A NEW ITERATIVE LEARNING CONTROL SCHEME FOR LINEAR TIME-VARYING DISCRETE SYSTEMS

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Abstract: In this paper we use a repetitive process setting to develop a new iterative learning control algorithm for plants modelled by a discrete linear time-varying state space model. As the next step in evaluating its performance, the results of a simulation based study are given where the plant model used is that obtained from frequency responses tests on a gantry robot.

Keywords: Linear repetitive process, iterative learning control

1. INTRODUCTION

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive (or pass-to-pass) mode with the requirement that a reference trajectory $r(t)$ defined over a finite interval $0 \leq t \leq T$ is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task to high precision, chemical batch processes or, more generally, the class of tracking systems.

Since the original work (Arimoto *et al.*, 1984) in the mid 1980's, the general area of ILC has been the subject of intense research effort. One possible initial source for the literature here is the survey paper (Hyo-Sung Ahn *et al.*, 2007). One approach in ILC is to construct the input to the plant or process from the input used on the last trial plus an additive increment which is typically a function of the past values of the measured output error, i.e. the difference between the achieved output on the current pass and the desired plant output. As such, it places the analysis of ILC schemes firmly outside standard (or 1D) control theory — although it is still has a significant role to play in certain cases of practical interest.

One approach to ILC is to use 2D systems theory where the directions of information propagation are from trial-to-trial and along the trials respectively. As the trial length is finite, this makes the theory of repetitive processes (see the references in (Rogers and Owens, 1992)) a natural setting for analysis and control law design. This is basis for the analysis in this paper and note that repetitive processes (unlike some other 2D systems models) have many control related applications.

In this paper, we first show how ILC schemes can be designed for a class of discrete linear time-varying systems by, in effect, extending tech-

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niques developed for 1D systems, e.g. (Siljak and Stipanovic, 2000) using the framework of linear repetitive processes. As a first step in evaluating its performance, the results of a simulation based study (for time-invariant dynamics) are given where the plant model has been constructed from frequency response tests on a gantry robot which has also been extensively used in other ILC research where experimental verification of the resulting control algorithms is also undetaken — see, for example, (Ratcliffe *et al.*, 2006)).

Throughout this paper, the symbols $M \succ 0$, $M \prec 0$, $M \succeq 0$, $M \preceq 0$ are used to denote symmetric matrices which are positive definite, negative definite, positive semi-definite and negative semi-definite respectively.

2. BACKGROUND AND INITIAL ANALYSIS

The discrete linear time-varying systems considered in this paper are described by the following state-space model

$$\begin{aligned} x(p+1) &= Ax(p) + \mu E[p]Hx(p) + Bu(p) \\ y(p) &= Cx(p) \\ p &= 0,1,\ldots,\alpha-1 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^m$ denotes the state, input and output vectors respectively. The initial state state vector is taken as $x(0) = d_0$. Moreover, we deal with the case when these systems operate over a finite duration α. In the time-varying term $\mu E[p]Hx(p)$, μ is a constant positive scalar, the normalizing matrix $H \in \mathbb{R}^{h\times n}$ has constant entries and it is $E[p] \in \mathbb{R}^{n\times h}$ which brings in the time-varying dynamics. This last matrix is assumed to satisfy

$$E[p]^T E[p] < I \quad \forall p = 0,1,\ldots,\alpha-1 \tag{2}$$

To introduce the ILC setting, we use the integer subscript $k \geq 0$ to denote the current trial and rewrite the model of (1) as

$$\begin{aligned} x_k(p+1) &= (A+\mu\Psi(p))x_k(p) + Bu_k(p) \\ y_k(p) &= Cx_k(p) \end{aligned} \tag{3}$$

where

$$\Psi(p) = E[p]H \tag{4}$$

Here we aim to control the ILC dynamics by a linear controller and it is also required that the transient behavior is 'acceptable'. Hence a first major question to be asked now is: how 'large' can the time varying component be before a linear control law cannot be used. As argued below, this is equivalent to obtaining the upper bound on the acceptable values of $\mu > 0$ in (1).

Consider now a so-called discrete linear repetitive process (Rogers and Owens, 1992) described by the following state-space model over $p = 0,1,\cdots,p-1, k \geq 1$

$$\begin{aligned} x_k(p+1) &= \hat{A}x_k(p) + \hat{B}u_k(p) + \hat{B}_0 y_{k-1}(p) \\ y_k(p) &= \hat{C}x_k(p) + \hat{D}u_k(p) + \hat{D}_0 y_{k-1}(p) \end{aligned} \tag{5}$$

where (on pass k) $x_k(p) \in \mathbb{R}^n$, $u_k(p) \in \mathbb{R}^r$, $y_k(p) \in \mathbb{R}^m$ are the state, input and pass profile vectors respectively. Consider also (3) in the case when $\mu\Psi(p) = 0$ and define the error on trial k as

$$e_k(p) = y_r(p) - y_k(p) \tag{6}$$

where $y_r(p)$ denotes the reference signal to be learnt. Then a known result is that, in its strongest form, convergence of the ILC scheme is equivalent to the stability property of linear constant pass length repetitive processes known as stability along the pass. In the repetitive process case, this is equivalent to uniform bounded input bounded output stability (defined in terms of the norm on the underlying function space), i.e. independent of the pass length.

We now develop results which are used in the next section to extend this last result to the case when $\mu\Psi(p) \neq 0$. To begin, first rewrite the first equation of the model of (3) in the form

$$x_k(p) = (A+\mu\Psi(p-1))x_k(p-1) + Bu_k(p-1)$$

and introduce

$$\begin{aligned} \eta_{k+1}(p+1) &= x_{k+1}(p) - x_k(p) \\ \Delta u_{k+1}(p) &= u_{k+1}(p) - u_k(p) \end{aligned} \tag{7}$$

Then we have

$$\begin{aligned} \eta_{k+1}(p+1) = (A + \mu\Psi(p-1))\eta_{k+1}(p) \\ + B\Delta u_{k+1}(p-1) \end{aligned}$$

Consider also a control law of the form

$$\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2 e_k(p+1) \tag{8}$$

and hence

$$\begin{aligned} \eta_{k+1}(p+1) = (A + \mu\Psi(p-1) + BK_1)\eta_{k+1}(p) \\ + BK_2 e_k(p) \end{aligned} \tag{9}$$

Also $e_{k+1}(p) - e_k(p) = y_k(p) - y_{k+1}(p)$ and we then obtain

$$\begin{aligned} &e_{k+1}(p) - e_k(p) = Cx_k(p) - Cx_{k+1}(p) \\ &= C(A + \mu\Psi(p-1))(x_k(p-1) - x_{k+1}(p-1)) \\ &\qquad + CB(u_k(p-1) - u_{k+1}(p-1)) \end{aligned}$$

Note also that the time varying term does not depend on k and hence (using (7)) and taking into account (8) we obtain

$$\begin{aligned} &e_{k+1}(p) = -C(A + BK_1)\eta_{k+1}(p) \\ &- C\mu\Psi(p-1)\eta_{k+1}(p) + (I - CBK_2)e_k(p) \end{aligned} \tag{10}$$

Introduce now

$$\begin{aligned} \hat{A} &= A + BK_1 & \hat{B}_0 &= BK_2 \\ \hat{C} &= -C(A+BK_1) & \hat{D}_0 &= (I - CBK_2) \\ \hat{\Psi}(p) &= \Psi(p-1) & \hat{\Upsilon}(p) &= C\Psi(p-1) \end{aligned} \tag{11}$$

Then clearly (9) and (10) can be written as

$$\begin{aligned} \eta_{k+1}(p+1) &= [\hat{A} + \mu\hat{\Psi}(p)]\eta_{k+1}(p) + \hat{B}_0 e_k(p) \\ e_{k+1}(p) &= [\hat{C} - \mu\hat{\Upsilon}(p)]\eta_{k+1}(p) + \hat{D}_0 e_k(p) \end{aligned} \tag{12}$$

which is of the form (5) and hence the repetitive process stability theory can be applied.

3. MAIN RESULT

In order to prove the main result, we will require the well known Schur's complement formula and also the following result (whose proof is also well known).

Lemma 1. Assume that Σ, Θ, Ξ, Π and $\bar{\Delta}$ are real matrices, with Ξ symmetric and $\Pi \succ 0$, such that $\bar{\Delta}^T \Pi \bar{\Delta} \preceq \Pi$. Then

$$\Sigma + \Theta\bar{\Delta}\Xi + \Xi^T\bar{\Delta}^T\Theta^T \prec 0 \tag{13}$$

if and only if there exists a scalar $\lambda > 0$ satisfying

$$\Sigma + \lambda\Theta\Pi\Theta^T + \lambda^{-1}\Xi^T\Pi\Xi \prec 0$$

Theorem 1. The ILC scheme described by (12) is stable along the pass for all time-varying uncertainties satisfying (2) if there exist $R_1 \succ 0$, $R_2 \succ 0$, $X_1 \succ 0$, $X_2 \succ 0$ and a scalar $\lambda > 0$ such that the following Generalized Eigenvalue Problem (GEVP) is feasible

minimize; $\eta > 0$
subject to:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & CC^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \prec \eta\Omega$$

where

$$\Omega = \begin{bmatrix} X_1 & 0 & -X_1A^T - R_1^TB^T & X_1A^TC^T + R_1^TB^TC^T & -X_1H^T \\ 0 & X_2 & -R_2^TB^T & -X_2 + R_2^TB^TC^T & 0 \\ -AX_1 - BR_1 & -BR_2 & X_1 & 0 & 0 \\ CAX_1 + CBR_1 & -X_2 + CBR_2 & 0 & X_2 & 0 \\ -HX_1 & 0 & 0 & 0 & 0.5\lambda I \end{bmatrix}$$

and $\Omega \succ 0$. Also an upper bound for the parameter μ in the time-varying uncertainty description is given by

$$\mu_{\max} = \sqrt{\frac{1}{\eta\lambda}} \tag{14}$$

If this condition holds, the control law matrices K_1 and K_2 are given by $K_1 = R_1X_1^{-1}$ and $K_2 = R_2X_2^{-1}$.

Proof:
Numerous conditions for stability along the pass exist but here we use the co-called 2D Lyapunov equation approach (Rogers and Owens, 1992) and hence (12) is stable along the pass if there exists $P = \text{diag}\{P_1, P_2\} \succ 0$ such that

$$\Phi[p]^TP\Phi[p] - P \prec 0 \; \forall p = 0, 1, \ldots, \alpha - 1 \tag{15}$$

where

$$\Phi[p] = \begin{bmatrix} \hat{A} + \mu\hat{\Psi}(p) & \hat{B}_0 \\ \hat{C} - \mu\hat{\Upsilon}(p) & \hat{D}_0 \end{bmatrix}$$

Now apply the Schur's complement formula to (15) with $W = -P$, $L = \Phi[p]$ and $V = P$, then set $X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$ and finally pre- and post multiply the result from this second step by $\text{diag}\left\{ X_1,\ X_2,\ I,\ I \right\}$ to obtain

$$M = \begin{bmatrix} -X_1 & 0 & X_1\hat{A}^T + X_1\mu\hat{\Psi}(p)^T & X_1\hat{C}^T - X_1\mu\hat{\Upsilon}(p)^T \\ 0 & -X_2 & X_2\hat{B}_0^T & X_2\hat{D}_0^T \\ \hat{A}X_1 + \mu\hat{\Psi}(p)X_1 & \hat{B}_0X_2 & -X_1 & 0 \\ \hat{C}X_1 - \mu\hat{\Upsilon}(p)X_1 & \hat{D}_0X_2 & 0 & -X_2 \end{bmatrix} \prec 0 \tag{16}$$

The problem now is that (16) contains the time-varying terms $\hat{\Psi}(p)$ and $\hat{\Upsilon}(p)$ and hence severe design difficulties. What we require here is a control law design algorithm which ensures stability along the pass under the constraint (2) and also gives the bound on the maximum value of μ allowed. Next show how this outcome can be obtained using the results of Lemma 1.

First write M of (16) as

$$M = \Sigma + F = \begin{bmatrix} -X_1 & 0 & X_1\hat{A}^T & X_1\hat{C}^T \\ 0 & -X_2 & X_2\hat{B}_0^T & X_2\hat{D}_0^T \\ \hat{A}X_1 & \hat{B}_0X_2 & -X_1 & 0 \\ \hat{C}X_1 & \hat{D}_0X_2 & 0 & -X_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & X_1\mu\hat{\Psi}(p)^T & -X_1\mu\hat{\Upsilon}(p)^T \\ 0 & 0 & 0 & 0 \\ \mu\hat{\Psi}(p)X_1 & 0 & 0 & 0 \\ -\mu\hat{\Upsilon}(p)X_1 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Using (4) and (11), F in (17) can now be written as $F = \Theta\bar{\Delta}\Xi + \Xi^T\bar{\Delta}^T\Theta^T$, where

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu I & 0 \\ 0 & 0 & 0 & -\mu C \end{bmatrix}, \Pi = I,$$
$$\bar{\Delta} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ E[p-1] & 0 \\ -E[p-1] & 0 \end{bmatrix}, \Xi = \begin{bmatrix} HX_1 & 0 & 0 & 0 \\ HX_1 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Moreover, the requirement that $\bar{\Delta}^T\Pi\bar{\Delta} \preceq \Pi$ (see Lemma 1) holds due to the assumption (2).

Return now to $M = \Sigma + \lambda\Theta\Pi\Theta^T + \lambda^{-1}\Xi^T\Pi\Xi$. Then substituting (18) into (13) and using (11) we obtain (after some routine manipulations which are omitted here)

$$M = \begin{bmatrix} -X_1 & 0 & X_1A^T + X_1K_1^TB^T & -X_1A^TC^T - X_1K_1^TB^TC^T \\ 0 & -X_2 & X_2K_2^TB^T & X_2 - X_2K_2^TB^TC^T \\ AX_1 + BK_1X_1 & BK_2X_2 & -X_1 + \lambda\mu^2 I & 0 \\ -CAX_1 - CBK_1X_1 & X_2 - CBK_2X_2 & 0 & -X_2 + \lambda\mu^2CC^T \end{bmatrix}$$

$$
+ \begin{bmatrix} X_1 H^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2\lambda^{-1} I \end{bmatrix} \begin{bmatrix} HX_1 & 0 & 0 & 0 \end{bmatrix} \prec 0
$$

or, on noting the fact that M can be written as $M = \hat{W} + \hat{L}^T \hat{V} \hat{L}$ and then applying the Schur's complement formula,

$$
\begin{bmatrix}
-X_1 & 0 \\
0 & -X_2 \\
AX_1 + BK_1X_1 & BK_2X_2 \\
-CAX_1 - CBK_1X_1 & X_2 - CBK_2X_2 \\
HX_1 & 0
\end{bmatrix}
$$

$$
\begin{matrix}
X_1A^T + X_1K_1^TB^T \\
X_2K_2^TB^T \\
-X_1 + \lambda\mu^2 I \\
0 \\
0
\end{matrix}
$$

$$
\left.
\begin{matrix}
-X_1A^TC^T - X_1K_1^TB^TC^T & X_1H^T \\
X_2 - X_2K_2^TB^TC^T & 0 \\
0 & 0 \\
-X_2 + \lambda\mu^2 CC^T & 0 \\
0 & -0.5\lambda I
\end{matrix}
\right] \prec 0 \qquad (19)
$$

The final step is to obtain the GEVP problem (which is suitable for numerical calculations). This follows immediately on setting $R_1 = K_1X_1$, $R_2 = K_2X_2$ and $\gamma = \lambda\mu^2$ in (19), multiplying both sides of resulting matrix inequality by $\eta = \gamma^{-1}$, and then rearranging the outcome of these steps to obtain the GEVP problem of the theorem and also (14). This completes the proof. ■

The condition of Theorem 1 is sufficient but not necessary and hence could lead to an over-conservative design. However, unlike all other known necessary and sufficient conditions for stability along the pass of discrete linear repetitive processes, it does lead directly to control law design. An obvious topic for further research is to obtain less conservative versions of this result.

4. A SIMULATION EXAMPLE

Other work, e.g. (Ratcliffe *et al.*, 2006) has used a gantry robot facility to experimentally verify ILC designs. These results are based on approximate linear models of the dynamics of each axis obtained by frequency response tests. Here we use the X axis model which with the matched pole-zero method and a sampling period of $T_s = 0.005[s]$ yields the following z-transfer function description of the dynamics

$$
G_x(z) = \frac{14.8651z(z - 0.08201)}{(z-1)(z^2 - 1.738z + 0.8869)} \cdot \frac{(z^2 - 1.676z + 0.9485)(z^2 - 1.25z + 0.8959)}{(z^2 - 1.44z + 0.8986)(z^2 + 0.9389z + 0.4979)} \qquad (20)
$$

and hence the state-space model matrices

$$
A = \begin{bmatrix}
-0.469 & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.277 & -0.469 & -0.377 & -0.742 & 0.676 & 2.03 & 1.62 \\
0 & 0 & 0.72 & 1 & 0 & 0 & 0 \\
0 & 0 & -0.38 & 0.72 & 0.215 & 0.645 & 0.515 \\
0 & 0 & 0 & 0 & 0.869 & 1 & 0 \\
0 & 0 & 0 & 0 & -0.132 & 0.869 & 1.32 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 3.28 & -5.03 & -0.868 & -1.71 & 1.55 & 4.66 & 3.72 \end{bmatrix}
$$

$$
D = 0
$$

At this stage, the theory developed in this paper can be used to design an ILC control law for the case when unmodelled time varying dynamics are present which can be modelled in the form of (3). Space limitations prohibit a full investigation and hence we only give results for the case when

$$
H = \begin{bmatrix}
0.123 & 0.933 & 0.00624 & 0.096 & 0.909 & 0.0978 & 0.652 \\
0.591 & 0.126 & 0.3 & 0.424 & 0.574 & 0.034 & 0.445 \\
0.297 & 0.658 & 0.763 & 0.939 & 0.329 & 0.173 & 0.485
\end{bmatrix}
$$

The control law matrices in this case are

$$
K_1 = \begin{bmatrix} -0.0371 & -0.189 & -0.0645 & -0.0605 & 0.131 & 0.17 & -0.225 \end{bmatrix}
$$

$$
K_2 = 0.00847
$$

and the maximum allowed nonlinearity parameter μ is 0.0014.

Suppose now that the trial length is $\alpha = 400$ (in the case of the gantry robot this would correspond to a trial duration of 2 seconds) and the reference signal is given by Figure 1. Moreover, the term

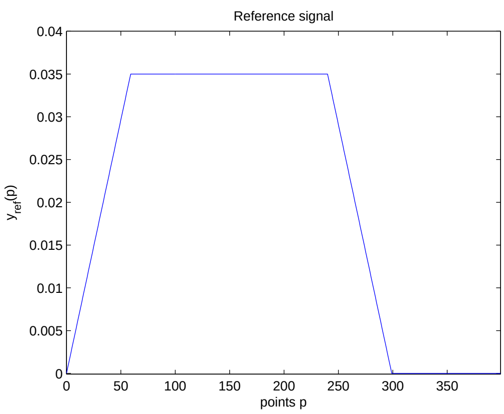

Fig. 1. Reference signal

$E(p)$ is taken as pseudo-random (uniform distribution with values ranging from 0 to 1).

Figure 2 shows the response of the controlled process and by trial $k = 30$ the error has become very small.

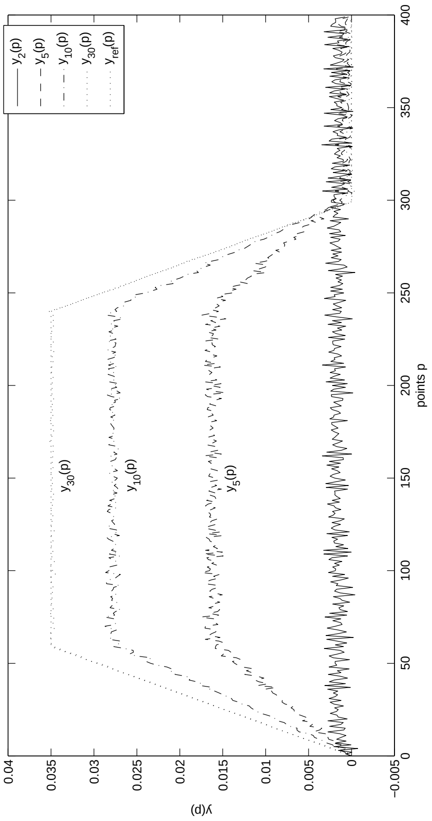

Fig. 2. Evolution of the controlled dynamics

5. CONCLUSIONS

This paper has developed a new design method for ILC for the case of systems described by linear time-varying models. These have been developed by formulating the ILC design problem in the linear repetitive process setting. These results are the first which extend the repetitive process approach to ILC design beyond the case of linear time invariant plant dynamics and much further research is required before its effectiveness can be fully assessed. This is the subject of ongoing work and will be reported on in due course.

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