Structural stability of singular systems under proportional and derivative feedback

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Abstract— **We consider quadruples of matrices (E, A, B, C), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$ and $C \in M_{p\times n}(C)$, under proportional and derivative feedback, and proportional and derivative output injection.**

In this work study the equivalence relation as a Lie group action that permit see the equivalence classes as differentiable manifolds and studying the tangent space to the orbits we obtain a characterization of the structural stability of quadruples of matrices, in terms of numerical invariants.

I. Introduction

We consider generalized time-invariant linear systems given by the matrix equations $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ where $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$ and $C \in M_{p\times n}(C)$ that we will represent by quadruples of matrices (E, A, B, C). These equations arise in theoretical areas as differential equations on manifolds as well as in applied areas as systems theory and control, e.g. they are used in modelling of mechanical multibody systems [5].

The aim of this paper is to characterize the structural stability of a quadruple of matrices, with regard the equivalence relation defined by the following elementary transformations: basis change in the state space, input space, output space, proportional and derivative feedback, and proportional and derivative output injection. Characterization is given in terms of a certain numerical invariants presented in the paper.

Structurally stable elements are those whose behavior does not change when applying small perturbations. The concept of structural stability, in the qualitative theory of dynamical systems has been widely studied by several authors in control theory (see [2], [3], [4], [8], for example).

The equivalence relation considered may seen as induced by a Lie group action, therefore the equivalent classes are orbits with regard the action. So, in order to characterize structural stability we can use geometrical techniques describing the tangent space to the orbits.

II. Equivalence relation

Let $\mathcal{M}$ be the set of matrices $\mathcal{M} = \{(E, A, B, C) \mid E, A \in M_n(C), B \in M_{n\times m}(C)\}$, $C \in M_{p\times n}(C)$ representing singular time-invariant linear systems.

In order to consider quadruples in a simpler form preserving qualitative properties as controllability-observability, controllability non observable, observability non controllable, finite and infinite zeros among others, we define the following equivalence relation.

Definition 1: Two quadruples (E', A', B', C') and (E, A, B, C) in $\mathcal{M}$ are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n; C)$, $R \in Gl(m; C)$, $S \in Gl(p; C)$, $F_E^B, F_A^B \in M_{m\times n}(C)$ and $F_E^C, F_A^C \in M_{n\times p}(C)$ such that

$$\begin{array}{l} E' = QEP + QBF_E^B + F_E^C CP, \\ A' = QAP + QBF_A^B + F_A^C CP, \\ B' = QBR, \\ C' = SCP, \end{array}$$

or in a matrix form

$$\begin{pmatrix} E' & B' & & \\ C' & & & \\ & & A' & B' \\ & & C' & \end{pmatrix} =$$

$$\begin{pmatrix} Q & F_E^C & & \\ & S & & \\ & & Q & F_A^C \\ & & & S \end{pmatrix}\begin{pmatrix} E & B & & \\ C & & & \\ & & A & B \\ & & C & \end{pmatrix}\begin{pmatrix} P & & & \\ F_E^B & R & & \\ & & P & \\ & & F_A^B & R \end{pmatrix}.$$

It is easy to check that this relation is an equivalence relation.

Systems $(E, A, B, C) \in \mathcal{M}$, for which there exist matrices F_E^B, F_E^C, F_A^B and F_A^C such that the pencil $\lambda(E + BF_E^B + F_E^C C) + (A + BF_A^B + F_A^C C)$ is regular, are called regularizable. Remember that regular systems are those such that there exists a unique solution for some consistent initial condition.

Obviously, regularizable character is invariant under equivalence relation considered.

The equivalence relation permit us to reduce regularizable systems to the following reduced form.

Proposition 1: Let $(E, A, B, C) \in \mathcal{M}$ be a n-dimensional m-input regularizable singular system. Then, it can be reduced to (E_c, A_c, B_c, C_c) with $E_c = \begin{pmatrix} I_r & \\ & N_1 \end{pmatrix}$, $A_c = \begin{pmatrix} A_1 & \\ & I_{n-r} \end{pmatrix}$, $B_c = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$, $C_c = \begin{pmatrix} C_1 & 0 \end{pmatrix}$ where (A_1, B_1, C_1) is in its Kronecker canonical form as a triple representing a standard system (see [4]), and N_1 is a nilpotent matrix in its Jordan reduced form.

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III. Equivalence relation as a Lie group action

Let us consider the following Lie group $\mathcal{G} = Gl(n;C) \times Gl(n;C) \times Gl(m;C) \times Gl(p;C) \times M_{m\times n}(C) \times M_{m\times n}(C) \times M_{n\times p}(C) \times M_{n\times p}(C)$, acting on $\mathcal{M}$. The product $\star$ in $\mathcal{G}$ is given by

$$\begin{array}{l}(Q_1, P_1, R_1, S_1, F^B_{E\,1}, F^B_{A\,1}, F^C_{E\,1}, F^C_{A\,1})\star\\ \qquad (Q_2, P_2, R_2, S_2, F^B_{E\,2}, F^B_{A\,2}, F^C_{E\,2}, F^C_{A\,2}) =\\ (Q_2Q_1, P_1P_2, R_1R_2, S_2S_1, F^B_{E\,1}P_2 + R_1F^B_{E\,2}, F^B_{A\,1}P_2+\\ \qquad R_1F^B_{A\,2}, Q_2F^C_{E\,1} + F^C_{E\,2}S_1, Q_2F^C_{A\,1} + F^C_{A\,2}S_1)\end{array}$$

being $e = (I_n, I_n, I_m, I_p, 0, 0, 0, 0)$ its unit element.

The action of the Lie group $\mathcal{G}$ on $\mathcal{M}$

$$\alpha : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$$

where

$$\alpha((P, Q, R, S, F^B_E, F^B_A, F^C_E, F^C_A), (E, A, B, C)) = (E_1, A_1, B_1, C_1)$$

with

$$\begin{array}{l}E_1 = QEP + QBF^B_E + F^C_E CP,\\ A_1 = QAP + QBF_A + F^C_A CP,\\ B_1 = QBR,\\ C_1 = SCP.\end{array}$$

give rise to the equivalence relations in $\mathcal{M}$ which defined in §1.

From now on, we will make use of the following notation: $g = (P, Q, R, S, F^B_E, F^B_A, F^C_E, F^C_A) \in \mathcal{G}$, and $x = (E, A, B, C) \in \mathcal{M}$.

Given a quadruple $x_0 = (E_0, A_0, B_0, C_0) \in \mathcal{M}$ we define the maps

$$\alpha_{x_0}(g) = \alpha(g, x_0).$$

The equivalence class of the quadruple x_0 with respect to the $\mathcal{G}$-action, called the $\mathcal{G}$-orbit of x_0, is the range of the function α_{x_0} and is denoted by

$$\mathcal{O}(x_0) = \operatorname{Im}\alpha_{x_0} = \{\alpha_{x_0}(g) \mid g \in \mathcal{G}\}.$$

Remark 1: The maps α_{x_0} are clearly differentiable, and $\mathcal{O}(x_0)$, are smooth submanifolds of $\mathcal{M}$.

So, we have the following proposition.

Proposition 2:

$$T_{x_0}\mathcal{O}(x_0) = \operatorname{Im} d\alpha_{x_0} \subset T_{x_0}\mathcal{M}.$$

A. Description of Tangent Space to the Orbits

Let us denote by $T_e\mathcal{G}$ the tangent space to the manifold $\mathcal{G}$ at the unit element e. It is known that

$$\begin{array}{l}T_e\mathcal{G} =\\ M_{n\times n}(C) \times M_{n\times n}(C) \times M_{m\times m}(C) \times M_{p\times p}(C)\times\\ \times M_{m\times n}(C) \times M_{m\times n}(C) \times M_{n\times p}(C) \times M_{n\times p}(C)\\ T_{x_0}\mathcal{M} = \mathcal{M}\end{array}$$

Proposition 3: Let $d\alpha_{x_0} : T_e\mathcal{G} \longrightarrow \mathcal{M}$ be the differential of α_{x_0} at the unit element e. Then

$$d\alpha_{x_0}((Q, P, R, S, F^B_E, F^B_A, F^C_E, F^C_A)) = (E_1, A_1, B_1, C_1)$$

with

$$\begin{array}{l}E_1 = EP + QE + BF^B_E + F^C_E C\\ A_1 = AP + QA + BF^B_A + F^C_A C\\ B_1 = BR + QB\\ C_1 = SC + CP\end{array}$$

Proof: Taking into account that α_{x_0} is differentiable it suffices to compute the first order approximation of the map.

$$\begin{array}{l}\alpha_{x_0}(e + \varepsilon g) =\\ (E, A, B, C) + \varepsilon(E_1, A_1, B_1, C_1) + O(\varepsilon^2)\end{array}$$

with

$$\begin{array}{l}E_1 = EP + QE + BF^B_E + F^C_E C,\\ A_1 = AP + QA + BF^B_A + F^C_A C,\\ B_1 = BR + QB,\\ C_1 = SC + CP\end{array}$$

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Using Kronecker products and vector valued function (see [6] for definitions and properties) the tangent space can be described in the following form:

$$\begin{pmatrix} E_1 & A_1 & B_1 & C_1 \end{pmatrix}^t = MX$$

with $M =$

$$\begin{pmatrix} I_n \otimes E\,E^t \otimes I_n & 0 & 0 & I_n \otimes BC^t \otimes I_n & 0 & 0 \\ I_n \otimes A\,A^t \otimes I_n & 0 & 0 & 0 & 0 & I_n \otimes B\,C^t \otimes I_n \\ 0 & B^t \otimes I_n I_m \otimes B & 0 & 0 & 0 & 0 & 0 \\ I_n \otimes C & 0 & 0 & C^t \otimes I_p & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $X =$

$$(vec(P)\, vec(Q)\, vec(R)\, vec(S)\, vec(F^B_E)\, vec(F^C_E)\, vec(F^B_A)\, vec(F^C_A))^t$$

In this notation we may say that the tangent space is generated for the vector columns of the matrix M:

$$\begin{bmatrix} I_n \otimes E\,E^t \otimes I_n & 0 & 0 & I_n \otimes B\,C^t \otimes I_n & 0 & 0 \\ I_n \otimes A\,A^t \otimes I_n & 0 & 0 & 0 & 0 & I_n \otimes B\,C^t \otimes I_n \\ 0 & B^t \otimes I_n\, I_m \otimes B & 0 & 0 & 0 & 0 & 0 \\ I_n \otimes C & 0 & 0 & C^t \otimes I_p & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can conclude that

$$\dim T\mathcal{O}(x_0) = \operatorname{rank} M$$

IV. Structural stability

In this Section we will recall the definition of structural stability, according to that appearing in the paper by Willems (see [8]), as well as equivalent conditions.

Let X be a topological space and consider an equivalence relation defined on it.

Definition 2: An element $x \in X$ is *structurally stable* if and only if there exists an open neighborhood $U \subset X$ of x such that all the elements in it are equivalent to x.
Let us assume that X is a differentiable manifold and the equivalence relation in X is that induced by the action of a Lie group $\mathcal{G}$ which acts on X, giving rise to orbits which are also differentiable manifolds.

Let us denote by $T_x\mathcal{O}(x)$ the tangent space in $x \in X$ to the orbit of x, $\mathcal{O}(x)$.

Proposition 4: Under the assumptions above, the following conditions are equivalent:

a) $x \in X$ is structurally stable,
b) $\dim \mathcal{O}(x) = \dim X$,
c) $\dim T_x\mathcal{O}(x) = \dim X$,

Proof: An element $x \in X$ is structurally stable if and only if there exists an open neighborhood contained in its orbit. Thus its orbit should be an open submanifold and therefore its dimension equal to $\dim X$.

$\blacksquare$

In our particular setup we have the following proposition.

Proposition 5: A quadruple $x_0 = (E, A, B, C) \in \mathcal{M}$ is structurally stable if and only if

$$\text{rank } M = 2n^2 + nm + np$$

The homogeneity property of the orbits permit us to consider an equivalent quadruple in a simpler form to compute the rank of the matrix M.

So given a quadruple (E, A, B, C), there exist matrices $Q \in Gl(n; C)$, $R \in Gl(m; C)$ such that $B' = PBR = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ and we can consider the equivalent quadruple (E', A', B', C') with $E' = \begin{pmatrix} 0 & 0 \\ E_1 & E_2 \end{pmatrix}$, $A' = \begin{pmatrix} 0 & 0 \\ A_1 & A_2 \end{pmatrix}$, $B' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $C' = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$.

In this form it is easy to observe that a necessary condition for stability is that the quadruple being standardizable.

Finally we prove the main theorem characterizing the structurally stable quadruples in terms of structural invariants.

Theorem 1: A quadruple $(E, A, B, C) \in M$ is structurally stable depending on the order relation with n, m, p,

i) If $\min(m, p) \geq n$, if and only if, $\text{rank}\, B = \text{rank}\, C = n$.

ii) If $p \geq n > m$, $\text{rank}\, B = m$, and $\text{rank}\, C = n$.

iii) If $m \geq n > p$, $\text{rank}\, B = n$, and $\text{rank}\, C = p$.

iv) If $n > m = p$ there are not structurally stable quadruples

v) If $n > m > p$, if and only if, $\text{rank}\, B = m$, $\text{rank}\, C = p$, $r_i^{co} = (n + p)(i + 1)$ and $r_i^c = in + \min(ip + n, (i + 1)m)$ where $r_0^{co} = \text{rank}\begin{pmatrix} E & B \\ C & 0 \end{pmatrix}$,

$$r_i^{co} = \text{rank}\begin{pmatrix} E & B & 0 & & & & & \\ C & 0 & 0 & & & & & \\ A & 0 & E & B & & & & \\ & & & & \ddots & & & \\ & & & & & E & B & & \\ & & & & & C & 0 & & \\ & & & & & A & 0 & E & B \\ & & & & & & & C & 0 \end{pmatrix},$$

$$r_1^c = \text{rank}\begin{pmatrix} A & B & 0 \\ C & 0 & 0 \\ E & 0 & B \end{pmatrix} \text{ and}$$

$$r_i^c = \text{rank}\begin{pmatrix} A & B & & & & & \\ C & 0 & & & & & \\ E & 0 & B & A & & & \\ & & & & \ddots & & \\ & & & & & A & B & \\ & & & & & C & 0 & \\ & & & & & E & 0 & B \end{pmatrix}$$

vi) If $n > p > m$ if and only if, $\text{rank}\, B = m$, $\text{rank}\, C = p$, $r_i^{co} = (n + m)(i + 1)$ and $r_i^o = in + \min(im + n, (i + 1)p)$ where $r_0^{co} = \text{rank}\begin{pmatrix} E & B \\ C & 0 \end{pmatrix}$,

$$r_i^{co} = \text{rank}\begin{pmatrix} E & B & & & & & & \\ C & 0 & & & & & & \\ A & 0 & E & B & & & & \\ & & & & \ddots & & & \\ & & & & & E & B & & \\ & & & & & C & 0 & & \\ & & & & & A & 0 & E & B \\ & & & & & & & C & 0 \end{pmatrix},$$

$$r_1^o = \text{rank}\begin{pmatrix} A & B & E \\ C & 0 & 0 \\ 0 & 0 & B \end{pmatrix} \text{ and}$$

$$r_i^o = \text{rank}\begin{pmatrix} A & B & E & & & & & \\ C & 0 & & & & & & \\ 0 & 0 & B & A & E & & & \\ & & & & & \ddots & & \\ & & & & & & A & B & E \\ & & & & & & C & 0 & 0 \\ & & & & & & 0 & 0 & C \end{pmatrix}$$

Proof: Taking into account that a necessary condition for stability is the standardization, if the quadruple is structurally stable it can be reduced to (I_n, A_1, B_1, C_1). So, after to prove that these ranks are invariant under equivalence relation, it suffices to observe that $r_i^{co} - n(i + 1)$ are the ρ^{co} numbers $r_i^c - ni$ are ρ^c numbers a,d $r_i^o - ni$ are ρ^o numbers of the triple (A_1, B_1, C_1) in the canonical form of the quadruple (E, A, B, C) it suffices to apply [4]. $\blacksquare$

V. Conclusion

We have obtained a list of numerical invariants in terms of certain ranks of matrices that it permit to deduce a characterization of structural stability of quadruples of matrices.

VI. References

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