VERSAL DEFORMATIONS OF BILINEAR SYSTEMS UNDER OUTPUT-INJECTION EQUIVALENCE

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Abstract
Bilinear systems under output injection equivalence are considered. The aim of this paper is to study what happens when a slight perturbation affects the coefficients of the matrices defining a bilinear equation in $\mathbb{C}^n$. For this goal we will use the Arnold's techniques, that is to say we will construct a versal deformation of a differentiable family of bilinear systems which are the orthogonal linear varieties to the orbits of output injection equivalent bilinear systems. Versal deformations provide a special parametrization of bilinear systems space, which can be applied to perturbation analysis and investigation of complicated objects like singularities and bifurcations in multi-parameter bilinear systems.

Key words
Bilinear systems, versal deformations, equivalence relation, output injection.

1 Introduction

A control bilinear system is a control system which is described by linear differential equations in such a way that the control inputs appear as coefficients

$$\begin{aligned} \dot{x} &= (A + uN)x + bu \\ y &= cx \end{aligned} \tag{1}$$

with the state vector $x \in \mathbb{C}^n$ and input vector $u \in \mathbb{C}$, that we will write as a 4-tuple of matrices $(A, N, b, c) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n\times 1}(\mathbb{C}) \times M_{1\times n}(\mathbb{C})$, we will denote by $\mathcal{M} = \{(A, N, b, c)\}$ the space of bilinear systems. Notice that, the set of linear systems constitute the subclass of bilinear systems for which $N = 0$.

The bilinear control systems have been studied with growing interest in the last years, because of the arising problems which challenging practical interest. Its methods and applications cross interdisciplinary on the borderland between physics and control, proving useful in areas as diverse as spin control in quantum physics and the study of Lie semigroups [3].

The stabilization problem analyze the existence of a constant control which renders the resulting linear system to have at least one eigenvalue in the open left half of the complex plan. Remember that $\lambda(u) \in \mathbb{C}$ is an eigenvalue of (A, N, b, c) under output injection, if rank $\begin{pmatrix} A+uN-\lambda(u)I \\ c \end{pmatrix} < n$.

The eigenstructure of (A, N, b, c) is quite sensitive to perturbations in the matrices defining the system and one wants therefore to accurately describe how that structure can change when small variations are applied to these coefficients. A manner to approach this problem can be by means so called versal deformations of the eigenstructure of the bilinear systems.

Versal deformation, of a differentiable family of square matrices under similarity [1] was constructed by V.I. Arnold and has been generalized by several authors to matrix pencils under the strict equivalence [4], [9], pairs or triples of matrices under the action of the general linear group [11], pairs of matrices under the feedback similarity [8]. Versal deformations provide a special parametrization of matrix spaces, which can be effectively applied to local perturbation analysis and investigation of complicated objects like singularities and bifurcations in dynamical systems [1], [2], [4], [8].

The theory of miniversal deformations is used to determine which classes of bilinear systems (A, N, b, c) under a previously defined equivalence relation can be found in all small neighbourhood of a given system.

2 Equivalence relation

Let us consider the space of bilineal systems $\mathcal{M} = \{(A, N, b, c) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n\times 1}(\mathbb{C}) \times M_{1\times n}(\mathbb{C})\}$.

Many of the works in bilinear systems has concentrated on bilinear systems up to output injection (see [10], for example). For that we consider the following equivalence relation that we will call output injection equivalence.

Definition 2.1. *Two 4-tuples* (A_1, N_1, b_1, c_1) *and* (A_2, N_2, b_2, c_2) *in* $\mathcal{M}$ *are output injection equivalent if and only if*

$$\begin{aligned} A_2 &= PA_1P^{-1} + Vc_1P^{-1}, \\ N_2 &= PN_1P^{-1} + Wc_1P^{-1}, \\ b_2 &= Pb_1, \\ c_2 &= c_1P^{-1} \end{aligned} \tag{2}$$

for some nonsingular square matrix $P \in \mathrm{Gl}(n; \mathbb{C})$, $V, W \in M_{n\times 1}(\mathbb{C})$.

Remark 2.1. *For* $V = 0$ *and* $W = 0$ *the equivalence relation coincides with the similarity equivalence.*

From this definition, it results easily, the following proposition.

Proposition 2.1. *Eigenvalues of the bilinear systems* (A, N, b, c) *under output injection are invariant under this equivalence relation.*

Proof.

$$\begin{aligned} &\mathrm{rank} \begin{pmatrix} A_1 + uN_1 - \lambda(u)I \\ c_2 \end{pmatrix} = \\ &\mathrm{rank} \begin{pmatrix} P & V + uW \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 + uN_1 - \lambda(u)I \\ c_1 \end{pmatrix} P^{-1} = \\ &\mathrm{rank} \begin{pmatrix} PA_1P^{-1} + uPN_1P^{-1} - \lambda(u)I + Vc_1P^{-1} + uWc_1P^{-1} \\ cP^{-1} \end{pmatrix} = \\ &\mathrm{rank} \begin{pmatrix} A_2 + uN_2 - \lambda(u)I \\ c_2 \end{pmatrix}. \end{aligned}$$

It is easy to prove that in this case and for $n = 2$, the systems can be reduced in the following form.

Proposition 2.2. *Let* (A, N, b, c) *be a 4-tuple. Then*

a) *If* $c \neq 0$

a1)

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right)$$

a2)

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right)$$

a3)

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right)$$

In all cases if $b_1 \neq 0$ *then* $b_1 = 1$*, and if* $b_1 = 0$ *but* $b_2 \neq 0$ *then* $b_2 = 1$.

b) *If* $c = 0$

b1) *If matrices A and N have a unique common one dimensional invariant subspace, then the system is equivalent to*

$$\left(\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \mu_1 & \beta \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix} \right),$$

and

i) *if* $\lambda_1 \neq \lambda_2$ *then* $\alpha = 0$ *and* $\beta \neq 0$

ii) *if* $\lambda_1 = \lambda_2$ *and* $A \neq \lambda_1 I$ *then* $\alpha = 1$ *and* $\beta \neq 0$

iii) *if* $\lambda_1 = \lambda_2$ *and* $A = \lambda_1 I$ *then* $\alpha = 0$ *and* $\beta \neq 0$

b2) *If matrices A and N have no common invariant subspaces, then the system is equivalent to*

$$\left(\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ \beta & \mu_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix} \right),$$

with $\alpha, \beta \neq 0$.

b3) *If matrices A and N have two common complementary one dimensional invariant subspaces, then the system is equivalent to*

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix} \right),$$

and as in the case a), if $b_1 \neq 0$ *then* $b_1 = 1$*, and if* $b_1 = 0$ *but* $b_2 \neq 0$ *then* $b_2 = 1$. *If* $b_1 \neq 0$ *then* $b_1 = 1$*, and if* $b_1 = 0$ *but* $b_2 \neq 0$ *then* $b_2 = 1$.

The case $c = 0$ the possible reduced forms are an extension to 4-tuples (A, N, b, c) from a classification theorem for couple of matrices (A, N) given but not explicit in [6]. The paper is focused on the case $c \neq 0$ where the input injection is allowed.

For $c \neq 0$ we have the following result

Proposition 2.3 ([7]). *Let* (A, N, b, c) *be a 4-tuple with* $c \neq 0$*, if* (A, c) *and* (N, c) *are observable pairs and*

$$\begin{aligned} &\mathrm{rank} \begin{pmatrix} c \\ c(A-N) \end{pmatrix} = 1, \\ &\mathrm{rank} \begin{pmatrix} c \\ cA \\ c(A^2-N^2) \end{pmatrix} = 2, \\ &\vdots \\ &\mathrm{rank} \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-2} \\ c(A^{n-1}-N^{n-1}) \end{pmatrix} = n-1. \end{aligned} \tag{3}$$

Then, the 4-tuple (A, N, b, c) *is equivalent under equivalence relation considered to the following 4-tuple*

(A_r, N_r, b_r, c_r). *where*

$$
\begin{aligned}
A_r &= \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \\
N_r &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n-1} & 0 \\ 1 & \alpha_{2,2} & \dots & \alpha_{2,n-1} & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \\
b_r &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \\
c_r &= \begin{pmatrix} 0 & 0 \dots & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

3 Tangent and normal spaces

Equivalence relation defined in (2) may be seen as induced by the action of the Lie group $\mathcal{G} = \mathrm{Gl}(n; \mathbb{C}) \times M_{n\times 1}(\mathbb{C}) \times \mathbb{C}_0 \times \mathbb{C}_0$ in the following manner:

$$
\begin{aligned}
&\alpha(g, x) = x_1 \\
&\text{where} \\
&x_1 = (A_1, N_1, b_1, c_1) \\
&A_1 = PAP^{-1} + VcP^{-1}, \\
&N_1 = PNP^{-1} + WcP^{-1}, \\
&b_1 = Pb, \\
&c_1 = cP^{-1}) \in \mathcal{M}, \\
&g = (P, V, W) \in \mathcal{G},\ x = (A, N, b, c) \in \mathcal{M}.
\end{aligned} \tag{4}
$$

Let us fix a 4-tuple $x_0 = (A_0, N_0, b_0, c_0) \in \mathcal{M}$ and define the mapping

$$\alpha_{x_0}(P, V, W) = \alpha((P, V, W), x_0) \in \mathcal{M}. \tag{5}$$

The equivalence class of the 4-tuple x_0 with respect to the action of $\mathcal{G}$ is called the *orbit* of x_0 and denoted by

$$\mathcal{O}(x_0) = \{\alpha_{x_0}(P, V, W) \mid \forall (P, V, W) \in \mathcal{G}\}. \tag{6}$$

The mapping α_{x_0} is differentiable, and $\mathcal{O}(x_0)$ is a smooth submanifold of $\mathcal{M}$.

Let $T_e\mathcal{G}$ be the tangent space to the manifold $\mathcal{G}$ at the unit element $e = (I, 0, 0) \in \mathcal{G}$. Since $\mathcal{G}$ is an open subset of $M_n(\mathbb{C}) \times M_{n\times 1}(\mathbb{C}) \times M_{n\times 1}(\mathbb{C})$, we have

$$T_e\mathcal{G} = M_n(\mathbb{C}) \times M_{n\times 1}(\mathbb{C}) \times M_{n\times 1}(\mathbb{C})$$

and, since $\mathcal{M}$ is a linear space,

$$T_{x_0}\mathcal{M} = \mathcal{M}.$$

Let $d\alpha_{x_0} : T_e\mathcal{G} \longrightarrow \mathcal{M}$ be the differential of α_{x_0} at the unit elements e. Using (5), we find the following proposition.

Proposition 3.1. *Let* $x_0 = (A, N, b, c) \in \mathcal{M}$

$$
\begin{aligned}
&d\alpha_{x_0}(g) = \\
&([P, A] + Vc, [P, N] + Wc, Pb, -cP) \in \mathcal{M} \\
&\quad \forall g = (P, V, W) \in T_e\mathcal{G}
\end{aligned} \tag{7}
$$

Proof. It suffices to compute the first approximation of $\alpha_{x_0}(I + \varepsilon P, \varepsilon V, \varepsilon W)$:

$$
\begin{aligned}
&\alpha_{x_0}(e + \varepsilon) \approx \\
&x_0 + \varepsilon([P, A] + Vc, [P, N] + Wc, Pb, -cP) \\
&\text{where } e + \varepsilon = (I + \varepsilon P, \varepsilon V, \varepsilon W).
\end{aligned}
$$

The mapping $d\alpha_{x_0}$ provides a simple description of the tangent space $T_{x_0}\mathcal{O}(x_0)$, and as a corollary its normal complement $(T_{x_0}\mathcal{O}(x_0))^\perp$ for any inner scalar product defined on $\mathcal{M}$.

Proposition 3.2. *The tangent space to the orbit of the 4-tuple* x_0 *is given by*

$$
\begin{aligned}
&T_{x_0}\mathcal{O}(x_0) = \\
&\{([P, A] + Vc, [P, N] + Wc, Pb, -cP) \mid \\
&\quad \forall (P, V, W) \in T_e\mathcal{G}\}
\end{aligned} \tag{8}
$$

The Euclidean scalar product in the spaces $\mathcal{M}$ considered in this paper is defined as follows

$$\langle x_1, x_2\rangle = \mathrm{tr}(A_1A_2^*) + \mathrm{tr}(N_1N_2^*) + \mathrm{tr}(b_1b_2^*) + \mathrm{tr}(c_1c_2^*), \tag{9}$$

with $x_i = (A_i, N_i, b_i, c_i) \in \mathcal{M}, i = 1, 2$ and where A^* denotes the conjugate and transpose of a matrix A and tr the trace of corresponding matrices.

Having defined a scalar product we can compute the orthogonal space to the tangent space.

Theorem 3.1. *Let* $x_0 = (A, N, b, c) \in \mathcal{M}$ *be a 4-tuple, then* $(X, Y, z, t) \in (T_{x_0}\mathcal{O}(x_0))^\perp$ *if and only if*

$$
\left.\begin{aligned}
[A, X^*] + [N, Y^*] + bz^* - t^*c = 0 \\
cX^* = 0 \\
cY^* = 0
\end{aligned}\right\} \tag{10}
$$

Proof. Calling $\alpha = ([P, A] + Vc, [P, N] + Wc, Pb, -cP)$ and $\beta = (X, Y, z, t)$, then

$$
\begin{aligned}
&< \alpha, \beta >= \\
&\mathrm{tr}\,([P, A]X^* + VcX^*) + \mathrm{tr}\,([P, N]Y^* + WcY^*) + \\
&\quad \mathrm{tr}\, Pbz^* + \mathrm{tr}\, -cPt^* = \\
&\mathrm{tr}\,([A, X^*] + [N, Y^*] + bz^* - t^*c)P + \\
&\quad \mathrm{tr}\, cX^* + \mathrm{tr}\, cY^* = \mathrm{tr}\,\mathbf{MQ}
\end{aligned}
$$

where

$$\mathbf{M} = \begin{pmatrix} [A, X^*] + [N, Y^*] + bz^* - t^*c) & & \\ & cX^* & \\ & & cY^* \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} P & & \\ & V & \\ & & W \end{pmatrix}.$$

Obviously, the condition of the theorem is sufficient. For the necessity, remark that in fact we have

$$\operatorname{tr} \mathbf{MQ} = \operatorname{tr} \mathbf{M}_1 \mathbf{Q}_1$$

where

$$\mathbf{M}_1 = \begin{pmatrix} [A, X^*] + [N, Y^*] + bz^* - t^* c) & & \\ & cX^* & \\ & & cY^* \end{pmatrix}$$

$$\mathbf{Q}_1 = \begin{pmatrix} P & M_1 & M_2 \\ M_3 & V & M_4 \\ M_5 & M_6 & W \end{pmatrix}.$$

because matrices M_i, $i = 1, \ldots, 6$ do not appear in the computation. Therefore, the first matrix should be the null matrix.

So, the proof is concluded.

Example 3.1. *Let us consider a bilinear system*

$$\begin{array}{l} x_0 = (A_0, N_0, b_0, c_0) = \\ \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right). \end{array} \tag{11}$$

According to Theorem 3.1, the elements $(X, Y, z, t) \in (T_{x_0}\mathcal{O}(x_0))^\perp$ can be found by solving the linear system (10). *As a result, we obtain a general element of $(T_{x_0}\mathcal{O}(x_0))^\perp$ in the form (X, Y, z, t) with*

$$\begin{array}{ll} X = \begin{pmatrix} 0 & x_3 \\ 0 & x_4 \end{pmatrix}, & Y = \begin{pmatrix} 0 & -x_3 \\ 0 & -x_4 \end{pmatrix}, \\ z = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, & t = \begin{pmatrix} 0 & x_4 + z_1 \end{pmatrix} \end{array} \tag{12}$$

where $x_3, x_4, z_1 \in \mathbb{C}$ are arbitrary parameters; so $\dim(T_{x_0}\mathcal{O}_2(x_0))^\perp = 3$.

4 Versal deformation

Let $\mathcal{M}$ be a differential manifold with the equivalence relation defined by the action of a Lie group $\mathcal{G}$. The $\mathcal{G}$-action is described by the mapping $(g, x) \longrightarrow \alpha(g, x)$, where x, $\alpha(g, x) \in \mathcal{M}$ and $g \in \mathcal{G}$. Let $\mathcal{U}_0$ be a neighborhood of the origin of $\mathbb{C}^\ell$. A *deformation* $x(\lambda)$ of x_0 is a smooth mapping

$$x : \mathcal{U}_0 \longrightarrow \mathcal{M}$$

such that $x(0) = x_0$. The vector $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{U}_0$ is called the parameter vector. The deformation $x(\lambda)$ is also called the *family* of 4-tuples of matrices. The deformation $x(\gamma)$ of x_0 is called *versal* if any deformation $y(\mu)$ of x_0, where $\mu = (\mu_1, \ldots, \mu_k) \in \mathcal{U}'_0 \subset \mathbb{C}^k$ is the parameter vector, can be represented in some neighborhood of the origin in the following form

$$y(\mu) = \alpha(g(\mu), x(\phi(\mu))), \quad \mu \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \tag{13}$$

where $\phi : \mathcal{U}''_0 \longrightarrow \mathbb{C}^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0) = e$. Expression (13) means that any deformation $z(\xi)$ of x_0 can be obtained from the versal deformation $x(\gamma)$ of x_0 by an appropriate smooth change of parameters $\gamma = \phi(\xi)$ and equivalence transformation $g(\xi)$ smoothly dependent on parameters. The versal deformation with minimal possible number of parameters ℓ is called *miniversal*.

The following result, proved by Arnold [1] for $\mathrm{Gl}(n; \mathbb{C})$ acting on $M_{n\times n}(\mathbb{C})$, and generalized by Tannenbaum [11] for a Lie group acting on a complex manifold, provides the relation between the versal deformation of x_0 and the local structure of the orbit of x_0.

Theorem 4.1. *1. A deformation $x(\lambda)$ of x_0 is versal if and only if it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.*

2. Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in $\mathcal{M}$, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.

In our particular setup.

Let us denote by $\{c_1, \ldots, c_\ell\}$ a basis of an arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ and by $\{n_1, \ldots, n_\ell\}$ a basis of $T_{x_0}\mathcal{O}(x_0)^\perp$;

Corollary 4.1. *The deformation*

$$x(\lambda) = x_0 + \sum_{j=1}^{\ell} c_j \lambda_j \tag{14}$$

is a miniversal deformation.

If we take $c_j = n_j$, $j = 1, \ldots, \ell$, in (14), then the corresponding miniversal deformation is called *orthogonal*.

Example 4.1. *We explicit a basis $\{c_1, \ldots, c_\ell\}$ for the case*

$$\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \right)$$

A simplest miniversal deformation is $(A, N, b, c) + (X, Y, z, t)$ with

$$\begin{array}{l} (X, Y, z, t) = \\ \left(\begin{pmatrix} 0 & x_6 & x_7 \\ 0 & 0 & x_8 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_4 & y_7 \\ 0 & y_5 & y_8 \\ 0 & 0 & y_9 \end{pmatrix}, \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right) \end{array}$$

The basis is obtained placing a 1 in a single parameter and 0 otherwise, for each one of them.

Given a bilinear system $x_0 = (A, N, b, c)$, the homogeneity of the orbits allow us to consider canonical reduced forms to write down explicitly the bases $\{c_1, \dots, c_\ell\}$ and $\{n_1, \dots, n_\ell\}$.

5 Structural stability

In a similar way to [5], from the miniversal deformation in Section 3.5 we can deduce conditions for a 4-tuple of matrices to be structurally stable, according to the usual definition.

Definition 5.1. *A 4-tuple of matrices $(A, N, b, c) \in \mathcal{M}$ is called structurally stable if and only if it has a neighborhood formed by 4-tuples equivalent to it-that is to say, if (A, N, b, c) is an interior point of its orbit.*

Because of homogeneity of the orbits, we have:

Proposition 5.1. *A 4-tuple of matrices $(A, N, b, c) \in \mathcal{M}$ is structurally stable if and only if so are all other 4-tuples in its orbit.*

Structural stability is equivalent to the nonexistence of deformations in the following sense:

Proposition 5.2. *A 4-tuple of matrices $x_0 = (A, N, b, c) \in \mathcal{M}$ is structurally stable if and only if*

$$\dim T_{x_0}\mathcal{O}(x_0)^\perp = 0.$$

Proof. Analogous to ([5], proposition (4.3)).

From Sections 3 and 4 it is immediate to see how Theorem 3.1 can be used to characterize the structural stability of a 4-tuple of matrices, because the above dimension is zero if and only if the only solution of the system (10) in Section 3 is the zero one.

In our particular setup all reduced forms presented have continuous invariants so, all miniversal deformation are not zero. We can consider strata defined as the infinite union of orbits of the 4-tuples having the same type of reduced form varying only on the values of continuous invariants. It is obvious that the strata are invariant under equivalence defined and we have the following proposition

Proposition 5.3. *For $n = 2$.*

i) The stratum consisting of 4-tuples of the type a1) in 2.2

$$\mathcal{S}(x_0) = \left\{ \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right) \right\}$$

with $\beta_1 \neq 0$ is stable under the equivalence relation considered.

ii) The remaining strata consisting of 4-tuples of the type a1) with $\beta_1 = 0$, a2), a3), b1), b2), b3) in 3.2 are not stable under the equivalence relation considered.

Proof. i) It suffices to compute a miniversal deformation of the 4-tuple

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right),$$

obtaining

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_1 + y_1 \\ 0 & \beta_2 + y_2 \end{pmatrix}, \begin{pmatrix} 1 \\ b_2 + z_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right)$$

that obviously, all 4-tuples on the deformation are not in the orbit but in the same stratum.

ii) It is easy to observe that any small perturbation of the 4-tuples in these strata contain 4-tuples belonging in the stratum of the type a1).

References

V. I. Arnold, *On matrices depending on parameters*, Russian Math. Surveys, 26: 2 (1971), pp. 29–43.

J. V. Burke and M. L. Overton, *Stable perturbations of nonsymmetric matrices*, Linear Algebra Appl., 171 (1992), pp. 249–273.

E. David, "Bilinear Control Systems" Applied Mathematical Sciences, Vol. 169 (2009).

A. Edelman, E. Elmroth, and B. Kågström, *A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 653–692.

J. Ferrer, Mª I. García-Planas, *Structural Stability of Quadruples of Matrices*. Linear Algebra and its Applications **241/243** (1996), pp. 279–290.

Sh. Friedland, *Similarity of families of matrices*, An article in CRC Handbook of Linear Algebra. (2005).

Mª I. García-Planas, *Similarity of pairs of linear maps defined modulo a subspace*. Linear and Multilinear Algebra. vol. 54, (5), (2006).

Mª I. García-Planas, A. Mailybaev, *Reduction to versal deformations of matrix pencils and matrix pairs with application to control theory*. SIAM Journal on Matrix Analysis. vol. 24, **4**, pp. 943-962, (2003).

M. I. Garcia-Planas and V. V. Sergeichuk, *Simplest miniversal deformations of matrices, matrix pencils, and contragredient matrix pencils*, Linear Algebra Appl., 302–303 (1999), pp. 45–61.

P. M. Pardalos, V. Yatsenko, "Optimization and control of bilinear systems: theory, algorithms, and applications", Ed. Springer, (2008).

A. Tannenbaum, *Invariance and System Theory: Algabraic and geometric Aspects*, Lecture Notes in Math. 845, Springer-Verlag 1981.