DYNAMICAL AND STATISTICAL ANALYSIS OF A NEW LOZI FUNCTION FOR RANDOM NUMBERS GENERATION

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Abstract
This paper presents the first results of the statistical and dynamical analysis of a new function showing random properties firstly proposed by Lozi. The phase plane analysis via the critical lines tool allowed to delimit analytically the holes in the chaotic attractor and to follow their evolution. In addition, the results of the statistical NIST tests for pseudo-randomness showed to be successful and significantly improved after an under-sampling of the output signal.

Key words
Lozi map, chaotic map, Random Number Generator, NIST test

1 Introduction

The incessantly increasing demand for secure data storage and transmission (e-banking, e-payments, personal data encoding...) motivates the research for newer and more secure data encryption techniques. The latter are classically performed via Pseudo Random Number Generators (PRNG) which, besides being highly reliable, should be able to generate as many different encoding sequences (hidden in the encryption keys) as possible. The encryption keys lie in the system parameters, since the structure is always supposed to be known by the pirates. For this reason, during the last decade, there has been a plethora of papers devoted to the nonlinear maps used for chaotic encryption.

Indeed, the well known intrinsic sensitivity to small parameter changes and initial conditions exhibited by the chaotic maps makes them perfect candidates for encryption. Thus, for each - even infinitesimal - parameter change, a different chaotic sequence will be generated, so in theory an infinite number of encoding sequences can be obtained - and therefore, an infinite number of keys (if we make abstraction of the quantization). Nevertheless, designing a chaotic PRNG remains a very tough problem, because chaoticity is only a necessary, but not a sufficient feature. Indeed, the encrypting sequence has to exhibit also a set of statistical properties [3], [4] and therefore not all chaotic maps are suitable for encryption purposes. However, most of the authors simply neglected the statistical properties, which have to be satisfied by the chaotic map, if used as PRNG. This is typically the case when the basin of attraction is not dense or exhibits holes, so the state variables are not equidistributed.

The most widely and universally used test to validate PRNG is the National Institute of Standards and Technology Test, known as NIST tests.

2 System Definition

The system under consideration has been proposed first by Lozi in [1] who emphasized its random features. It is defined on the p-dimensional torus $T^p = [-1,1[^p$ by the map $M_p : T^p => T^p$

$$M_p : \begin{cases} x^1_{n+1} = 1 - 2\left|x^1_n\right| + k^1 \times x^2_n \\ x^2_{n+1} = 1 - 2\left|x^2_n\right| + k^2 \times x^3_n \\ \vdots \\ x^p_{n+1} = 1 - 2\left|x^p_n\right| + k^p \times x^1_n \end{cases} \quad (1)$$

where the parameters $k^i = (-1)^{i+1}$ or $k^i = 1$, the latter case being considered hereafter. The flow x is contained on the torus:

$$\begin{aligned} &if \quad x^j_{n+1} = 1 - 2\left|x^j_n\right| + k^j \times x^{j+1}_n < -1 \\ &\quad add \quad 2 \\ &if \quad x^j_{n+1} = 1 - 2\left|x^j_n\right| + k^j \times x^{j+1}_n \geq 1 \\ &\quad substract \quad 2 \end{aligned} \quad (2)$$

$|x_n|$ denotes the absolute value of x_n, therefore the map (1) is a noninvertible map (i.e. the backward iterates are not unique, or do not exist).

Hereafter we deal with the dynamical analysis of the second order system M$_2$ on the torus T^2, here simply denoted as M.

Therefore, considering that $|x|$ can take two different values, there are four regions in T^2 with locally linear behaviour (two for x^1, and two for x^2). Thus the map can also be considered as a piece-wise linear one.

2.1 Analysis of the 2D-System

Singularities of type 1. Fixed and periodic points.

The fixed points are to be studied independently in the four regions of T^2.They are defined by:

$$M(x) = x \qquad (3)$$

Keeping in mind that $x = (x^1, x^2)$ and $x^i \in$[-1,1[, for $x^1 > 0$, $x^2 > 0$, the fixed point is located at (x^1, x^2) = (0.5,0.5). It is unstable with eigenvalues (λ_1, λ_2) = (-3,-1). But it is numerically stable because of the structure of the floating point numbers.

For $x^1 < 0$, $x^2 < 0$, each point of the line $x^2 = -1 - x^1$ is an unstable fixed point with positive eigenvalues (λ_1, λ_2) = (3,1).

For $x^1 \geq 0$, $x^2 < 0$, there is only one fixed point at (x^1, x^2) = (0, -1), which is unstable. The eigenvalues are (λ_1, λ_2) = ($\sqrt{5}, -\sqrt{5}$).

For $x^1 < 0$, $x^2 \geq 0$, the single fixed point (x^1, x^2) = (-1, 0) has the same eigenvalues as before, and is also unstable.

Due to the piece-wise linear nature of (Eq1– Eq.2), closed formulas can be found for every periodic solution, see fig.1 with all period-2 cycles:

$$x_n = x_{n+2}$$

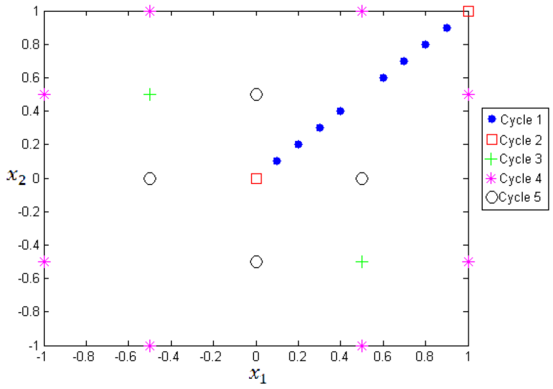


Figure 1. Period 2 solutions of the map M$_2$ (1) on the torus T^2 = [-1,1[^{2}

Singularities of type 2. Critical lines.

The critical lines CL [2] are singularities of dimension 1 and represent an important tool for the analysis of noninvertible maps. By definition, the critical lines separate regions of the phase space with different number of preimages (backward iterates). In the case of piece-wise linear maps, they are the first iterates of the lines of discontinuity CL$_{-1}$ of the system.

For the two dimensional system M$_2$ there are four groups of critical lines CL with preimages CL$_{-1}$ given by:

Critical Lines A

For $CL_{-1}^{A} : x^1 = 0$, we have:

$$CL_1^{A1} : x^2 = -2\,x^1 - 1 \quad if\ x^2 > 0$$
$$CL_1^{A2} : x^2 = 2\,x^1 - 1 \quad if\ x^2 < 0$$

Critical Lines B

For $CL_{-1}^{B} : x^1 = -1$

$$CL_1^{B1} : x^2 = 2\,x^1 \quad if\ x^2 < 0,\ x^1 \in [0,0.5]$$
$$CL_1^{B2} : x^2 = -2\,x^1 - 2 \quad if\ x^2 > 0,\ x^1 \in [-1,-0.5]$$
$$CL_1^{B3} : x^2 = 2\,x^1 - 2 \quad if\ x^2 < 0,\ x^1 \in [0.5,1[$$
$$CL_1^{B4} : x^2 = -2\,x^1 \quad if\ x^2 > 0,\ x^1 \in [-0.5,0]$$

Critical Lines C

For $CL_{-1}^{C} : x^2 = 0$

$$CL_1^{C1} : x^2 = -\frac{1}{2}\left(x^1 + 1\right) \quad if\ x^1 > 0$$
$$CL_1^{C2} : x^2 = \frac{1}{2}\left(x^1 + 1\right) \quad if\ x^1 < 0$$

Critical Lines D

For $CL_{-1}^{D} : x^2 = -1$

$$CL_1^{D1} : x^2 = \frac{x^1}{2} \quad if\ x^1 < 0,\ x^2 \in [0,0.5]$$
$$CL_1^{D2} : x^2 = -\frac{x^1}{2} - 1 \quad if\ x^1 > 0,\ x^2 \in [-1,-0.5]$$
$$CL_1^{D3} : x^2 = -\frac{x^1}{2} \quad if\ x^1 > 0,\ x^2 \in [-0.5,0]$$
$$CL_1^{D4} : x^2 = \frac{x^1}{2} + 1 \quad if\ x^1 < 0,\ x^2 \in [0.5,1[$$

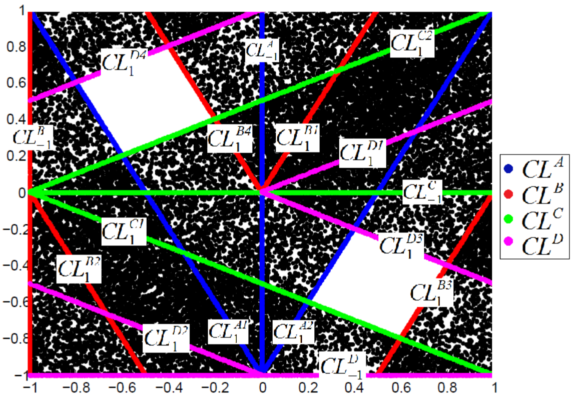


Figure 2. Critical lines of the map M_2 (1) on the torus $T^2 = [-1,1[^{\,2}$

Fig.2 shows the invariant measure associated to the chaotic mapping for the second order system (the transient of the first 10^6 iterations has been cut off). It can be observed that this measure is not constant everywhere and vanishes on two diamond-like holes. The latter are completely delimited by segments of the critical lines CL_1^{A1}, CL_1^{B4}, CL_1^{C2}, CL_1^{D4}, and CL_1^{A2}, CL_1^{B3}, CL_1^{C1}, CL_1^{D3} so there evolution can be analytically followed under parameter or system order variation. Moreover, as the holes are symmetrical, the signal could be considered as symmetrically distributed as well (i.e. there are as many points in the half plane x<0 than in the half plane x>0). In the case where k^1=1 and k^2=-1, the invariant measure does not present the same pattern, see fig. 3, and needs a more sophisticated study.

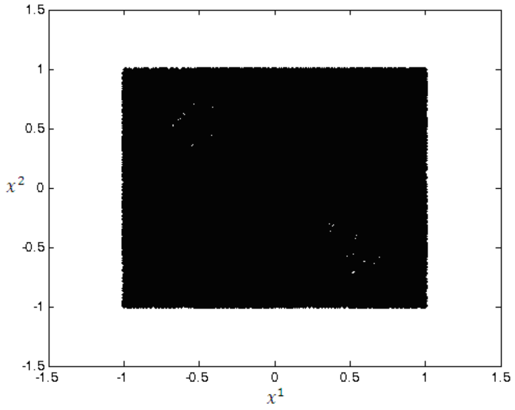


Figure 3. Invariant measure associated to the map M_2 (1) on the torus $T^2 = [-1,1[^{\,2}$ where k^1=1 and k^2=-1

3 NIST Tests for the 4D-System

The measure of the holes decreases with the increase of the system dimension, and becomes neglectable for the four dimensional Lozi system M_4 (1) which shall be considered hereafter. As an example Fig 4 (resp.5) shows the projection of the invariant measure of 3D (resp.4D) - systems onto the (x^1, x^2) coordinates for values of x^3 and x^4 chosen inside the diamond-like holes, $0.49 \le x^3 \le 0.51$ (resp. $0.49 \le x^3, x^4 \le 0.51$).

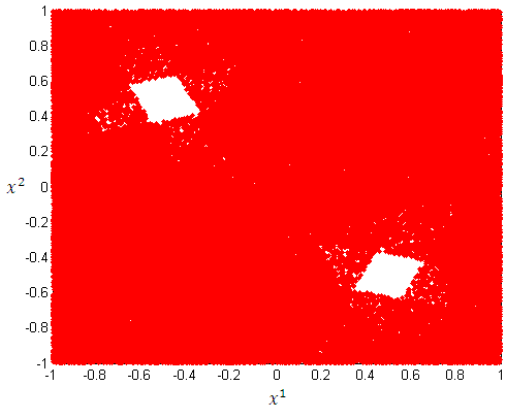


Figure 4. Invariant measure associated to the map M_3 (1) on the torus T^3 = [-1, 1[^{3}

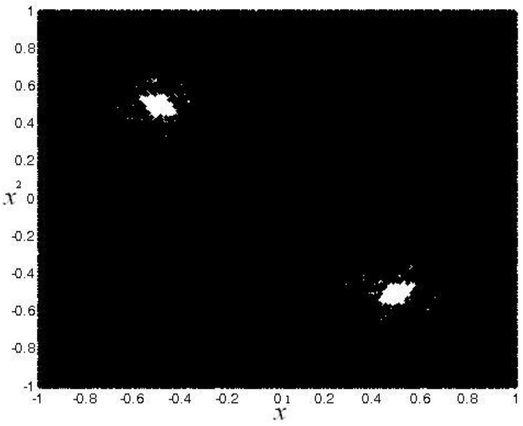


Figure 5. Invariant measure associated to the map M_4 (1) on the torus T^4 = [-1, 1[^{4}

The output of the system has been arbitrary chosen as:

$$y_n = x_n^1$$

The NIST tests proposed by the National Institute of Standards and Technology [4] require a binary signal, whereas the generated chaotic output contains real values (floating point numbers), therefore an appropriate binarization has to be performed prior to applying the tests. Several types of floating-point binarization have been tested, such as the standard IEEE754 [5] that allows the conversion in single and double formats (32 or 64 bits). However, as the map (1) generates only numbers between $[-1,1[$, a significant number of bits remain always constant. Thus, periodic patterns appear necessarily in the sequence.

Finally, the selected approach in this paper has been to choose a threshold value, and classify all the numbers above this threshold as ones, and all the others as zeros.

The optimal threshold is a tricky problem, for the system (1) it has been naturally chosen as:

$$\begin{array}{ll} if\ y_n \geq 0 & b_n = 1 \\ else & b_n = 0 \end{array} \quad (4)$$

For the 4th order system M_4, the holes become much smaller, and it can be assumed that the output signal is equidistributed.

Length of the original sequence: 10^8 bits.

Initial condition (randomly chosen)

$$x_0 = [0.6324,-0.0975,0.2785,-0.5469]$$

The statistical test evaluates the randomness of the sequence: the null hypothesis (H0) assumes that it is random, and the alternative hypothesis (Ha) assumes that it is not random.

For a successful test, the sequence must be accepted as being a random. The P-value (table 1) is a complex quantifier used to measure if the zeroes and the ones in the sequence can be considered as uniformly distributed.

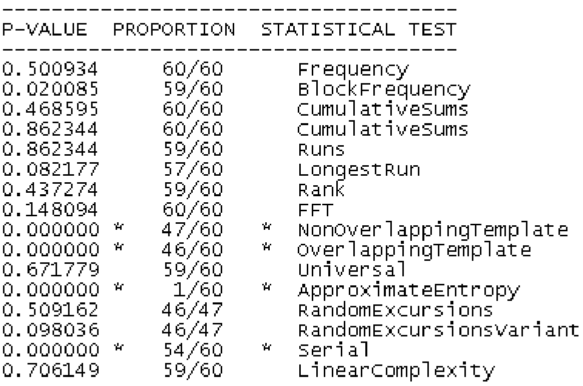
A test is successful if the p-value is superior to the significance level (for this case 0.01). In addition, the minimum pass rate for each statistical test with the exception of the random excursion (variant) test is approximately = 57 for a sample size = 60 binary sequences (approximately = 96 for a sample size = 100). The minimum pass rate for the random excursion (variant) test is approximately = 46 for a sample size = 49 binary sequences (approximately = 64 for a sample size = 68). The failing tests are denoted by an asterisk (*).

For the NIST test, each bit stream is considered as a different sequence, so in order to evaluate the results, different lengths bit stream have been tested and compared.

For the first test, the original sequence has been divided into ten sub-sequences (or bit strings) having a length of 10^7 points.

Test 1 (shown in Table one)

Length of bit string: 1.666.666
Quantity of bit strings: 60



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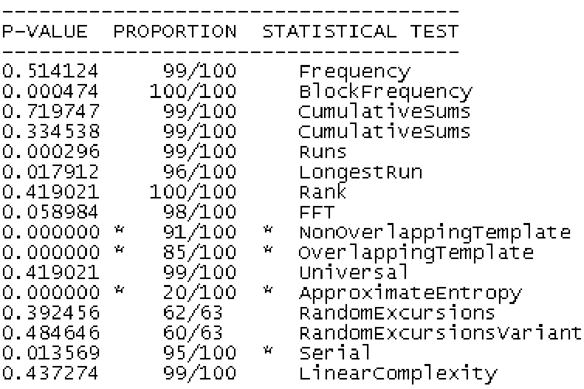
------------------------------------
P-VALUE   PROPORTION   STATISTICAL TEST
------------------------------------
0.500934      60/60       Frequency
0.020085      59/60       BlockFrequency
0.468595      60/60       CumulativeSums
0.862344      60/60       CumulativeSums
0.862344      59/60       Runs
0.082177      57/60       LongestRun
0.437274      59/60       Rank
0.148094      60/60       FFT
0.000000 *    47/60    *  NonOverlappingTemplate
0.000000 *    46/60    *  OverlappingTemplate
0.671779      59/60       Universal
0.000000 *     1/60    *  ApproximateEntropy
0.509162      46/47       RandomExcursions
0.098036      46/47       RandomExcursionsVariant
0.000000 *    54/60    *  Serial
0.706149      59/60       LinearComplexity
```

Table 1.

Conclusion: four of the tests fail.

For the second test, the original sequence has been divided into 100 bit strings, each of length 10^6.

Test 2 (shown in Table two)
Length of bit string: 1M
Quantity of bit strings: 100



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------------------------------------
P-VALUE   PROPORTION   STATISTICAL TEST
------------------------------------
0.514124      99/100      Frequency
0.000474     100/100      BlockFrequency
0.719747      99/100      CumulativeSums
0.334538      99/100      CumulativeSums
0.000296      99/100      Runs
0.017912      96/100      LongestRun
0.419021     100/100      Rank
0.058984      98/100      FFT
0.000000 *    91/100   *  NonOverlappingTemplate
0.000000 *    85/100   *  OverlappingTemplate
0.419021      99/100      Universal
0.000000 *    20/100   *  ApproximateEntropy
0.392456      62/63       RandomExcursions
0.484646      60/63       RandomExcursionsVariant
0.013569      95/100   *  Serial
0.437274      99/100      LinearComplexity
```

Table 2.

Conclusion: again, four of the tests fail.

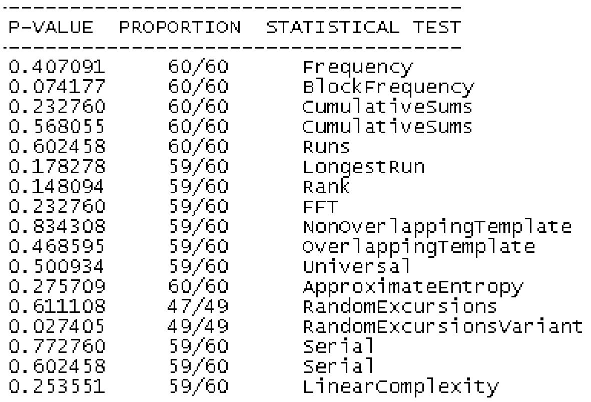
To improve the results, we applied an under-sampling which has been shown to improve the statistical properties of the signal [3]. For a sequence *S*, we take one bit out of ten, periodically.

$$S_{(10k)},\ k \in \mathbb{N}$$

Same initial condition:

$$x_0 = [0.6324,-0.0975,0.2785,-0.5469]$$

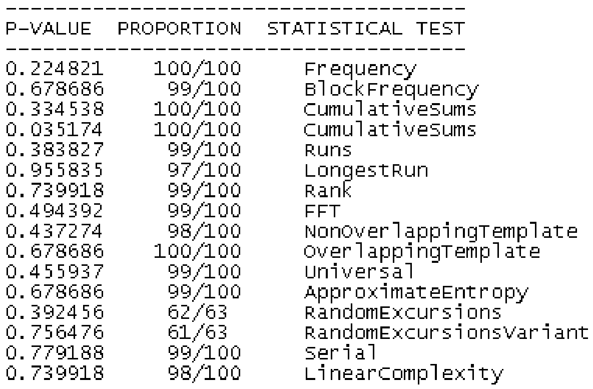
Test 1
Length of bit string: 1.666.666
Quantity of bit strings: 60



```
-------------------------------------
P-VALUE  PROPORTION  STATISTICAL TEST
-------------------------------------
0.407091     60/60      Frequency
0.074177     60/60      BlockFrequency
0.232760     60/60      CumulativeSums
0.568055     60/60      CumulativeSums
0.602458     60/60      Runs
0.178278     59/60      LongestRun
0.148094     59/60      Rank
0.232760     59/60      FFT
0.834308     59/60      NonOverlappingTemplate
0.468595     59/60      OverlappingTemplate
0.500934     59/60      Universal
0.275709     60/60      ApproximateEntropy
0.611108     47/49      RandomExcursions
0.027405     49/49      RandomExcursionsVariant
0.772760     59/60      Serial
0.602458     59/60      Serial
0.253551     59/60      LinearComplexity
```

Table 3.

Test 2
Length of bit string: 1M
Quantity of bit strings: 100



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P-VALUE  PROPORTION  STATISTICAL TEST
-------------------------------------
0.224821    100/100     Frequency
0.678686     99/100     BlockFrequency
0.334538    100/100     CumulativeSums
0.035174    100/100     CumulativeSums
0.383827     99/100     Runs
0.955835     97/100     LongestRun
0.739918     99/100     Rank
0.494392     99/100     FFT
0.437274     98/100     NonOverlappingTemplate
0.678686    100/100     OverlappingTemplate
0.455937     99/100     Universal
0.678686     99/100     ApproximateEntropy
0.392456     62/63      RandomExcursions
0.756476     61/63      RandomExcursionsVariant
0.779188     99/100     Serial
0.739918     98/100     LinearComplexity
```

Table 4.

Now, all tests are statistically successful; moreover it should be emphasized that the results do not depend on the bit strings quantity

4 Conclusion

Dynamical and statistical analysis demonstrated the efficiency of a new map firstly proposed by Lozi. The NIST tests carried out have been improved using a constant under-sampling. Future work with chaotic under-sampling has to be envisaged. The main difference between this model and other chaotic pseudo-random number generators is that this map not only provides one single stream of pseudo-random number but several uncorrelated parallel streams of numbers. This property is very useful in the case of simulation of multi-agent complex problem.

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