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CONTROLLABILITY OF SECOND-ORDER SWITCHED LINEAR SYSTEMS

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Abstract

We consider second-order switched linear systems and obtain sufficient conditions for such systems to be controllable.

Key words

Switched linear system, controllability.

1 Introduction

Many second order linear systems appear in mechanical and electrical engineering: forced mass-spring damper systems and LRC circuits, for example, exhibit second order behaviour. Controllability of these systems was studied by the authors in [Clotet and García-Planas, 2006], [García-Planas, 2007] and [García-Planas, 2008].

Switched systems constitute a particular kind of hybrid systems which have been studied with growing interest in the last years. In particular, different authors have studied second-order switched systems. In [Sun and Ge, 2005], the controllability of switched linear systems is characterized. The conjecture which states that Hurwitz stability of the convex hull of a set of Metzler matrices is a necessary and sufficient condition for the asymptotic stability of the associated switched linear system under arbitrary switching is proved to be true for systems constructed from a pair of second-order Metzler matrices, systems constructed from an arbitrary finite number of second-order Metzler matrices and that the conjecture is in general false for higher order systems. In [Xuping and Antsaklis, 1999], necessary and sufficient conditions for stabilizability of second-order LTI systems are found. In [Zhang, Yangzhou and Pingyuan, 2005], a necessary and sufficient condition for the origin to be asymptotically stable under the predesigned switching law is obtained.

In this paper, we obtain sufficient conditions for a second-order switched linear system to be controllable.

2 Second-order Switched Linear Systems

Roughly speaking, a switched system is a set of several of continuous-time (or discrete-time) dynamical subsystems and a rule (switching law) orchestrating the switching between them, specifying at any time instant which subsystem is active.

We deal with *switched second-order linear systems*, that is to say, switching systems consisting of several second-order linear subsystems.

Let us consider a well-defined *switching path* $\theta : [t_0, T) \longrightarrow M$, $t_0 < T \leq \infty$, for some index set M, initial time t_0, and a *switching sequence* of θ over $[t_0, T)$, $\{(t_0, \theta(t_0^+)), (t_1, \theta(t_1^+)), \dots, (t_\ell, \theta(t_\ell^+))\}$.

Definition 2.1. *A singular second-order switched linear system is a system which consists of several linear second-order subsystems and a switching well-defined path σ taking values into the index set $M = \{1, \dots, \ell\}$ which indexes the different subsystems.*

$$\boxed{\begin{cases} E_\sigma \delta^2(x) = A_\sigma \delta x + B_\sigma x + C_\sigma u \\ \qquad y = D_\sigma x \end{cases}} \tag{1}$$

where $E_\sigma, A_\sigma, B_\sigma \in M_n(\mathbb{R})$, $C_\sigma \in M_{n\times m}(\mathbb{R})$, $D_\sigma \in M_{p\times n}(\mathbb{R})$.

In the continuous case, such a system can be mathematically described by

$$\begin{cases} E_\sigma \ddot{x}(t) = A_\sigma \dot{x}(t) + B_\sigma x(t) + C_\sigma u(t) \\ \quad y(t) = D_\sigma x(t) \end{cases} \tag{2}$$

where $E_\sigma, A_\sigma, B_\sigma \in M_n(\mathbb{R})$, $C_\sigma \in M_{n\times m}(\mathbb{R})$, $D_\sigma \in M_{p\times n}(\mathbb{R})$.

In the sequential case, by

$$\begin{cases} E_\sigma x(k+2) = A_\sigma x(k+1) + B_\sigma x(k) + C_\sigma u(k) \\ \qquad y(k) = D_\sigma x(k) \end{cases} \tag{3}$$

where $E_\sigma, A_\sigma, B_\sigma \in M_n(\mathbb{R})$, $C_\sigma \in M_{n\times m}(\mathbb{R})$, $D_\sigma \in M_{p\times n}(\mathbb{R})$.

Calling $z = \delta x$, we can linearize a second-order switched linear system thus obtaining:

$$\begin{pmatrix} I_n & \\ & E_\sigma \end{pmatrix} \begin{pmatrix} \delta x \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ B_\sigma & A_\sigma \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ C_\sigma \end{pmatrix} u$$

and will write

$$\boxed{\mathbb{E}\delta X = \mathbb{A}X + \mathbb{B}u} \tag{4}$$

where $X = \left(\begin{smallmatrix} x \\ \delta x \end{smallmatrix} \right)$.

In the case where $E_i = I_n$, for all $i \in M$ the switched system will be called non-singular. Obviously systems where $\operatorname{rk} E_i = n$ for all $i \in M$ can be reduced to the non-singular case.

3 Controllable States. Controllability

We will denote, as usual, by $\Phi(t, t_0, x_0, u, \sigma)$ the state trajectory at time t of system (1) starting from t_0 with initial value x_0, input u and switching well-defined path σ.

Definition 3.1. *A switched singular second-order linear system is said to be controllable when for any $t_0 \in \mathbb{R}$, x_f and $w \in \mathbb{R}^n$, there exists a real number $t_f > t_0$, a switching well-defined path $\sigma : [t_0, t_f] \longrightarrow M$ and an input $u : [t_0, t_f] \longrightarrow \mathbb{R}^m$ such that*

$$\Phi(t_f, t_0, x_0, u, \sigma) = x_f, \dot{\Phi}(t_f, t_0, x_0, u, \sigma) = w$$

Remark 3.1. *Obviously, when one of the subsystems of system*

$$\left\{ \begin{array}{r} E_\sigma \delta^2(x) = A_\sigma \delta x + B_\sigma x + C_\sigma u \\ y = D_\sigma x \end{array} \right. \tag{1}$$

is controllable, the system is also controllable.

We remember that we can determine whether any subsystem is controllable using the following characterizations.

Proposition 3.1. ([Clotet and García-Planas, 2006]) *A second order singular linear system*

$$\left\{ \begin{array}{l} E\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t) \\ \quad y(t) = Dx(t) \end{array} \right.$$

is controllable if, and only if, both matrices

$$\begin{array}{ll} i) & \begin{pmatrix} E \; C \end{pmatrix} \\ ii) & \begin{pmatrix} s^2E - sA - B \; C \end{pmatrix} \end{array} \tag{5}$$

have full rank for all $s \in \mathbb{C}$.

Proposition 3.2. ([García-Planas, 2007]) *A second order singular linear system*

$$\left\{ \begin{array}{l} E\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t) \\ \quad y(t) = Dx(t) \end{array} \right.$$

is controllable if, and only if, the following $2n^2 \times ((2n-2)n + 2nm)$-matrix

$$\mathcal{C} = \underbrace{\begin{pmatrix} -E & 0 & \dots & 0 \\ -A & -E & \dots & 0 \\ B & -A & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & -E \\ 0 & 0 & \dots & -A \\ 0 & 0 & \dots & B \end{pmatrix}}_{(2n-2)n} \underbrace{\begin{matrix} C & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & C & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & C & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & C & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & C & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & C \end{matrix}}_{2nm}$$

has full rank.

From here we deduce the first sufficient conditions for controllability of system (1).

Proposition 3.3. *Sufficient conditions for controllability of system (1) are the following ones.*

1. *There exists $i \in \{1, \dots, \ell\}$ such that*

$$\begin{array}{ll} i) & \text{rk} \begin{pmatrix} E_i \; C_i \end{pmatrix} = n, \quad \text{and} \\ ii) & \text{rk} \begin{pmatrix} s^2E_i - sA_i - B_i \; C_i \end{pmatrix} = n, \; \forall s \in \mathbb{C}. \end{array}$$

2. *There exists $i \in \{1, \dots, \ell\}$ such that matrix*

$$\mathcal{C}_i = \left(\underbrace{\begin{matrix} -E_i & 0 & \dots & 0 \\ -A_i & -E_i & \dots & 0 \\ B_i & -A_i & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & -E_i \\ 0 & 0 & \dots & -A_i \\ 0 & 0 & \dots & B_i \end{matrix}}_{(2n-2)n} \underbrace{\begin{matrix} C_i & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & C_i & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & C_i & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & C_i & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & C_i & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & C_i \end{matrix}}_{2nm} \right)$$

   has full rank.

We are interested in the non-trivial case where all subsystems are not controllable. From now on, we will consider all subsystems are non-controllable.

Proposition 3.4. *A switched second-order linear system*

$$\left\{ \begin{array}{r} E_\sigma \delta^2(x) = A_\sigma \delta x + B_\sigma x + C_\sigma u \\ y = D_\sigma x \end{array} \right. \tag{1}$$

is controllable if, and only if, the linearized switched linear system $\mathbb{E}_\sigma \delta X = \mathbb{A}_\sigma X + \mathbb{B}_\sigma u$:

$$\begin{pmatrix} I_n & \\ & E_\sigma \end{pmatrix} \begin{pmatrix} \delta x \\ \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ B_\sigma & A_\sigma \end{pmatrix} \begin{pmatrix} x \\ \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ \\ C_\sigma \end{pmatrix} u$$

is controllable.

Let us consider the following matrix sequence.
$\mathcal{N}_0 = \begin{pmatrix}\mathbb{B}_1 \dots \mathbb{B}_\ell\end{pmatrix}$

$\mathcal{N}_k = \mathcal{A}\cdot \mathrm{diag}(\mathcal{N}_{k-1},\dots,\mathcal{N}_{k-1})$ for $k > 0$, where $\mathcal{A} = \begin{pmatrix} I_{2n} \ \mathbb{A}_1 \dots \mathbb{A}_1^{2n-1} \dots \mathbb{A}_\ell \dots \mathbb{A}_\ell^{2n-1}\end{pmatrix}$.

Remark 3.2. *Note that*

$$\mathrm{rk}\mathcal{N}_0 \leq \mathrm{rk}\mathcal{N}_1 \leq \dots \leq \mathrm{rk}\mathcal{N}_{2\mathrm{n}} = \mathrm{rk}\mathcal{N}_{2\mathrm{n}+1} = \dots$$

Proposition 3.5. *A necessary and sufficient condition for controllability of singular second-order switched linear system (2) is*

$$\mathrm{rk}\mathcal{N}_{2n} = 2n$$

Proof. It follows from the algebraic characterization of controllable switched linear systems in [Sun and Ge, 2005].

Corollary 3.1. *If there exists $j \in \{1,\dots,2n\}$ such that $\mathcal{N}_j$ has full rank, then the singular second-order switched linear system (2) is controllable.*
Then the following sufficient condition for controllability may be stated.
In particular, if

$$rk\ \begin{pmatrix}\mathbb{B}_1 \dots \mathbb{B}_\ell\end{pmatrix} = n$$

then system (2) *is controllable.*

For all $(i_1,\dots,i_\ell)$ permutations of $\{1,\dots,\ell\}$, let us denote by

$$\mathcal{A}_i = \begin{pmatrix} \mathbb{A}_{i_1} & \mathbb{A}_{i_2} \dots \mathbb{A}_{i_\ell} & & \\ 0 & I_{2n} & & \\ & & \ddots & \\ & & & I_{2n}\end{pmatrix},$$

$$\mathcal{B}_i = \begin{pmatrix} \mathbb{B}_{i_1}\ \mathbb{B}_{i_2} \dots \mathbb{B}_{i_\ell} & & \\ & \ddots & \\ & & \mathbb{B}_{i_1}\ \mathbb{B}_{i_2} \dots \mathbb{B}_{i_\ell}\end{pmatrix}$$

Theorem 3.1. *A sufficient condition for controllability of the singular second-order switched linear system is that there exists $i \in \{1,\dots,\ell\}$ such that*

$$\mathrm{rk}\begin{pmatrix}\mathcal{B}_i\ \mathcal{A}_i\mathcal{B}_i \dots \mathcal{A}_i^{2n-1}\mathcal{B}_i\end{pmatrix} - 2n =$$

$$= (1-m)\mathrm{rk}\ \begin{pmatrix}\mathbb{B}_{i_1} \dots \mathbb{B}_{i_\ell}\end{pmatrix}$$

4 Controllability and feedback transformations

Let us consider a subsystem

$$\ddot{x} = A_i\dot{x} + B_i x + C_i u \tag{6}$$

for some $i \in M$.
We introduce state and derivative feedbacks:

$$u = F_i\dot{x} + G_i x + v$$

so that the system is transformed into

$$\boxed{\ddot{x} = (A_i + C_iF_i)\dot{x} + (B_i + C_iG_i)x + C_i v} \tag{7}$$

Theorem 4.1. *The system* (6) *is controllable if and only if system* (7) *is.*

Corollary 4.1. *Suppose that the system* (6) *verifies $C_j = C_i$, $A_j = A_i + C_iF_i$ and $B_j = B_i + C_iG_i$ for some couple $1 \leq i < j \leq \ell$. Then, the system is controllable if and only if $\ddot{x} = A_{\bar\sigma}\dot{x} + B_{\bar\sigma}x + C_{\bar\sigma}u$ is, where $\bar\sigma$ is the path taking values into the index set $\bar{M} = \{1,\dots,j-1,\hat{j},j+1,\dots,\ell\}$.*

Corollary 4.2. *Suppose that the system* (6) *verifies $C_i = C$, $A_{i+1} = A_i + C_iF_i$ and $B_{i+1} = B_i + C_iG_i$ for all $i = 1,\dots,\ell$. Then, the system is controllable if, and only if, $\ddot{x} = A_1\dot{x} + B_1x + C_1u$ is.*

Remark 4.1. *Condition rk $\begin{pmatrix}E_i & C_i\end{pmatrix} = n$ ensures that there exists a second order derivative feedback F_i such that $E_i + C_iF_i$ is regular and pre-multiplying the system by $(E_i + C_iF_i)^{-1}$ the new system is standard.*
We conclude that in this case we can apply sufficient conditions for controllability obtained for standard systems.

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