MOTION PLANNING AND DIFFERENTIAL FLATNESS

Cutberto Romero-Meléndez
Basic Sciences Department
Universidad Autónoma Metropolitana-Azcapotzalco
México
cutberto@correo.azc.uam.mx

Abstract
We study the motion planning problem for a under actuated vibratory mechanical system and for the nonholonomic car-like a robot with a trailer, by means of the differential flatness and the sub-Riemann nilpotent approximation approaches, respectively. We show two techniques for the solution of the motion planning problem.

Key words
Motion planning, differential flatness based control, nilpotent approximation.

1 Introduction

The motion planning problem has been extensively studied and it's a relevant problem both theoretically and in applications in areas such as robotics and constructive controllability. In general, a robot is completely described by a non-linear system over a n-dimensional manifold $\mathcal{X}^n$

$$\dot{x} = f(x, u), \quad x \in \mathcal{X}^n, \quad u \in \mathcal{U}^m \quad m \leq n \tag{1}$$

with nonholonomic constraints given by a r-distribution Δ over the manifold $\mathcal{X}^n$. Roughly speaking, the motion planning problem consists in finding a collision-free admissible path for the system, for steering the robot from an initial position and velocity, to a goal position and velocity. Moreover, we can request for the trajectory to have an optimal cost. In the control theory formalism, the motion planning problem is formulated through a controllable control system, together with an arbitrary non-admissible but feasible (collision-free) trajectory, determined, for instance, by computational geometric methods. The motion planning reduces then to the design of control strategies approximating the reference curve by means of admissible curves within appropriate tubular neighborhoods. We look the trajectories like $(x(t), u(t))$, where $x(t)$ is a feasible trajectory and $u(t)$ is an open-loop control generating $x(t)$. The solution of the motion planning problem allows the planification of the robot's trajectories to avoid obstacles.

We will consider two systems.

Firstly, we consider a vibratory system inspired in a robotic mechanism, called the Elasto-Robot, consisting of a prismatic pair coupled with a revolute and containing an oscillating end-effector (Figure 1). Here we follow the approach of non-linear control of closed loop systems, and more specifically we use the so-called flatness techniques which describe the control systems whose trajectories can be parameterized by a finite number of functions and their time-derivatives.

The concept of flat differential systems finds its mathematical foundations in D. Hilbert's 22th problem about the uniformization of analytic relations by means of meromorphic functions [Hilbert, 1902] and the equivalence method for differential systems of E. Cartan [Cartan, 1914]. That is a technique in differential geometry for determining whether two geometrical structures are the same up to a diffeomorphism. The equivalence method is an essentially algorithmic procedure that has been successfully applied in differential geometry and control theory. More recently flat differential systems have been extensively studied within the non-linear control literature, see for instance M. Fliess et al. [Fliess, Lévine, Martin, Martin, Rouchon, 1992] and P. Rouchon treatment of control of oscillators [Rouchon, 2005].

Secondly, we consider the archetype system nonholonomic car-like robot towing a trailer. We are interested in the trajectories for a car with a trailer. More exactly, we are interested in the optimal trajectories (the car's trajectories which minimize the energy and time), admissible, joining two points given of the configuration space. The configuration in the space $\mathbb{R}^2 \times S \times S$ of the nonholonomic car is given by the position $(x, y) \in \mathbb{R}^2$ of the mid-point of the rear wheels, the angle θ between the main direction of the car and the X-axis, and the angle φ between the front wheels and the X-axis. The controls u and v allow to the car displacements forward and backward, and turning, respectively. There are two constraint nonholonomics, displacements forward and backward, and displacements without to slide. The model is given by the following system:

$$\begin{aligned}\dot{x} &= u(t)\cos\theta \\ \dot{y} &= u(t)\sin\theta \\ \dot{\theta} &= u(t)\tan\varphi \\ \dot{\varphi} &= v(t)\end{aligned}$$

Although is possible to show that the system is differentially flat following a coordinates change, we prefer exploit the geometrical techniques to obtain the nilpotent approximation, close to the chained form.

2 Differentially flat systems

In this section we present the main definitions concerning flatness, we restrict ourselves to the basic statements leaving aside formal demonstrations, we refer the reader to the book [Sira-Ramírez and Agrawal, 2004].

A differential system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \leq n$$

is said to be differentially flat if there is a vector $y \in \mathbb{R}^m$ such that

1. $y, \dot{y}, \ddot{y}, \ldots$ are linearly independent: they are not related by any differential equation.
2. y is a function of x and a finite number of derivatives of u
3. There are two smooth maps Θ and Ψ such that

$$x = \Theta(y, \dot{y}, \ldots, y^{(\alpha)}), \quad u = \Psi(y, \dot{y}, \ldots, y^{(\alpha+1)}),$$

for certain multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ and

$$y^{(\alpha)} = \left(\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \ldots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}}\right)$$

Roughly speaking, a control system is flat if we can find functions (flat outputs) of the state and control variables and their time-derivatives, so that the state and the control can be expressed in terms of that flat outputs and their derivatives. By consequence, the trajectories for y can be chosen freely.

3 Flatness and motion planning

Given the system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \leq n \tag{2}$$

and two configurations x_I, x_F in the space $\mathbb{R}^n$, the motion planning problem consists in finding an admissible trajectory $t \mapsto (x(t), u(t))$, $t \in [t_I, t_F]$ for the system (2), connecting those configurations, avoiding obstacles and with low cost.

For differentially flat systems it is possible to generate admissible paths joining two given states since there is a smooth 1-1 correspondence between solutions $x(t)$ of the system and the functions $y(t)$. Also, it suffices to control the flat outputs to control the whole system.

Once the terminal conditions over $x(t)$ and $u(t)$ are given, through the surjectivity of the mappings Θ and Ψ between sufficiently smooth trajectories of the output and feasible trajectories of the system, we can find a trajectory $t \mapsto y(t)$, sufficiently differentiable that satisfies the corresponding conditions for the flat output. To find a trajectory of the flat output satisfying the conditions

$$\left.\begin{aligned} y(t_I) = y_I, \quad \dot{y}(t_I) = 0, \quad \cdots y^{(r+1)}(t_I) = 0 \\ y(t_F) = y_F, \; \dot{y}(t_F) = 0, \quad \cdots y^{(r+1)}(t_F) = 0 \end{aligned}\right\} \tag{3}$$

we construct $(2r+3)$ *th* degree interpolation polynomials for the reference trajectories y_i of each variable of the flat output y:

$$\eta(t) = \eta_I + (\eta_I - \eta_F)\left(\frac{t-t_I}{t_F-t_I}\right)^{r+2} \sum_{j=0}^{r+1} a_j \left(\frac{t-t_I}{t_F-t_I}\right)^j \tag{4}$$

where $\eta_I = \eta(t_I)$, $\eta_F = \eta(t_F)$ and the coefficients a_j are independent of t_I, t_F, $\eta(t_I)$, $\eta(t_F)$ and [Levine, 2009] satisfy $r+2$ linear equations in $r+2$ unknown coefficients

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r+2 & r+3 & & 2r+3 \\ \vdots & \vdots & & \vdots \\ (r+2)! & \frac{(r+3)!}{2} & \cdots & \frac{(2r+3)!}{(r+2)!} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{r+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5}$$

We note that the above linear system always has a unique solution because the matrix have all its columns independent.

4 The Elasto-Robot

The Elasto-Robot is a mechanism consisting of a circular base body, which can perform freely two movements: rotation and translation, and a prismatic pair coupled with a revolute and containing an vibratory element in the end-effector, moving on a horizontal plane. The motion planning problem for this systems is to move the robot between any given initial and final configurations such that the vibrating can be controlled, see Figure 1. The parameters involved in this

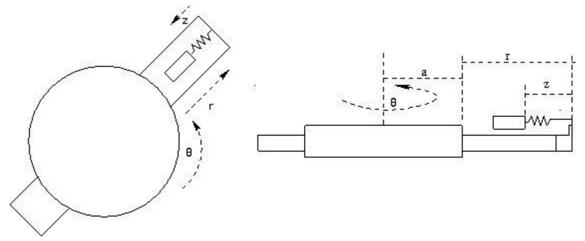

Figure 1. Robot with vibratory end-effector

model are the following: a is the disk radius of the circular base body; θ is the angular displacement of the circular base body; r is the parallel displacement of the end-effector arm; m_2 is the prismatic-pair mass; m_3 is the terminal-effector mass and z is the coordinate associated to the vibration. the base body have mass negligible, κ denotes the spring constant associated to the vibration and the rotational inertia I. The torque forces $(u, v) = (\tau_1, \tau_2)$ are control parameters. We consider the kinetic and potential energies for the revolute, prismatic pair and the terminal-effector, so the Lagrangian $\mathcal{L} = \mathcal{L}(\theta, r, z, \dot{\theta}, \dot{r}, \dot{z})$ of the system is

$$\left.\begin{array}{c} \mathcal{L} = I\dot{\theta}^2 + (m_2 + m_3)\dot{r}^2 + (m_2 + m_3)r^2\dot{\theta}^2 \\ +m_3\dot{z}^2 - r^2\kappa - z^2\kappa + 2rz\kappa \end{array}\right\} \quad (6)$$

By writing the Euler-Lagrange equations

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_1 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \tau_2 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0, \end{array}\right\} \quad (7)$$

defining the state variables

$$\begin{array}{l} x_1 = \theta, \;\; x_4 = \dot{x}_1, \\ x_2 = r, \;\; x_5 = \dot{x}_2, \\ x_3 = z, \;\; x_6 = \dot{x}_3, \end{array}$$

for the coordinates $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ in the manifold

$$\mathcal{M} = (0, 2\pi) \times (0, R) \times (0, Q) \times (0, 2\pi) \times (0, R) \times (0, Q)$$

for certain fixed values for R and Q, and by setting

$$I + m_2 x_2^2 + m_3(x_2 - x_3)^2 = J$$

with $J > I$, assuming $J = 1$, $m_2 = 1$, $a^2 - x^2 > 0$, with $a = \sqrt{J - I}$, we obtain ([Monroy-Romero, 2011]) the following non-linear control system

$$\dot{x} = X_0(x) + X_1(x)u + X_2(x)v, \quad (8)$$

for the drift vector field

$$X_0 = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -2x_2x_4x_5 - 2\sqrt{m_3}x_4(x_5 - x_6)\sqrt{a^2 - x_2^2} \\ x_2x_4^2 \\ \frac{\kappa}{\sqrt{m_3^3}}\sqrt{a^2 - x_2^2} + x_3x_4^2 \end{pmatrix}$$

and $X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ the control vector fields.

5 Flatness of the model

We know [Monroy-Romero, 2012] that the Elasto-robot is flat and the position $(\theta, r) = (x_1, x_2)$ of the base body and the end-effector arm is a flat output:

$$y = (y_1, y_2) = (x_1, x_2). \quad (9)$$

We get

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ x_3 &= y_2 - \frac{1}{m_3}\sqrt{a^2 - y_2^2} \\ x_4 &= \dot{y}_1 \\ x_5 &= \dot{y}_2 \\ x_6 &= \frac{\ddot{y}_1 - \ddot{y}_2 + 2y_2\dot{y}_1\dot{y}_2 + y_2y_1^2}{2\sqrt{m_3}\dot{y}_1\sqrt{a^2 - y_2^2}} + \dot{y}_2 \\ u &= \ddot{y}_1 - 4y_2\dot{y}_1\dot{y}_2 \\ v &= \ddot{y}_2 - y_2\dot{y}_1^2 \end{aligned} \quad (10)$$

The applied transformation

$$\begin{aligned} x &= \Theta(y, \dot{y}, \ddot{y}) \\ u &= \phi(y, \dot{y}, \ddot{y}) \\ v &= \psi(y, \dot{y}, \ddot{y}) \end{aligned} \quad (11)$$

is invertible, so y is a flat output of the Elasto-robot.

Now, we illustrate the solution of the the motion planning problem for the Elasto-Robot, in order to prevent

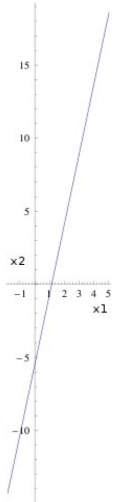

Figure 2. Motion of the point (x_1, x_2) for the Elasto-robot.

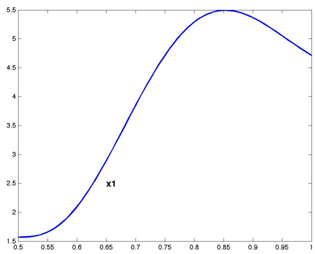

Figure 3. Reference trajectory for angle θ for the Elasto-robot.

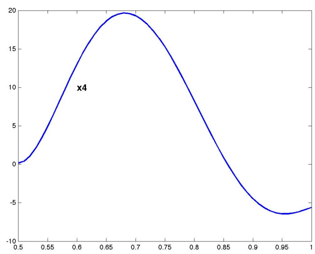

Figure 4. Coordinate x_4, angular velocity for the Elasto-robot.

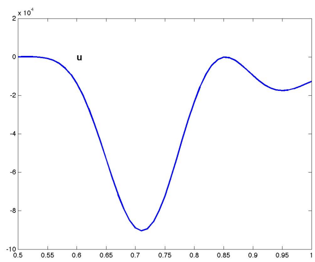

Figure 5. Control input u for the Elasto-robot.

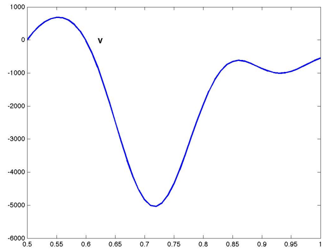

Figure 6. Control input v for the Elasto-robot

vibrations of the small mass. In this case we have the constraints

$$\left.\begin{array}{l} y(t_I) = y_I, \quad \dot{y}(t_I) = 0, \quad \ddot{y}(t_I) = 0 \\ y(t_F) = y_F, \; \dot{y}(t_F) = 0, \quad \ddot{y}(t_F) = 0 \end{array}\right\} \tag{12}$$

and the reference trajectory is

$$x_i(t) = x_i^I - (x_i^I - x_i^F)(\frac{t-t_I}{t_F-t_I})^4 \sum_{j=0}^{3} a_j (\frac{t-t_I}{t_F-t_I})^j \tag{13}$$

for the variables x_i, $i = 1,2$ of the flat output $y = (x_1, x_2)$, where $x_i^I = x_i(t_I)$, $x_i^F = x_i(t_F)$. The values of the coefficients a_j are $a_0 = 35$, $a_1 = -84$, $a_2 = 70$ and $a_3 = -20$. Then, by using the interpolation polynomials (13) for each variable x_i, $i = 1,2$ of the flat output $y = (x_1, x_2)$, we obtain, as solution for the motion planning problem for the Elasto-Robot, connecting the two rest-to-rest configurations $x_1^I = 0.5$, $x_1^F = \pi/2$, $x_2^I = 0.5$ and $x_2^F = 2$, a straight line trajectory for the point (x_1, x_2) (Figure 2). Figure 3 illustrates our reference trajectory solution $x_1(t)$. Figure 4 shows the angular velocity of the Elasto-robot. Figure 5 and Figure 6 show the control outputs.

6 Car-like robot towing a trailer

The model for the system is:

$$\begin{aligned} \dot{x} &= u(t)\cos\theta \\ \dot{y} &= u(t)\sin\theta \\ \dot{\theta} &= u(t)\tan\varphi \\ \dot{\varphi} &= v(t) \end{aligned}$$

Following the coordinates change

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= \tan\theta \\ x_4 &= \tan\varphi\sec\theta \\ u_1 &= \cos\varphi\cos\theta \\ v_1 &= 3u\sec^3\theta\sin\varphi\tan\varphi\tan\theta + v\sec^3\theta\sec^2\varphi \end{aligned}$$

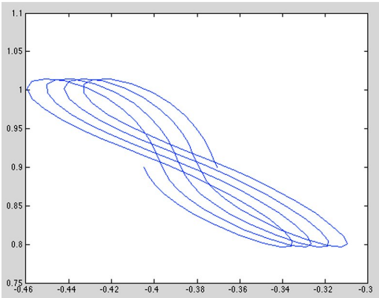

Figure 7. Motion of the point (x, y) of the car.

we obtain the chained form for the system

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= x_3 u_1\\ \dot{x}_3 &= x_4 u_1\\ \dot{x}_4 &= u_1\end{aligned}$$

and then, the flat outputs are $(y_1, y_2) = (x_1, x_2)$. However, rather than dealing with the flat system, we will use the nilpotent approximation, based on [Berret, 2006]. If we consider φ as the angle measured between the front wheels and the main direction of the car, then the control system is

$$\dot{x} = Xu + Yv$$

where the coordinates of x and vector fields are

$$x = \begin{pmatrix} x\\ y\\ \theta\\ \varphi \end{pmatrix}, \; X = \begin{pmatrix} \cos\theta\\ \sin\theta\\ 0\\ -\sin\varphi \end{pmatrix}, \; Y = \begin{pmatrix} 0\\ 0\\ 1\\ 1 \end{pmatrix}.$$

We compute the Lie's brackets $Z = [X, Y]$ and $W = [X, Z]$:

$$Z = \begin{pmatrix} \sin\theta\\ -\cos\theta\\ 0\\ \cos\varphi \end{pmatrix}, W = \begin{pmatrix} 0\\ 0\\ 0\\ 1 \end{pmatrix}.$$

So, $\{X, Y, Z, W\}$ is a basis of the tangent space at every point (x, y, θ, φ) of the space $\mathbb{R}^2 \times S \times S$.

We compute the nilpotent approximation for the system near from one reference parametrized trajectory $\Gamma = \begin{pmatrix} 0\\ 0\\ 0\\ t \end{pmatrix}$.

The nilpotent approximation along Γ is obtained by means of normal coordinates by taking in the Taylor expansions of X and Y, the terms of homogeneous degree -1. We get

$$\dot{x} = \hat{X}u + \hat{Y}v$$

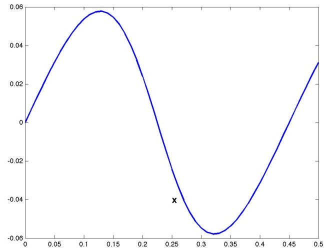

Figure 8. Coordinate x for the Car-like robot.

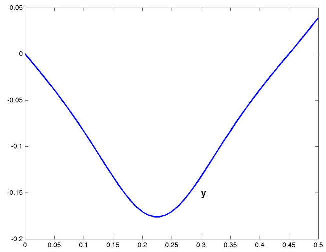

Figure 9. Coordinate y for the Car-like robot.

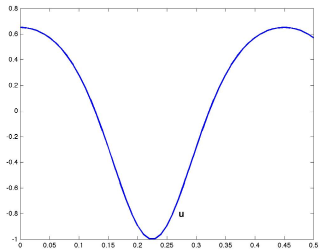

Figure 10. Control u for the Car-like robot.

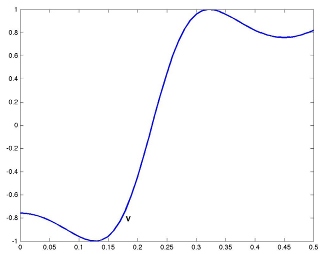

Figure 11. Control v for the Car-like robot.

where $x = \begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix}$, $\hat{X} = \begin{pmatrix} 1\\ 0\\ \frac{y}{2}\\ \frac{y^2}{2} \end{pmatrix}$ $\hat{Y} = \begin{pmatrix} 0\\ 1\\ -\frac{x}{2}\\ \frac{xy}{2} \end{pmatrix}$.

That system is nilpotent of order 3. Returning to the issue of the motion planning, to go from the configuration $(-0.4, 0.9, 0, 0)$ to the configuration $(-0.37, 0.9, 0, 0)$ in a minimal time, we use the Pontryagin's Maximum Principle for a fixed time. The Hamiltonian of the system is

$$H = \frac{p_0}{2}(y^2 v - xyv) + qu + pv + \frac{r}{2}(yv - xu),$$

with $(x, \psi) \in T^*(\mathbb{R}^2 \times S \times S)$, where $\psi = (p, q, r, p_0)$ and $p_0 \leq 0$ is an adjoint variable of Hamiltonian H. By using the two coordinate systems and the necessary conditions for the optimality of H, we get

$$\begin{aligned} \dot{x} &= u(t) = -\cos\varphi \\ \dot{y} &= v(t) = -\sin\varphi \end{aligned}$$

We obtain according to ([Love, 1927]) the solutions for $x(t)$, $y(t)$, and therefore for controls $u(t)$, $v(t)$, in terms of Jacobi elliptic function. So, the geodesic curves are the famous Euler's elastic curves. In Figure 7 we show the path followed by (x, y). Figure 8 and Figure 9 illustrate our reference trajectory solution for $x(t)$ and $y(t)$. Figure 10 and Figure 11 show the control outputs. We have obtained the solution for the motion planning problem for the Car-like robot towing a trailer, connecting the configuration $(-0.4, 0.9, 0, 0)$ to the configuration $(-0.37, 0.9, 0, 0)$.

7 Conclusion

In this paper we have illustrated the solution of the motion planning problem for two control systems, a vibratory mechanical system and the nonholonomic car-like a robot with a trailer, by means of differential flatness and sub-Riemann nilpotent approximation approaches, respectively. Both control systems are flats, however only in the first instance we have used that property to obtain the trajectories and the corresponding controls that solve the problem, computing the reference trajectory solution by elementary interpolation, without the integration of the model equations. Thus, simple solutions have been obtained. In the second case, we have used the sub-Riemannian nilpotent approximation to obtain the optimal trajectories that solve the motion planning problem, due to intrinsic geometric nature of the model. So, the differential flatness may be useful or more relevant in some cases in the solution of the motion planning problem.

8 Appendix

In this section we give some definitions and results related to different approaches or frameworks to differential flatness .

8.1 Differential fields

It is a commutative ring $\mathcal{R}$ with a derivation $\frac{d}{dt} : \mathcal{R} \to \mathcal{R}, \quad a \mapsto \frac{d}{dt}(a) =: \dot{a}$

$$\left.\begin{aligned} \tfrac{d}{dt}(a+b) &= \dot{a} + \dot{b} \\ \tfrac{d}{dt}(ab) &= \dot{a}\, b + a\, \dot{b} \end{aligned}\right\} \tag{14}$$

An element $c \in \mathcal{R}$ is a constant if $\dot{c} = 0$.

$L/K$for two given fields $K \subset L$, in such a way that the derivation of L in K coincides with the derivation of K.

An element $\xi \in L$ is differentially K-algebraic, if there exists a $p \in K[x_1, \ldots, x_n]$ such that

$$p(\xi, \dot{\xi}, \ldots, \xi^{(n)}) = 0 \tag{15}$$

The extension L/K is said to be algebraic if all the elements in L are K-algebraic.

$\xi \in L$ is K-transcendent if and only if is not K-algebraic. The extension L/K is said to be transcendent if there exist at least an element L that is transcendent.

A set $\{\xi_i\}_{i\in I}$ is differentially K- algebraic independent if $\{\xi_i^{(\nu)} \mid \nu \in \mathbb{N}\}_{i\in I}$ is K-algebraic independent.

Maximal independent sets with respect to the inclusion. The cardinality of a basis is the transcendence differential degree of the extension. Let K be a differential field then

$$K\left[\frac{d}{ds}\right] = \left\{ \sum_{finita} a_\nu \frac{d^\nu}{ds^\nu} \right\} \tag{16}$$

is a principal ideals ring. It is commutative if and only if K is a field of constants.

8.2 Field of differential operators

Let $\mathcal{C} = \{f : [0, +\infty) \longrightarrow \mathbb{C}\}$ be a ring of functions with respect to sum and convolution

$$(f \star g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau \tag{17}$$

$\mathcal{C}$ has no zero divisors (). The field of differential operators is the quotient field of $\mathcal{C}$.

1. Identity element: Dirac in $t = 0$
2. The inverse of the Heaviside function: is the derivation operator

$$\mathbf{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{18}$$

8.3 Equivalence

Let M be a differential manifold and let $F \in C^\infty(TM, \mathbb{R}^{n-m})$, an implicit system is written as follows

$$F(x, \dot{x}) = 0, \quad \text{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m \qquad (19)$$

Any system $\dot{x} = f(x, u)$ can be taken into this form: rank $\left(\frac{\partial f}{\partial u}\right) = m$ implies $u = \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)$, for then

$$F_i(x, \dot{x}) = \dot{x}_i - f_i(x, \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)) \quad (20)$$

Two systems $(M, F), (N, G)$ with rank $\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ and rank $\left(\frac{\partial G}{\partial \dot{y}}\right) = p - q$ are equivalent in $x_0 \in M$ and $y_0 \in N$ if:

1. There is $\Phi = (\varphi_1, \varphi_2, \ldots) \in C^\infty(N, M)$ such that

$$\Phi(y_0) = x_0, \quad \frac{d\varphi_i}{dt} = \varphi_{i+1} \qquad (21)$$

   and any solution $t \mapsto y(t)$ of $G(y, \dot{y}) = 0$ satisfies $F(\varphi_1(y(t)), \varphi_2(y(t))) = 0$
2. There is $\Psi = (\psi_1, \psi_2, \ldots) \in C^\infty(M, N)$ such that

$$\Psi(x_0) = y_0, \quad \frac{d\psi_i}{dt} = \psi_{i+1} \qquad (22)$$

   and any solution $t \mapsto y(t)$ of $F(x, \dot{x}) = 0$ satisfies

$$G(\psi_1(x(t)), \psi_2(x(t))) = 0 \qquad (23)$$

If two systems are equivalent then they have the same co-ranks $m = q$.

Given a trajectory $t \mapsto x(t)$ of system $F(x, \dot{x}) = 0, x \in M$ and $\xi \in TM$, the implicit system

$$\left(\frac{\partial F}{\partial x}(x, \dot{x})\right) \xi(t) + \left(\frac{\partial F}{\partial \dot{x}}(x, \dot{x})\right) \dot{\xi}(t) = 0 \qquad (24)$$

is called *the linear approximation* around x

If two systems are equivalent then the corresponding linear approximations are also equivalent.

(M, F) is flat in x_0 if it is equivalent to $(\mathbb{R}^m, 0)$, that is, if trajectories $t \mapsto x(t)$ are the image of a trivialization Φ, such that, $\Phi(y_0) = x_0$. Equivalently, for each curve $t \mapsto y(t)$

$$x(t) = (x, \dot{x}, \ldots) = \Phi(\varphi_1(y(t)), \varphi_2(y(t)), \ldots) \quad (25)$$

If a system is flat then it is equivalent to its linear approximation.

If (M, F) is flat in x_0, then

1. Its linear approximation is controllable.
2. If x_0 is an equilibrium point, the system is locally controllable around x_0.

References

Berret, B., Gauthier, J-P., Zakalyukin, V. (2006). Nonholonomic Interpolation: a general methodology for motion planning in robotics, International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan.

Cartan, E. (1914). Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes, *Bull. Soc. Math. France* 42, pp. 12–48.

Fliess, M., Lévine, J., Martin, Ph. and Rouchon, P. (1992). Sur les systèmes non linéaires différentiellement plats, *C. R. Acad. Sci. Paris Sér. I Math.* 315, no. 5, pp. 619–624.

Hilbert, D. (1902). Mathematical problems, *Bulletin of the American Mathematical Society*, vol. 8, no. 10, pp. 437-479. Earlier publications (in the original German) appeared in Göttinger Nachrichten, 1900, pp. 253-297, and Archiv der Mathematik und Physik, 3d. ser., vol. 1 (1901), pp. 44–63, 213–237.

Lévin, J. (2009). Analysis and control of non linear systems. A flatness-based approach. Springer-Verlag, Analysis and Control of Non Linear Systems Series.

Love A. A treatise on the mathematical theory of elasticity. Cambridge University Press.

Monroy-Pérez, F, Romero-Meléndez, C., Vázquez-González, B. (2011). Control of vibratory systems: a flatness approach, *Physcon 2011*.

Monroy-Pérez, F, Romero-Meléndez, C. (2012). Controllability and motion planning of vibratory systems: a flatness approach, *Cybernetics and Physics, vol. 2, No. 1, 2012*.

Rouchon, P. (2005). Flatness based control of oscillators, *ZAMM Z. Angew. Math. Mech.* 85, no. 6, pp. 411–421.

Sira-Ramírez, H. and Agrawal, S.K. (2004). *Differentially Flat Systems*, Marcel Dekker, Control Engineering Series.

Tongue, B.H. (2002). *Principles of Vibration, Oxford University press.*