta^2(t)+1+a_\theta^2(t)(1-2y)^2\big)+\frac{1}{4\eta^2}+ +\xi\Big(\big(a_\theta(t)+b_\theta(t)\big)y-\dot b_\theta(t)x-a_\theta(t)y^2\Big), \quad (52)$$

where $\eta\ne0$ is another varying parameter.

After some transformations we get for all arguments the inequality

$$h_\theta(t,x,y;\xi)\le \le\eta^2\big(a_\theta^2(t)+b_\theta^2(t)+1\big)+\frac{1}{4\eta^2}-\xi\dot b_\theta(t)x- -\Big(\xi a_\theta(t)-4\eta^2a_\theta^2(t)\Big)\Big(y+\frac{4\eta^2a_\theta^2(t)-\xi\big(a_\theta(t)+b_\theta(t)\big)}{\xi a_\theta(t)-4\eta^2a_\theta^2(t)}\Big)^2+ +\frac{\Big(4\eta^2a_\theta^2(t)-\xi\big(a_\theta(t)+b_\theta(t)\big)\Big)^2}{4\Big(\xi a_\theta(t)-4\eta^2a_\theta^2(t)\Big)}. \quad (53)$$

Let us take ξ and η so that

$$\xi a_\theta(t)-4\eta^2a_\theta^2(t)>0$$

for all $t\in[0,\tau]$ and $\theta\in\mathcal{H}(f)$. (54)

This is possible under our conditions for sufficiently small $\varepsilon>0$.

Using (53) and (54) we get

$$h_\theta(t,x,y;\xi)\le\eta^2\big(a_\theta^2(t)+b_\theta^2(t)+1\big)+\frac{1}{4\eta^2}- -\xi\dot b_\theta(t)x+\frac{\Big(4\eta^2a_\theta^2(t)-\xi\big(a_\theta(t)+b_\theta(t)\big)\Big)^2}{4\Big(\xi a_\theta(t)-4\eta^2a_\theta^2(t)\Big)}. \quad (55)$$

Since $\mathrm{Pr}_x\widetilde{K}$ is compact there exists an $m>0$ such that

$$|x|\le m \quad \text{for all } x\in\mathrm{Pr}_x\widetilde{K}. \quad (56)$$

Let us choose now the parameters as

$$\xi:=4\eta^2a_0\frac{a_0+2b_0}{a_0+b_0},$$
$$\eta^2:=\frac{1}{2\sqrt{(a_0+2b_0)^2+b_0^2+1}}. \quad (57)$$

Substituting these values into (55), taking a number of direct calculations and using the estimates (38), (39) and (56) we finally get the estimate

$$h_\theta(t,x,y;\xi)\le\sqrt{(a_0+2b_0)^2+b_0^2+1}+\varepsilon\cdot C$$

for all $t\in[0,\tau]$, $\theta\in\mathcal{H}(f)$ and $(x,y)\in\mathrm{Pr}_{x,y}\widetilde{K}$, (58)

where C is a term which can be directly calculated by means of the parameters a_0, b_0, ε, l and m of the system and which is bounded from above for all small $\varepsilon>0$.

In order to use Theorem 2 effectively we need to find the minimal s for which the inequality (50) still holds. Thus, from (50), (58) and Theorem 2 it follows that

$$\dim_H Z(\theta) \le 3 - \frac{2b_\theta(t)}{b_\theta(t)+h_\theta(t,x,y;\xi)} \le$$
$$\le 3 - \frac{2(1-\varepsilon)b_0}{(1+\varepsilon)b_0+\sqrt{(a_0+2b_0)^2+b_0^2+1} + \varepsilon \cdot C} \ . \quad (59)$$

It is clear that if we turn back to the autonomous Rössler system, i.e. tend $\varepsilon \to 0$, we will get the already known Hausdorff dimension estimate for a compact invariant set of the Rössler system

$$\dim_H \widetilde{K} \le 3 - \frac{2b_0}{b_0+\sqrt{(a_0+2b_0)^2+b_0^2+1}} \qquad (60)$$

(cf. [Boichenko, Leonov and Reitmann, 2005).

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