ASYMPTOTIC HARMONIC GENERATOR DESIGN VIA MODIFICATION OF VAN DER POL OSCILLATOR

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Abstract: A well-known Van der Pol oscillator is modified to be introduced into the synthesis as an asymptotic harmonic generator of the periodic motion. The proposed modification possesses a limit cycle, producing a single harmonic as opposed to multi-harmonics of a standard harmonic oscillator. The parameters of the asymptotic harmonic generator are shown to specify damping, amplitude, and frequency of the limit cycle production. While being used as a reference model, the proposed Van der Pol modification proves to be well-suited for a model orbit stabilization of a two-link pendulum robot (Pendubot). The quasihomogeneous control synthesis is utilized to design a variable structure controller that drives the actuated link of the Pendubot to a periodic reference orbit in finite time. Performance issues of the controller constructed are illustrated in an experimental study of the laboratory Pendubot.

Keywords: Variable structure control, orbital stabilization, Van der Pol equation.

1. INTRODUCTION

A modification of the Van der Pol oscillator is introduced into the orbitally stabilizing synthesis of a double pendulum (Pendubot) as a reference model. This modification is made to shape the oscillator limit cycle to a harmonic one. Moreover, the limit cycle production of the modified Van der Pol oscillator possesses a single harmonic (as opposed to multi-harmonics of a standard harmonic oscillator) and the oscillator parameters specify amplitude, frequency, and damping of oscillation. Amplitude, frequency, and damping can thus be modified dynamically by simply changing the oscillator parameters. Due to this, the proposed oscillator is superior to that producing a non-harmonic response in its limit cycle (e.g., as is in Example 3.13 from (Slotine and Li, 1991)).

Along this line, the proposed Van der Pol modification serves as an *asymptotic harmonic generator* of the to-be-enforced motion of the Pendubot. It is introduced in a conceptually different way to the integral form (Shiriaev *et al.*, 2005) of the reference model that possesses a center, whose neighborhood is filled with cycles, and the target orbit is *a priori* to be specified among the cycles. Another example of an asymptotic harmonic generator (nearly the only one available in the literature) is the variable structure Van der Pol oscillator from (Sira, 1987). However, it is hardly possible to use that oscillator for generating a reference signal because the system response would be contaminated by high frequency oscillations (a so-

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called chattering effect) caused by fast switching the structure of the Van der Pol oscillator.

The control law, enforcing the system to slide along a periodic orbit of the phase space, and an asymptotic harmonic generator of this orbit, being coupled together, yield a novel unified framework for orbital stabilization of the Pendubot. The resulting closed-loop system possesses its own limit cycle, producing a prescribed harmonic whose frequency and amplitude can be modified dynamically at our will. The proposed synthesis is thus expected to yield desired robustness properties against the discrepancy between the real friction and that described in the model. Capabilities of the proposed orbitally stabilizing synthesis are illustrated in an experimental study of the laboratory Pendubot.

The rest of the paper is organized as follows. Section 2 is focused on the modification of the Van der Pol oscillator to be used in Section 3 as an asymptotic harmonic generator in the quasihomogeneous orbital stabilization of the Pendubot. Section 4 presents experimental results and Section 5 finalizes the paper with some conclusions.

2. ASYMPTOTIC HARMONIC GENERATOR

The Van der Pol equation, whose general representation is given by the second order scalar nonlinear differential equation

$$\ddot{x} + \varepsilon[(x - x_0)^2 - \rho^2]\dot{x} + \mu^2(x - x_0) = 0 \qquad (1)$$

with positive parameters ε, ρ, μ, is a special case of the Lienard equation (see, e.g., (Khalil, 2002))

$$\ddot{v} + r(v)\dot{v} + g(v) = 0 \qquad (2)$$

where the functions $r(v)$ and $g(v)$ are continuously differentiable.

The Van der Pol equation is a fundamental example in nonlinear oscillation theory. It possesses a periodic solution that attracts every other solutions except the unique equilibrium point $(x, \dot{x}) = (x_0, 0)$. Such a periodic solution is typically referred to as a stable limit cycle (Khalil, 2002). The parameter ρ controls the amplitude of this limit cycle, the parameter μ controls its frequency, the parameter ε controls the speed of the limit cycle transients, and the parameter x_0 is for the offset of x (see (Wang and Krstic, 2000) for details).

For later use, we present a modification of the Van der Pol equation

$$\ddot{x} + \varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x = 0, \qquad (3)$$

recently announced in (Orlov *et al.*, 2004), where in contrast to (1) no offset of x is admitted, i.e., the parameter $x_0 = 0$ is used, and the additional term $\frac{\varepsilon}{\mu^2}\dot{x}^3$ is involved.

As opposed to the Van der Pol equation (1), the proposed modification (3) has nothing to do with the Lienard equation (2). Meanwhile, it still possesses a stable limit cycle, being expressible in the explicit form

$$x^2 + \frac{\dot{x}^2}{\mu^2} = \rho^2 \qquad (4)$$

(unlike that of the Van der Pol oscillator, exhibiting a nonsinusoidal periodic response in its limit cycle!). The following result is in order.

Theorem 1. Consider the modified Van der Pol equation (3) with positive parameters ε, μ, ρ. Then this equation has a stable limit cycle, given by (4), so that every other solution except the equilibrium point $x = \dot{x} = 0$ converges to the limit cycle (4) as $t \to \infty$.

Proof. Proof is rather standard (cf., for instance, Example 3.13 from (Slotine and Li, 1991)) and it has appeared in (Orlov *et al.*, 2004). ■

Now it becomes clear that in equation (3) the parameter ρ stands for the amplitude of the limit cycle whereas μ is for its frequency. Furthermore, by substituting the orbit equation (4) into (3) we conclude that the limit cycle of the modified Van der Pol equation (3) is remarkably generated by a standard linear harmonic oscillator

$$\ddot{x} + \mu^2 x = 0, \qquad (5)$$

initialized on (4). Thus, we arrive at a *nonlinear asymptotic harmonic generator* (3) which naturally exhibits an ideal sinusoidal signal (5) in its limit cycle (4). The amplitude and frequency of this sinusoidal signal can be varied at will by tuning the parameters ρ and μ of the harmonic generator (3).

The modified Van der Pol oscillator (3), the phase portrait of which is shown in Fig.1 for the parameter values $\varepsilon = 1000$, $\rho = 0.01$, $\mu = 1$, still belongs to a class of damped systems. In the region of negative damping, occurring within the limit cycle where the signals are small, the damping increases the energy level of the response (see the proof of Theorem 1). Conversely, outside the limit cycle, the damping becomes positive, thus decreasing the energy of the output signal. As a result, the motion approaches the limit cycle whose energy is determined by its amplitude ρ and frequency μ and therefore a desired level of the energy can be attained by assigning appropriate values of the oscillator parameters ρ and μ.

3. ORBITAL STABILIZATION OF PENDUBOT

3.1 Problem Statement

The state equation of the Pendubot, depicted in Fig. 2, is given by (Utkin *et al.*, 1999, p. 55):

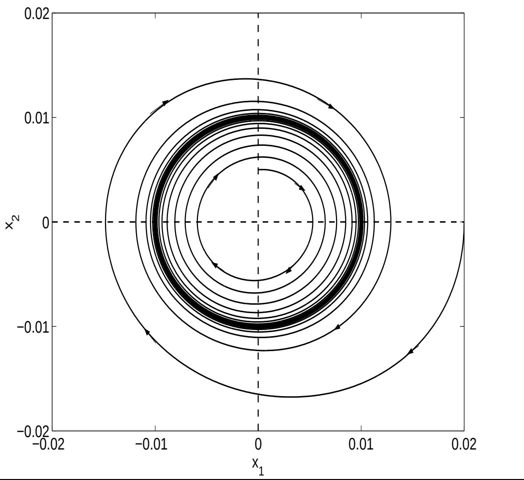

Fig. 1. Phase portrait of the modified Van der Pol equation

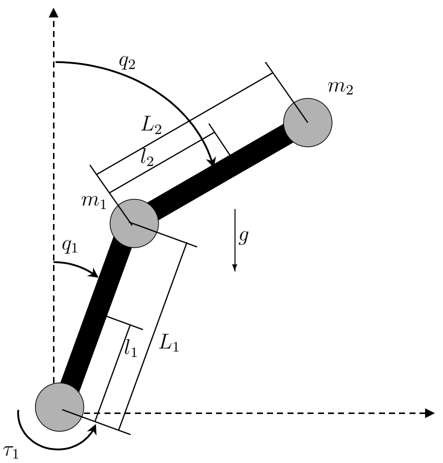

Fig. 2. Pendubot

$$\underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}}_{N(q,\dot{q})} = \underbrace{\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}}_{\tau} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{w} \tag{6}$$

where

$$m_{11} = \theta_1,\ m_{12} = \theta_3 \cos(q_1 - q_2),\ m_{22} = \theta_2 \tag{7}$$

$$\begin{aligned} N_1 &= \theta_3 \sin(q_1 - q_2)\dot{q}_2^2 - g\theta_4 \sin(q_1), \\ N_2 &= -\theta_3 \sin(q_1 - q_2)\dot{q}_1^2 - g\theta_5 \sin(q_2), \end{aligned} \tag{8}$$

$$\begin{aligned} &\theta_1 = m_1 l_1^2 + m_2 L_1^2 + J_1 + J_m,\ \ \theta_2 = m_2 l_2^2 + J_2, \\ &\theta_3 = m_2 L_1 l_2, \qquad\qquad \theta_4 = m_1 l_1 + m_2 L_1, \\ &\theta_5 = m_2 l_2. \end{aligned} \tag{9}$$

Here, m_1 is the mass of link 1, m_2 the mass of link 2, L_1 and L_2 are respectively the lengths of link 1 and link 2; l_1 and l_2 are the distances to the center of mass of link 1 and link 2; J_1 and J_2 are the moments of inertia of link 1 and link 2 about their centroids; J_m is the motor inertia, τ_1 is the control torque, w is the external disturbance, and g is the gravity acceleration.

We assume throughout that the external disturbance is of class $L_\infty(0,\infty)$ with an *a priori* known norm bound $K > 0$, i.e.,

$$ess \sup_{t\in[0,\infty)} \|w(t)\| \leq K \tag{10}$$

where $\|\cdot\|$ stands for the standard Euclidean norm.

Our objective is to design a controller that causes the actuated link of the Pendubot to track a reference trajectory

$$\lim_{t\to\infty} [q_1(t) - x(t)] = 0, \tag{11}$$

while attenuating the effect of an admissible external disturbance (10).

In order to present a control strategy that allows us to achieve the above objective let us partially linearize the Pendubot dynamics in accordance with (Spong and Praly, 1997). For this purpose, let us rewrite equation (6) in the form

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + N_1 = \tau_1 + w_1 \tag{12}$$

$$m_{12}\ddot{q}_1 + m_{22}\ddot{q}_2 + N_2 = w_2. \tag{13}$$

Then using (13), the following equation is derived:

$$\ddot{q}_2 = -m_{22}^{-1}[m_{12}\ \ddot{q}_1 + N_2 - w_2]. \tag{14}$$

Now substituting equation (14) into (12) yields

$$\begin{aligned} &(m_{11} - m_{12}m_{22}^{-1}m_{12})\ddot{q}_1 \\ &-m_{12}m_{22}^{-1}(N_2 - w_2) + N_1 = \tau_1 + w_1. \end{aligned} \tag{15}$$

Finally, setting $|M| = m_{11} - m_{12}m_{22}^{-1}m_{12}$ and

$$\tau_1 = |M|u - m_{12}m_{22}^{-1}N_2 + N_1 \tag{16}$$

where u is the new control input, the desired linearization is obtained:

$$\ddot{q}_1 = u + |M|^{-1}[w_1 - m_{12}m_{22}^{-1}w_2] \tag{17}$$

$$\begin{aligned} \ddot{q}_2 = &-m_{22}^{-1}\{m_{12}[u + |M|^{-1}(w_1 - m_{12}m_{22}^{-1}w_2)] \\ &+N_2 - w_2\}. \end{aligned} \tag{18}$$

In the above relations the positive definiteness of the inertia matrix $M(q)$ has been used to ensure that $|M| \neq 0$. Since system (17), (18) describes the linearized actuated joint model it is referred to as *collocated linearization* (Spong, 1995).

The control strategy is now formalized as follows. The control input (16) is composed by an inner loop controller, partially linearizing the Pendubot, and an outer loop controller u to be constructed. Given the system output

$$y(t) = q_1(t) - x(t), \tag{19}$$

that combines the actuated state $q_1(t)$ of the system and the reference variable $x(t)$ governed by the modified Van der Pol equation (3), the outer loop controller u is to drive the system output (19) to the surface $y = 0$ in finite time and maintain it there in spite of a uniformly bounded external disturbance w, affecting the system.

3.2 Switched Control Synthesis

Due to (3), (17), (19), the output dynamics is given by

$$\ddot{y} = u + |M|^{-1}[w_1 - m_{12}m_{22}^{-1}w_2] \\ +\varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x. \qquad (20)$$

The following control law

$$u = -\varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} - \mu^2 x \\ -\alpha sign(y) - \beta sign(\dot{y}) - hy - p\dot{y} \qquad (21)$$

with the parameters such that

$$h, p \geq 0, \ \alpha - \beta > (|M|^{-1} + |m_{12}m_{22}^{-1}|)K \qquad (22)$$

is proposed.

The closed-loop system (20), (21) is then feedback transformed to the one

$$\ddot{y} = |M|^{-1}[w_1 - m_{12}m_{22}^{-1}w_2] \\ -\alpha sign(y) - \beta sign(\dot{y}) - hy - p\dot{y} \qquad (23)$$

with piece-wise continuous right-hand side. Throughout, solutions of such a system are defined in the sense of Filippov (Filippov, 1988) as that of a certain differential inclusion with a multi-valued right-hand side.

Relating the quasihomogeneous synthesis from (Orlov, 2005b), the above controller has been composed of the nonlinear compensator

$$u_c = -\varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} - \mu^2 x, \qquad (24)$$

the homogeneous switching part (the so-called twisting controller from (Fridman and Levant, 2002))

$$u_h = -\alpha sign(y) - \beta sign(\dot{y}),$$

and the linear remainder

$$u_l = -hy - p\dot{y}$$

that vanishes in the origin $y = \dot{y} = 0$. By Theorem 4.2 from (Orlov, 2005a) the quasihomogeneous system (23) with the parameter subordination (22) is finite time stable regardless of which external uniformly bounded disturbance subject to (10) affects the system. The control objective is thus achieved.

In the remainder, capabilities of the proposed synthesis procedure are tested in an experimental study of the laboratory Pendubot.

4. EXPERIMENTAL RESULTS

4.1 Pendubot prototype

Performance issues of the quasihomogeneous synthesis are tested on the laboratory Pendubot, manufactured by Mechatronics Systems Inc. and installed in the CICESE Research Center. The values of the Pendubot parameters (9), supplied by the manufacturer (Mechatronics, 1998), are listed in Table 1.

Table 1. Parameters of the Pendubot.

Notation	**Value**	**Units**
θ_1	0.0308	kg m^2
θ_2	0.0106	kg m^2
θ_3	0.0095	kg m^2
θ_4	0.2087	kg m
θ_5	0.0630	kg m

4.2 Experimental verification

In our experimental study of the orbitally stabilizing synthesis (3), (16), (21), the controller gains were set to $\alpha = 140$, $\beta = 40$, $h = 0$, $p = 0$ whereas the reference parameters were tuned to $\varepsilon = 8.7$, $\rho = 0.013$, $\mu = 10$. The initial conditions of the Pendubot position and those of the modified Van der Pol oscillator, selected for all experiments, were $q_1(0) = 3.14$ rad, $q_2(0) = 3.14$ rad, and $x(0) = -3.14$ rad, whereas all the velocity initial conditions were set to zero.

In order to test the robustness of the controller constructed an external disturbance, similar to that of (Zhang and Tarn, 2002), was randomly added by lightly hitting the links of the Pendubot. For demonstrating the capability of the controller to move the Pendubot from one orbit to another by modifying the orbit parameters we then introduced a random time instant t_0 (it was $t_0 \approx 10s$ in the experiment), when the amplitude ρ of the model limit cycle was changed from its initial value $\rho = 0.013$ to the new one $\rho = 0.5$.

Experimental results for the resulting Pendubot motion, enforced by the orbitally stabilizing controller, are depicted in Fig. 3. This figure demonstrates that while being driven by the orbitally stabilizing controller, the closed-loop system generates a bounded, quasi-periodic motion and exhibits fast recovery of this motion when the quick disturbance is successively applied to each link of the Pendubot. As predicted by theory, the desired orbital transfer is achieved by simply changing the amplitude of the orbit limit cycle. Thus, good performance of the orbitally stabilizing controller is concluded from Fig. 3.

5. CONCLUSIONS

A well-known Van der Pol oscillator is modified to possess a stable limit cycle, governed by a standard linear oscillator equation. The proposed modification is introduced into the orbitally stabilizing synthesis of a double pendulum as an asymptotic harmonic generator of the periodic motion. The resulting closed-loop system is capable of moving from one orbit to another by simply changing the parameters of the modified Van der Pol oscillator. The quasihomogeneity-based

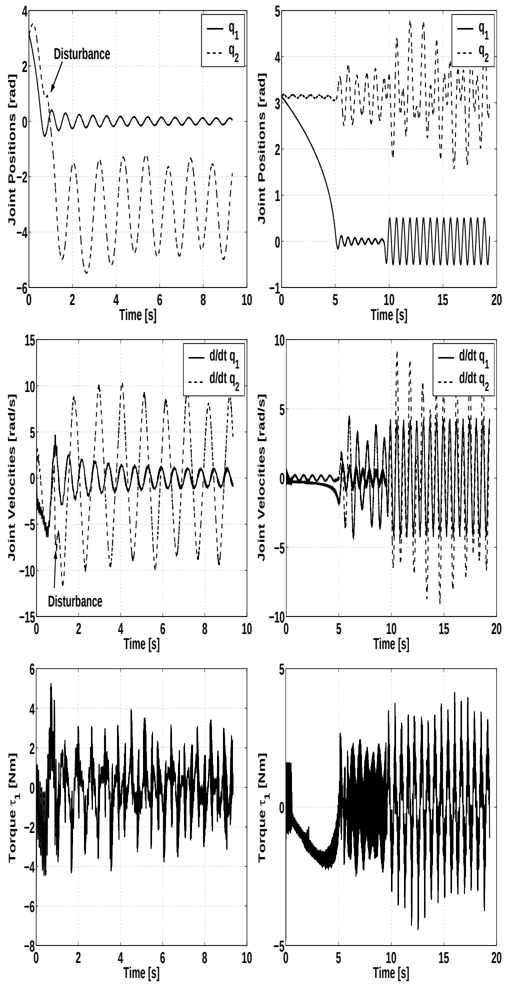

Fig. 3. Orbital stabilization of the Pendubot: left column for the orbital recovery under the randomly added quick disturbance, right column for the transfer from one orbit to another

control synthesis is utilized to design a variable structure controller that drives the actuated state of the Pendubot to a model orbit in finite time. The resulting closed-loop system is shown to track the model orbit in a sliding mode of the second order, even in the presence of external disturbances with an a priori known magnitude bound. Capabilities of the quasihomogeneous synthesis and its robustness against modelling errors and external disturbances are illustrated in an experimental study made for the laboratory Pendubot.

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