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NONLINEAR MODEL REDUCTION FOR INERTIALLY COUPLED NONLINEAR ELASTIC STRUCTURES

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Abstract
This work investigates nonlinear model reduction in excited elastic structures that contain essential inertial nonlinearities. To achieve high fidelity in nonlinear resonant response prediction, linear normal modes may not be sufficient for order reduction. Another natural way to obtain reduced order models is to use nonlinear normal modes to perform model reduction. These nonlinear normal modes allow for a master-slave separation of degrees of freedom where these modes are constructed based on the multiple time scale approximation. For the elastic structure studied, the nonlinear modes based reduced model shows an over-prediction of the softening nonlinearity compared to the complete system.

Key words
Nonlinear elastic structures, Inertial nonlinearities, Internal resonances, Reduced order models, Nonlinear normal modes.

1 Introduction

There are essentially two approaches to constructing reduced order models of nonlinear systems: approach based on linear transformation or linear modal basis functions, and approaches based on nonlinear normal modes or nonlinear invariant manifolds. For approaches based on some form of linear transformation, one can refer to Nayfeh et al. [2005] who, based on their implementation procedures, further classify these into two categories: node methods and domain methods. Both of these categories of reduced-order models are obtained by one of the following three linear processes: linear static reduction, proper orthogonal decomposition (POD) (also known as singular value decomposition (SVD), Karhunen-Loève decomposition (KLD), or principle component analysis (PCA)), and reduction based on linear undamped modal basis functions. A representative node model reduction method is "*Guyan reduction*" (Guyan [1965]). The domain methods eliminate spatial dependence in the PDEs using the Galerkin method. Here, the displacements are expressed as a linear combination of a complete set of basis functions. One approach to obtaining basis set is based on generating time series of snapshots describing the variation of states of the structure over time, and then applying proper orthogonal decomposition method (POD) to obtain 'optimal' mode shapes (e.g., see Hung et al. [1997]). The other approach is based on using quasi-comparison functions (Meirovitch [1997], Wang and Bajaj [2007]), or a direct solution of the linear undamped eigenvalue problem for the structure (Balachandran and Nayfeh [1990]).

The nonlinear modal reduction through nonlinear normal modes was first suggested by Shaw and Pierre [1993]. In the works of Pecheck et al. [2001, 2002], the nonlinear modal reduction of a nonlinear rotating beam through nonlinear normal modes was studied based on invariant manifolds approach. The rotating beam was modeled as a conservative system with cubic geometric nonlinearities.

Nonlinear model reduction in systems with excitations has also been considered in recent years. Sinha et al. [2005] studied nonlinear order reduction in parametrically excited systems, and presented three different methods: nonlinear projection via singular perturbations, post-processing approach and invariant manifold technique. Touzè and Amabili [2006] studied dynamics of a water-filled cylindrical shell with external resonant forcing and the reduced model was obtained by normal form theory.

In this work we discuss model reduction by nonlinear normal modes for a structure with quadratic inertial nonlinearities under forced excitation, using the method of multiple time scales. For constructing nonlinear normal modes by the method of multi time scales, see Nayfeh and Nayfeh [1994] and Wang et al. [2005]. The structural model obtained by a Galerkin discretization in the form of a set of ordinary differential equations is considered the full system (Wang and Bajaj [2007]). For the study of forced resonant response, models reduced through linear normal modes and the one obtained by reduction through nonlinear normal modes are compared to the full six degrees of freedom system. The nonlinearly reduced model is seen to over-predict softening characteristics present in response.

The present work is organized as follows. We first review the discretized model for a three-beam-tip-mass structure and discuss possible linear interactions. Then, the following cases are considered: nonlinear model reduction for external excitation without internal resonance; and nonlinear model reduction for external excitation with 1:2 internal resonance, including subharmonic and superharmonic external resonances. Finally, we summarize the results.

2 Equations of Motion and Linear Interactions

Consider the multi-beam structure shown in the Fig. 1. Each of the beams is modeled as an Euler-Bernoulli beam with welded joints and the last beam carries a tip mass. For this three-beam structure, Wang and Bajaj [2007] obtained the discretized equations of motion with essential inertial nonlinearities and system parameters. The Galerkin discretization was based on the works of Crespo da Silva [1998] and Meirovitch [1997] and used quasi-comparison functions. The degrees of freedom of the discrete system were determined by the number of the convergence global linear structural modes. Using 30 quasi-comparison functions, the first six modes and frequencies achieved convergence. This six degrees of freedom model (up to quadratic nonlinear terms) will serve as the full reference model for the present study, and it has the form:

$$\ddot{Q}_j + \omega_j^2 Q_j + 2\zeta_j \omega_j \dot{Q}_j + \Delta_m m_{D(j,r)} \ddot{Q}_r +$$
$$(m_{02(j,r,l)} + \Delta_m m_{D02(j,r,l)}) Q_r \ddot{Q}_l +$$
$$(m_{11(j,r,l)} + \Delta_m m_{D11(j,r,l)}) \dot{Q}_r \dot{Q}_l$$
$$= \ddot{x}_s \left(m_{s(j)} + \Delta_m m_{Ds(j)} + m_{s0(j,r)} Q_r \right), \qquad \text{(1a)}$$

where $j = 1, 2, \cdots, 6$ and Δ_m is the perturbation parameter of the mass ratio between the attached mass and the total mass of the three beams. The matrices in Eqs. (1a), m_D, m_{02}, …, and m_{s00} are numerical matrices determined by the combination of the three parameters: the length ratio of the three beams, the position of the attached mass, and the ratio between the attached mass and the three beams.

Wang and Bajaj [2007] illustrated the variation of first two natural frequencies as a function of the length ratios of the beams and the magnitude of the attached mass. More specifically, they obtained parameter combinations for 1:2 and 1:3 internal resonances between the first two modes. If the length ratios of the three beams and the particle mass ratio are fixed, and the position of the attached mass is varied, other linear resonances can arise. As an example, let the lengths be in ratios 5:3:2 and the mass ratio be 5:1. Then we vary the position of the attached mass from the bottom of first beam to the tip end of the third beam. The resulting variation of the lowest six calculated non-dimensional natural frequencies is shown in Figure 2. These results were

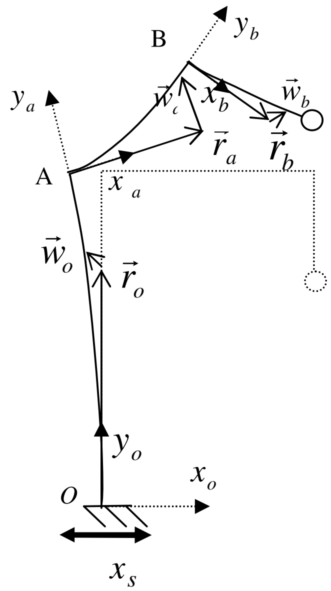

Fig. 1 The three-beam-tip mass structure with horizontal foundation excitation x_s.

obtained by the formulation developed in Wang and Bajaj [2007], and verified by the Finite Element Analysis by using the software ANSYS.

From Fig. 2, it is clear that there are many possible internal resonances. Thus, a proper combination of the three parameters, the length ratios of the three beams, the location of the attached mass, and the mass ratio between the attached mass and the three beams, can lead to various internal resonances among

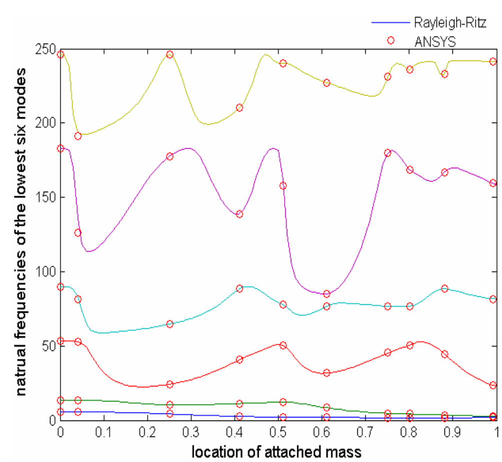

Fig. 2 Variations of the lowest six natural frequencies as a function of the location of attached mass. The length ratios of three beams are 5:3:2, the mass ratio between the attached mass and the three beams, $m_c/(M - m_c)$ is 5:1.

the lowest six modes. We do not list here the possible combinations of these three parameters and the resulting internal resonances, though we note that when using nonlinear normal modes for model reductions, we should take them into account.

3 Nonlinear Model Reduction through Nonlinear Normal Modes

The six degrees of freedom model in Eqs. (1a) will be further reduced through nonlinear normal modes constructed by the method of multiple time scale. The base excitation term provides possible external or parametric excitation based on the frequency ratio between the base coordinate x_s and the six generalized coordinates Q_n ($n = 1, 2, \cdots, 6$). Nonlinear normal mode theory can be modified to include forcing and modal damping (Agnes and Inman [2001]). The harmonic base excitation can also be introduced as another degree of freedom coupled to each of the six modes in Eqs. (1a):

$$\ddot{x}_s + \omega_f^2 x_s = 0. \tag{1b}$$

This seven degree of freedom full model can be reduced to two, three or several degree of freedom system depending on the number of generalized coordinates involved with internal resonances. Note that with the excitation also introduced as another degree of freedom, the system is autonomous.

To apply the method of multiple time scales, we introduce ε as a bookkeeping parameter and rescale the terms in equations as

$$Q_j = \varepsilon \hat{Q}_j,\ \zeta_j = \varepsilon \hat{\zeta}_j,\ \Delta_m = \varepsilon \hat{\Delta}_j,$$

$$\hat{Q}_j = q_{j0} + \varepsilon q_{j1} + O(\varepsilon^2),\ (j = 1, 2, \cdots, 6).$$

For the base coordinate, when studying the case of external excitation, we scale it as

$$x_s = \varepsilon^2 q_{f0} + \varepsilon^3 q_{f1} + O(\varepsilon^4), \tag{2a}$$

whereas for parametrical excitation, we scale the base motion as

$$x_s = \varepsilon q_{f0} + \varepsilon^2 q_{f1} + O(\varepsilon^3). \tag{2b}$$

For the nonlinear model reduction analysis here, we only discuss the resonant external excitation case, that is

$$\omega_f = \omega_k + \varepsilon\sigma. \tag{3}$$

Substituting Eqs. (2a) and (3) into Eq. (1), dropping the '^' notation, and applying the method of multiple time scales, we obtain

$\varepsilon^{(1)}$:

$$D_0^2 q_{j0} + \omega_j^2 q_{j0} = 0, \tag{4a}$$

$$D_0^2 q_{f0} + \omega_k^2 q_{f0} = 0, \tag{4b}$$

$\varepsilon^{(2)}$:

$$\begin{aligned} D_0^2 q_{j1} + \omega_j^2 q_{j1} = & -2D_0 D_1 q_{j0} - 2\zeta_j \omega_j D_0 q_{j0} \\ & - \Delta_m m_{D(j,r)} D_0^2 q_{r0} - m_{02(j,r,l)} q_{r0} D_0^2 q_{l0} \\ & - m_{11(j,r,l)} D_0(q_{r0}) D_0(q_{l0}) + D_0^2(q_{f0}) m_{s(j)}, \end{aligned} \tag{5a}$$

$$D_0^2 q_{f1} + \omega_k^2 q_{f1} = -2D_0 D_1 q_{j0} - 2\sigma\omega_k q_{f0}, \tag{5b}$$

with ($j = 1, 2, \cdots, 6$).

In the following subsections we consider two cases: a system without internal resonance and a case with 1:2 internal resonance between the first two modes.

3.1 System without Internal Resonance

It can be shown that for the system with length ratios 5:3:2, mass ratio $m_c/(M - m_c) = 0.4$, and the mass attached at the end of the third beam, there are no internal resonances. The numerical values of matrices m_D, m_{02}, …, and m_{s00} are set by the three parameters. For lack of space, explicit numerical elements of these matrices won't be given here.

To construct the nonlinear normal mode which tends to the k^{th} mode as $\varepsilon \to 0$, we let

$$q_{j0} = \begin{cases} A_k e^{i\omega_k T_0} + cc, & j = k, \\ 0, & j \neq k. \end{cases} \tag{6a}$$

$$q_{f0} = \frac{f}{2} e^{i\omega_k T_0} + cc, \tag{6b}$$

Substituting Eqs. (6) into Eqs. (5) and eliminating terms that will generate secular terms, we obtain

$$\begin{aligned} D_1 A_k = & -\frac{1}{2} i\omega_k \Delta_m m_{D(k,k)} A_k \\ & - \omega_k \zeta_k A_k - \frac{1}{4} i\omega_k m_{s(k)} f. \end{aligned} \tag{7}$$

We also obtain from Eqs. (5a):

$$q_{j1} = \begin{cases} q_{k1}\_\hom g + q_{k1}\_n\hom g, & j = k, \\ q_{j1}\_n\hom g, & j \neq k. \end{cases} \tag{8}$$

where $q_{k1}\_\hom$, a function of $A_k, \overline{A}_k$ and $e^{i\omega_k T_0}$, is the homogenous part of the solution of (5a), $q_{j1}\_\hom$, a nonlinear function of $A_k, \overline{A}_k$ and higher order harmonics of $e^{i\omega_k T_0}$, is the non-homogenous part of solution of (5a). In Nayfeh and Mook [1979], the homogenous part $q_{k1}\_\hom$ was always set to zero in order to make the solution form look exactly same as that obtained by other perturbation methods. Nayfeh [2005] introduced that the homogenous solution $q_{k1}\_\hom$ must be determined by an orthogonalilty condition. The explicit form of q_{j1} is again not given.

Now, substituting Eqs. (6), (7) and (8) into Eqs. (5), we obtain the dynamics of the k^{th} nonlinear normal mode. Considering the large size of this equation, here we only give the k^{th} nonlinear normal mode. We also set the damping ratio for each mode to be equal, that is $\zeta_j = \zeta$, ($j = 1, 2, \cdots, 6$).

First consider the case when $k = 1$, that is, the excitation frequency is approximately equal to the first natural frequency. Then the equation representing dynamics of the 1st nonlinear normal mode is of the form

$$\ddot{q}_{10} + \omega_1^2 q_{10} + \varepsilon\left(\omega_1\zeta_1\dot{q}_{10} - 2.08778\omega_1\Delta_m q_{10} + 1.44628\omega_1 f\cos(\omega_1 t)\right) = \ldots \qquad (9)$$

The corresponding first-order approximation of the invariant manifold, which represents the dynamics of 1st nonlinear normal mode, is obtained as

$$Q_2 = \varepsilon\left(-0.78832 f\cos(\omega_1 t) - 0.14516\Delta_m Q_1 - 3.95698 Q_1^2 + 0.062965\dot{Q}_1^2\right) + O\left(\varepsilon^2\right), \qquad (10a)$$

$$\vdots$$

$$Q_6 = \varepsilon\left(-0.63957e^{-4} f\cos(\omega_1 t) - 0.41572e^{-4}\Delta_m Q_1 - 4.4006e^{-4} Q_1^2 + 0.14885e^{-4}\dot{Q}_1^2\right) + O\left(\varepsilon^2\right). \qquad (10b)$$

Substituting the invariant manifold defined by Eqs. (10) into the first of Eqs. (1a) and ignoring the remaining equations, we obtain a one degree of freedom reduced model through nonlinear normal modes. For the one degree of freedom linear reduced model, one directly sets variables Q_2 to Q_6 to zero in Eqs. (1a). In Fig. 3 is shown a comparison between the system response with linear model reduction and with nonlinear model reduction. Clearly, the linear reduced model represents stiffened nonlinearities, while the nonlinear reduced model for the same system has overcorrected and shows softening.

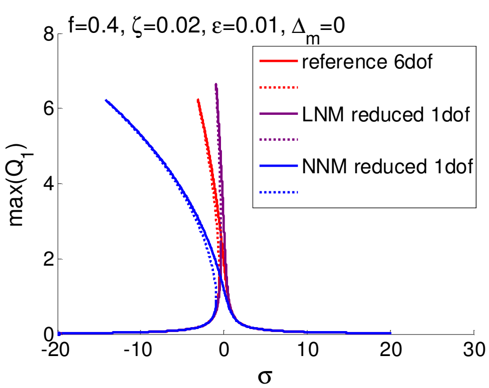

Fig. 3 Frequency response curves for resonant excitation of first mode with no internal resonance. Purple: LNM reduced 1dof system. Blue: NNM reduced 1dof system. $f = 0.4$, $\zeta = 0.02$, $\varepsilon = 0.01$, $\Delta_m = 0$.

Consider a second case with $k = 2$, that is, with excitation frequency approximately equal to the second natural frequency. Then, the equation representing the dynamics of the 2nd nonlinear normal mode has the form

$$\ddot{q}_{20} + \omega_2^2 q_{20} + \varepsilon\left(\omega_2\zeta_2\dot{q}_{20} - 6.29884\omega_2\Delta_m q_{20} + 6.16841\omega_2 f\cos(\omega_2 t)\right) = \ldots \qquad (11)$$

The corresponding first-order approximation of the invariant manifold, which represents the dynamics of 2nd nonlinear normal mode, is obtained as

$$Q_1 = \varepsilon\left(0.98798 f\cos(\omega_1 t) + 0.44376\Delta_m Q_2 - 2.16889 Q_2^2 - 0.02258\dot{Q}_2^2\right) + O\left(\varepsilon^2\right), \qquad (12a)$$

$$\vdots$$

$$Q_6 = \varepsilon\left(-1.95654e^{-4} f\cos(\omega_1 t) + 0.40464e^{-5}\Delta_m Q_2 - 4.74721e^{-3} Q_2^2 - 0.38224e^{-4}\dot{Q}_2^2\right) + O\left(\varepsilon^2\right), \qquad (12b)$$

In Fig. 4 is shown a comparison between the responses obtained with linear model reduction and nonlinear model reduction. One can again see that the linearly reduced model hardly shows any soften nonlinearities, while the nonlinearly reduced model provides extra softening characteristic.

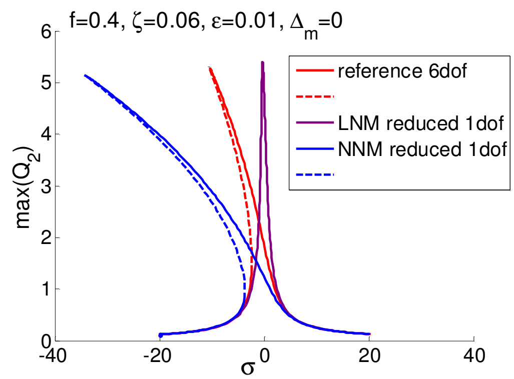

Fig. 4 Frequency response curves for resonant excitation of second mode with no internal resonance. Purple: LNM reduced 1dof system. Blue: NNM reduced 1dof system. $f = 0.4$, $\zeta = 0.06$, $\varepsilon = 0.01$, $\Delta_m = 0$.

3.2 System with 1:2 Internal Resonance

For the system with length ratios equal to 5:3:2 and attached mass at the end of the third beam, if we choose a mass ratio $m_c/(M - m_c) = 0.1539$, the system exhibits a 1:2 resonance between the first and the second linear modes. Note that the corresponding numerical values of matrices m_D, m_{02}, …, and m_{s00} will be different from those for the non-internal resonant case considered above.

For 1:2 internal resonance forced response, the model reduction for both cases, 1:2 subharmonic internal resonance and 1:2 superharmonic internal resonace, are studied.

3.2.1 The 1:2 Subharmonic Resonance

The subharmonic resonance case arises when the excitation frequency approximately equals the 2nd mode frequency or twice the 1st mode frequency. The detuning parameter σ accounts for this nearness,

$$\omega_f = 2\omega_1 + \varepsilon\sigma. \qquad (13)$$

In Eqs. (1), we also introduced a mass perturbation parameter Δ_m, and this parameter accounts for the mistuning between 1st and 2nd modes. Thus, we don't need to introduce another new parameter to denote 1:2 subharmonic resonance.

To construct solutions that reduce to a combination of the first two modes as $\varepsilon \to 0$, we let

$$q_{j0} = \begin{cases} A_1 e^{i\omega_1 T_0} + cc, & j = 1, \\ A_2 e^{i2\omega_1 T_0} + cc, & j = 2, \\ 0, & j \neq 1 \text{ or } 2. \end{cases} \tag{14a}$$

$$q_{f0} = \frac{f}{2} e^{i2\omega_1 T_0} + cc, \tag{14b}$$

Substituting Eqs. (14) into Eqs. (5) and eliminating secular terms, we obtain the slow flow equations for the resonant modes:

$$D_1 A_1 = i\omega_1 \left(-\frac{1}{2} m_{20(1,2)} - 2m_{20(1,7)} + m_{11(1,2)} \right) A_2 \overline{A}_1 - \omega_1 \left(\frac{i}{2} \Delta_m m_{D(1,1)} + \zeta \right) A_1, \tag{15a}$$

$$D_1 A_2 = -\frac{1}{4} i\omega_1 \left(m_{20(2,1)} + m_{11(2,1)} \right) A_1^2 - \frac{f}{2} i\omega_1 m_{s(2)} - \omega_1 \left(2\zeta + i\Delta_m m_{D(2,2)} \right) A_2 . \tag{15b}$$

At the same time, from Eqs. (5a) we also obtain the solutions for q_{1j}. Following the usual process, we then obtain the solutions for q_{0j} and q_{1j}, and finally the first-order approximation of invariant manifolds:

$$Q_3 = \varepsilon \big(0.85933 f \cos(\omega_1 t) - 0.16987 e^{-1} \Delta_m Q_1 - 0.14627 \Delta_m \dot{Q}_1 + 0.04263 Q_1^2 - 0.10853 Q_1 \dot{Q}_1 + 0.59589 \dot{Q}_1^2 + 0.11389 e^{-2} Q_2^2 - 0.21062 Q_2 \dot{Q}_2 + 0.33033 e^{-2} \dot{Q}_2^2 \big), \tag{16a}$$

$$\vdots$$

$$Q_6 = \varepsilon \big(0.46029 e^{-3} f \cos(\omega_1 t) - 0.10726 e^{-3} \Delta_m Q_1 - 0.35198 e^{-4} \Delta_m \dot{Q}_1 - 0.43916 e^{-3} Q_1^2 + 0.30335 e^{-3} Q_1 \dot{Q}_1 + 0.93327 e^{-2} \dot{Q}_1^2 + 0.22078 e^{-4} Q_2^2 - 0.9337 e^{-5} Q_2 \dot{Q}_2 + 0.51315 e^{-4} \dot{Q}_2^2 \big). \tag{16b}$$

In Fig. 5 is presented a comparison between the responses for the two degrees-of-freedom model obtained by linear model reduction and by nonlinear model reduction. The nonlinear reduced model through nonlinear normal modes approach also shows over softening while the linear model reduction shows stiffening more than the full model.

3.2.2 The 1:2 Superharmonic Resonance

In this case the excitation frequency approximately equals the 1st mode frequency, and thus

$$\omega_f = \omega_1 + \varepsilon\sigma . \tag{17}$$

Then, following the approach identical to that used in the subharmonic case, we get the modulation equations:

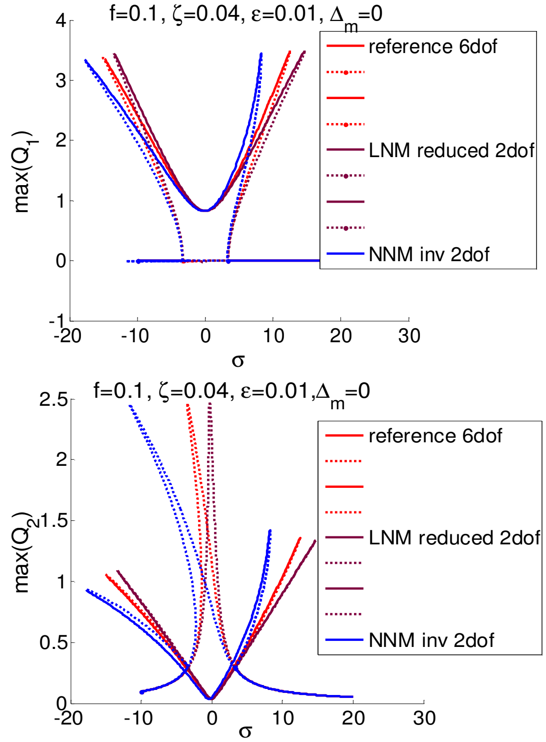

Fig. 5 Frequency response curves for subharmonic excitation with 1:2 internal resonance. Purple: LNM reduced 1dof system. Blue: NNM reduced 1dof system. $f = 0.1$, $\zeta = 0.04$, $\varepsilon = 0.01$, $\Delta_m = 0$.

$$D_1 A_1 = i\omega_1 \left(-\frac{1}{2} m_{20(1,2)} - 2m_{20(1,7)} + m_{11(1,2)} \right) A_2 \overline{A}_1 - \omega_1 \left(\frac{i}{2} \Delta_m m_{D(1,1)} + \zeta \right) A_1 - \frac{f}{4} i\omega_1 m_{s(1)}, \tag{18a}$$

$$D_1 A_2 = -\frac{1}{4} i\omega_1 \left(m_{20(2,1)} + m_{11(2,1)} \right) A_1^2 - \omega_1 \left(2\zeta + i\Delta_m m_{D(1,1)} \right) A_2 . \tag{18b}$$

The solution for q_{0j} and q_{1j} can then give us the first-order approximation of invariant manifolds for 1:2 superharmonic resonance:

$$Q_3 = \varepsilon \big(0.20263 e^{-1} f \cos(\omega_1 t) - 0.16987 e^{-1} \Delta_m Q_1 - 0.14627 \Delta_m \dot{Q}_1 + 0.04263 Q_1^2 - 0.10853 Q_1 \dot{Q}_1 + 0.59589 \dot{Q}_1^2 + 0.11389 e^{-2} Q_2^2 - 0.21062 Q_2 \dot{Q}_2 + 0.33033 e^{-2} \dot{Q}_2^2 \big), \tag{19a}$$

$$\vdots$$

$$Q_6 = \varepsilon \big(0.11493 e^{-3} f \cos(\omega_1 t) - 0.10726 e^{-3} \Delta_m Q_1 - 0.35198 e^{-4} \Delta_m \dot{Q}_1 - 0.43916 e^{-3} Q_1^2 + 0.30335 e^{-3} Q_1 \dot{Q}_1 + 0.93327 e^{-2} \dot{Q}_1^2 + 0.22078 e^{-4} Q_2^2 - 0.9337 e^{-5} Q_2 \dot{Q}_2 + 0.51315 e^{-4} \dot{Q}_2^2 \big). \tag{19b}$$

In Fig. 6 is shown a comparison between the responses obtained for the reduced models obtained via the two distinct approximations. Here again, the nonlinearly reduced model shows over softening and also a little bit smaller damping for the response.

4 Summary and Conclusion

Using the nonlinear normal modes obtained through the method of multiple time scales, we reduce a base excited nonlinear system with inertial quadratic

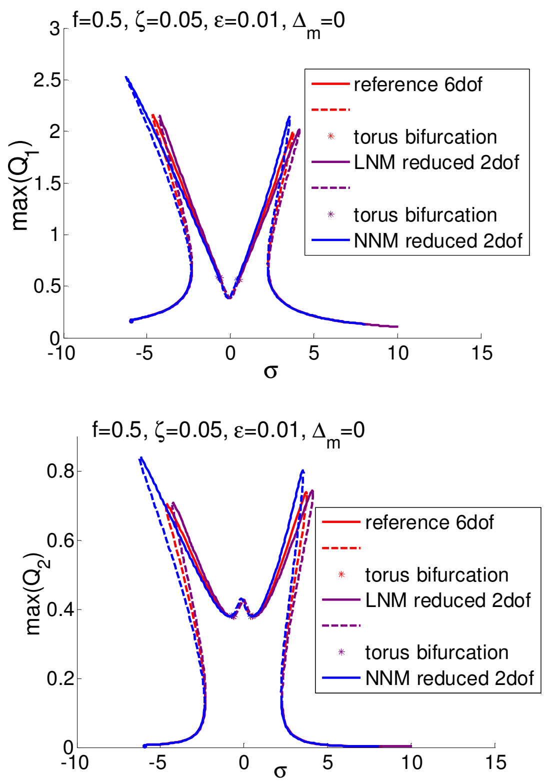

Fig. 6 Frequency response curves for superharmonic excitation and 1:2 internal resonance. Purple: LNM reduced 1dof system. Blue: NNM reduced 1dof system. $f = 0.5$, $\zeta = 0.05$, $\varepsilon = 0.01$, $\Delta_m = 0$.

nonlinearity to a one or two degree of freedom model depending on the existence of internal resonance. The frequency response for reduced models obtained using the nonlinear normal modes are compared with reduced models using linear modes as well as the full 6 degree of freedom reference model. The reduced models obtained through nonlinear normal modes show over softening of response, while the reduced models obtained through linear normal modes show stiffening of the response. For the nonlinear reduced models, there are two possible explanations to this over softening: (i) The method of multiple time scales always pushes the linear damping term in the slave coordinates to higher-order approximations. Thus, in computing the invariant manifolds by first-order approximation, these linear damping terms are neglected or under estimated. (ii) Our system is with quadratic inertial nonlinearities unlike most other studies that focus on geometric nonlinearities. The reasons of this over soften nonlinearities for nonlinear reduced model will be studied in more detail in our future work.

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