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ON ONE METHOD OF ANALYSIS OF LAGRANGE SYSTEMS

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Abstract

The paper discusses the problem of finding and qualitative investigation of invariant manifolds (IMs) of Lagrange systems with cyclic coordinates.

Key words

Routh function, invariant manifolds, cyclic coordinates, stability

1 Introduction

It is well known that adding the complete differential to the characteristic function under the integral of action does not provoke any change in the form of differential equations, which correspond to the given characteristic function. This is equivalent to adding the time derivative of the function, which is dependent on the generalized coordinates, to the characteristic function.

This is the technique by which we propose to use the "extended" Routh functions for purpose of deriving and qualitative investigation of invariant manifolds of the respective Routh and Lagrange systems of equations [Arnold, Kozlov, Neyshtadt, 1985] on the basis of the approach , which is similar to Routh-Lyapunov's method [Lyapunov, 1954], [Irtegov, 1985]. We use proposed method to find invariant manifolds (i) in Lagrange system with two positional and with two cyclic coordinates, (ii) for rigid body with a fixed point in Euler's case.

2 Invariant manifolds of the system, whose reduced system is linear

Consider the Lagrange system with two cyclic coordinates $(q_1,\ q_2)$ and two positional coordinates $(q_3,\ q_4)$, whose Routh function is quadratic with respect to all the phase variables:

$$R = \frac{1}{2}\big(-b_2 p_1^2 + 2b_4 p_1 p_2 - b_1 p_2^2 + B_1 q_3^2 + B_3 q_4^2 \\ + c_1 ({q_3}')^2 + 2c_3 {q_3}'{q_4}' + c_2({q_4}')^2 + 2q_3(B_2 q_4 \\ + A_1 {q_3}' + A_3 {q_4}') + q_4(A_2 {q_3}' + A_4 {q_4}')\big), \tag{1}$$

where $p_1,\ p_2$ are constant of cyclic integrals of the initial Lagrange system, coefficients A_i, B_i are also expressed via $p_1,\ p_2$ and via coefficients of the initial differential system:

$$A_3 = p_1\omega_{33} + p_2\omega_{43},\ A_4 = p_1\omega_{34} + p_2\omega_{44},\\ A_1 = p_1\omega_{13} + p_2\omega_{23},\ A_2 = p_1\omega_{14} + p_2\omega_{24},\\ B_1 = g - a_{23}p_1^2 + 2a_3 p_1 p_2 - a_{13}p_2^2,\\ B_3 = f - a_{24}p_1^2 + 2a_4 p_1 p_2 - a_{14}p_2^2, B_2 = h.$$

The Routh function (1) contains the addends with the coefficients A_1, A_4, that is to say contains the function $d(A_1 {q_3}^2 + A_4 {q_4}^2)/dt$. These are not included into the Routh differential equations. We call this Routh function an "extended" Routh function.

Consider the stationary conditions for the function (1)

$$\left.\begin{array}{r} \partial R/\partial q_3 = B_1 q_3 + B_2 q_4 + A_1 {q_3}' + A_3 {q_4}' = 0,\\ \partial R/\partial q_4 = B_2 q_3 + B_3 q_4 + A_2 {q_3}' + A_4 {q_4}' = 0,\\ \partial R/\partial {q_4}' = A_3 q_3 + A_4 q_4 + c_3 {q_3}' + c_2 {q_4}' = 0,\\ \partial R/\partial {q_3}' = A_1 q_3 + A_2 q_4 + c_1 {q_3}' + c_3 {q_4}' = 0. \end{array}\right\} \tag{2}$$

By imposing the constraints on the constant coefficients of function R, it is possible to make equations of system (2) dependent. Hence equations (2), which remain independent, define the invariant manifolds of the Routh equations.

We have conducted complete analysis of the conditions needed for making zero the determinant of system

(2), which is linear with respect to the phase variables. The determinant of system (2)

$$\Delta_4 = \det \begin{pmatrix} c_1 & c_3 & A_1 & A_2 \\ c_3 & c_2 & A_3 & A_4 \\ A_1 & A_3 & B_1 & B_2 \\ A_2 & A_4 & B_2 & B_3 \end{pmatrix} \qquad (3)$$

can be submitted as

$$\begin{array}{l} \Delta_4 = \Delta_{a1}^2 - \Delta_b \Delta_c - \Delta_{24}\Delta_{q13} + \Delta_{23}\Delta_{q14} \\ \quad +\Delta_{14}\Delta_{q23} - \Delta_{13}\Delta_{q24}, \end{array} \qquad (4)$$

here

$$\begin{array}{c} \Delta_{13} = A_3 c_1 - A_1 c_3,\ \Delta_{23} = -A_1 c_2 + A_3 c_3, \\ \Delta_{14} = A_4 c_1 - A_2 c_3,\ \Delta_{24} = -A_2 c_2 + A_4 c_3, \\ \Delta_{q24} = -A_4 B_2 + A_3 B_3,\ \Delta_{q14} = -A_2 B_2 + A_1 B_3, \\ \Delta_{q13} = -A_2 B_1 + A_1 B_2,\ \Delta_{q23} = -A_4 B_1 + A_3 B_2, \\ \Delta_{a1} = A_2 A_3 - A_1 A_4,\ \Delta_c = c_1 c_2 - c_3^2, \\ \Delta_b = B_2^2 - B_1 B_3 \end{array}$$

When considering different variants of turning the addends into zero Δ_4, so that $\Delta_4 = 0$, we obtain the condition of singularity of the matrix of linear system (2). There are right 44 variants. This has given us the possibility to identify the two groups of IMs: 1. The IM is described by two equations; 2. the IM is described by three equations. To investigate the stability of IMs obtained we have used the Routh function for the purpose of constructing the Lyapunov function .

2.1 Example 1

Under the constraints imposed on the coefficients in function (1)

$$\begin{array}{l} B_1 = \dfrac{A_3^2 c_1}{c_3^2},\ A_4 = \dfrac{A_2 c_2}{c_3},\ B_3 = \dfrac{A_2^2 c_2}{c_3^2}, \\ B_2 = \dfrac{A_2 A_3}{c_3},\ A_1 = \dfrac{A_3 c_1}{c_3} \end{array}$$

independent in (2) are only the two equations, which define the IM of the Routh equations

$$\Gamma_1 = \frac{A_3 q_3}{c_3} + {q_3}' = 0,\ \Gamma_2 = \frac{A_2 q_4}{c_3} + {q_4}' = 0 \qquad (5)$$

When represented in terms of variables Γ_i the Routh function (1) writes:

$$\begin{array}{l} \Delta R_1 = \dfrac{1}{2}\big(- b_2 p_1^2 + 2 b_4 p_1 p_2 - b_1 p_2^2 \\ + c_1 {\Gamma_1}^2 + 2 c_3 \Gamma_1 \Gamma_2 + c_2 {\Gamma_2}^2 \big). \end{array}$$

For the purpose of investigation of stability of IM (5) let us choose

$$\begin{array}{l} V_1 = \Delta R_1 + (b_2 p_1^2 - 2 b_4 p_1 p_2 + b_1 p_2^2)/2 = \\ \big(c_1 {\Gamma_1}^2 + 2 c_3 \Gamma_1 \Gamma_2 + c_2 {\Gamma_2}^2 \big)/2 \end{array}$$

in the capacity of the Lyapunov function, whose derivative due to the Routh equations

$$V_1' = \frac{A_3 c_1 {\Gamma_1}^2}{c_3} + \frac{(A_2 c_3 + A_3 c_3)\Gamma_1 \Gamma_2}{c_3} + \frac{A_2 c_2 {\Gamma_2}^2}{c_3}$$

is the quadratic form of its variables. IM (5) is asymptotically stable when the following conditions of definite positiveness of form V and definite negativity of V_1' hold:

$$\begin{array}{l} c_1 > 0,\ c_2 > 0, c_1 c_2 - c_3^2 > 0,\ \dfrac{A_3 c_1}{c_3} < 0, \\ -4 A_2 A_3 c_1 c_2 + (A_2 + A_3)^2 c_3^2 < 0. \end{array}$$

2.2 Example 2

If coefficients in (1) are bound up by the relations

$$\begin{array}{l} B_1 = \dfrac{B_2^2 (c_1 c_2 - c_3^2)}{A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3}, \\ A_1 = \dfrac{A_2 B_2 (c_1 c_2 - c_3^2)}{A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3}, \\ B_3 = \dfrac{A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3}{c_1 c_2 - c_3^2}, \\ A_3 = \dfrac{A_4 B_2 (c_1 c_2 - c_3^2)}{A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3}, \end{array}$$

then the Routh equations assume the IM:

$$\left. \begin{array}{l} \Psi_2 = {q_4}' + \Big(\big((A_4 c_1 - A_2 c_3) \big(B_2 (c_1 c_2 - c_3^2) q_3 \\ + \big(A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3 \big) q_4 \big) \big) / \\ \big((A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3) (c_1 c_2 - c_3^2) \big) \Big) = 0, \\ \Psi_1 = {q_3}' + \Big(\big((A_2 c_2 - A_4 c_3) \big(B_2 (c_1 c_2 - c_3^2) q_3 \\ + \big(A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3 \big) q_4 \big) \big) / \\ \big((A_4^2 c_1 + A_2^2 c_2 - 2 A_2 A_4 c_3) (c_1 c_2 - c_3^2) \big) \Big) = 0. \end{array} \right\} \qquad (6)$$

When represented in terms of variables (6) the Routh function (1) has the form

$$\begin{array}{l} \Delta R_2 = \dfrac{1}{2}\big(- b_2 p_1^2 + 2 b_4 p_1 p_2 - b_1 p_2^2 \\ + c_1 {\Psi_1}^2 + 2 c_3 \Psi_1 \Psi_2 + c_2 {\Psi_2}^2 \big). \end{array}$$

Introduce the function

$$\begin{array}{l} V_2 = \Delta R_2 - (-b_2 p_1^2 + 2 b_4 p_1 p_2 - b_1 p_2^2)/2 = \\ \big(c_1 {\Psi_1}^2 + 2 c_3 \Psi_1 \Psi_2 + c_2 {\Psi_2}^2 \big)/2 \end{array}$$

and compute its derivative due to the Routh differential equations:

$$V_2' = \big((A_2\Psi_1 + A_4\Psi_2)\big(B_2\big(c_1c_2 - c_3^2\big)\Psi_1 + \big(A_4^2c_1 + A_2^2c_2 - 2A_2A_4c_3\big)\Psi_2\big)\big)/ \big(A_4^2c_1 + A_2^2c_2 - 2A_2A_4c_3\big).$$

Unfortunately, this derivative cannot be sign-definite with respect to all its variables. Let $B_2 = 0, A_2 = 0$. Hence $V' = A_4\Psi_2^2$, and the IM under scrutiny is stable when the following conditions hold: $(c_1c_2 - c_3^2) > 0,\ c_1 > 0,\ A_4 < 0$.

2.3 Example 3

If coefficients in (1) are bound up by the relations

$$A_1 = \frac{1}{A_3(A_3^2B_3 - B_2^2c_2)}(A_3^4B_2 + A_3^2B_2B_3c_1 - B_2^3c_1c_2 - 2A_3^2B_2^2c_3 + B_2^3c_3^2),\ A_2 = A_3,$$
$$B_1 = -\frac{1}{A_3^2(A_3^2B_3 - B_2^2c_2)}(B_2^2(-A_3^4 - A_3^2B_3c_1 + B_2^2c_1c_2 + 2A_3^2B_2c_3 - B_2^2c_3^2),\ A_4 = \frac{A_3B_3}{B_2},$$

then the Routh equations assume the invariant manifold:

$$\left.\begin{aligned} &\Phi_1 = -B_2\big(-A_3^4 - A_3^2B_3c_1 + B_2^2c_1c_2 \\ &+2A_3^2B_2c_3 - B_2^2c_3^2\big)(B_2q_3 + A_3q_3') \\ &+A_3^2\big(A_3^2B_3 - B_2^2c_2\big)(B_2q_4 + A_3q_4'), \\ &\Phi_2 = B_2^2q_3 + B_2B_3q_4 + A_3B_2q_3' \\ &+A_3B_3q_4',\ \Phi_3 = A_3B_2q_3 + A_3B_3q_4 \\ &+B_2c_3q_3' + B_2c_2q_4' \end{aligned}\right\} \quad (7)$$

The differential equations for Φ_i are as follows:

$$\left.\begin{aligned} &\Phi_1' = (W_2\Phi_1 + A_3^2B_2(A_3^2 - B_2c_3)(-W_2\Phi_2 \\ &+A_3W_1\Phi_3))/(A_3(A_3^2B_3 - B_2^2c_2)(c_1c_2 - c_3^2)), \\ &\Phi_2' = \big((-B_2c_2 + B_3c_3)\Phi_1 + A_3^2B_2\big(A_3^2 \\ &-B_2c_3\big)(B_2c_2 - B_3c_3)\Phi_2 - A_3B_2W_1\Phi_3\big)/ \\ &\big(A_3B_2\big(-A_3^2B_3 + B_2^2c_2\big)\big(c_1c_2 - c_3^2\big)\big), \\ &\Phi_3' = \Phi_2, \end{aligned}\right\} \quad (8)$$

here

$$W_1 = \big(A_3^2B_3^2c_1 + A_3^2B_2^2c_2 - B_2^2B_3c_1c_2 - 2A_3^2B_2B_3c_3 + B_2^2B_3c_3^2\big),$$
$$W_2 = \big(A_3^4B_2c_2 + A_3^2B_2B_3c_1c_2 - B_2^3c_1c_2^2 - A_3^4B_3c_3 - A_3^2B_2^2c_2c_3 + B_2^3c_2c_3^2\big).$$

The Routh function is expressed by Φ_i

$$\Delta R_3 = \frac{1}{2}\big(-b_2p_1^2 + 2b_4p_1p_2 - b_1p_2^2\big) + \frac{B_3\Phi_1^2}{2A_3^2B_2^2(A_3^2B_3 - B_2^2c_2)W_1} - \frac{\Phi_1\Phi_2}{B_2W_1} + \big(\big(A_3^6B_3 - 2A_3^4B_2^2c_2 - A_3^2B_2^2B_3c_1c_2 + B_2^4c_1c_2^2 + 2A_3^2B_2^3c_2c_3 - B_2^4c_2c_3^2\big)\Phi_2^2\big)/\big(2\big(A_3^2B_3 - B_2^2c_2\big)W_1\big) + \frac{A_3B_2\Phi_2\Phi_3}{A_3^2B_3 - B_2^2c_2} - \frac{B_2^2\Phi_3^2}{2(A_3^2B_3 - B_2^2c_2)}.$$

For the purpose of investigation of stability of IM (7) let us choose

$$V_3 = \Delta R_3 - \big(-b_2p_1^2 + 2b_4p_1p_2 - b_1p_2^2\big)/2 \quad (9)$$

in the capacity of the Lyapunov function. The derivative of V_3 (9) due to the equations (8):

$$V_3' = \big(c_2\Phi_1^2 + A_3^4B_2^2\big(A_3^4c_2 + A_3^2B_3c_1c_2 - B_2^2c_1c_2^2 - 2A_3^2B_2c_2c_3 - A_3^2B_3c_3^2 + 2B_2^2c_2c_3^2\big)\Phi_2^2 - 2A_3^3B_2^2\big(A_3^4B_2c_2 + A_3^2B_2B_3c_1c_2 - B_2^3c_1c_2^2 - A_3^4B_3c_3 - A_3^2B_2^2c_2c_3 + B_2^3c_2c_3^2\big)\Phi_2\Phi_3 + A_3^4B_2^2W_1\Phi_3^2 - 2A_3^2B_2\Phi_1\big(c_2\big(A_3^2 - B_2c_3\big)\Phi_2 - A_3(B_2c_2 - B_3c_3)\Phi_3\big)\big)/ \big(A_3^3B_2\big(A_3^2B_3 - B_2^2c_2\big)^2\big(c_1c_2 - c_3^2\big)\big) \quad (10)$$

The Silvester conditions of sign-definiteness of the forms (9) and (10) cannot be satisfied simultaneously. If form (10) is sign-definite then satisfied are the conditions of Lyapunov's theorem on instability, and hence IM (7) is unstable.

The IMs of the Routh equations obtained may be "brought up" into the phase space of the respective Lagrange system with Lagrangian

$$L == \tfrac{1}{2}\Big(gq_3^2 + 2hq_3q_4 + fq_4^2 + c_1(q_3')^2 + 2c_3q_3'q_4' + c_2(q_4')^2 + \tfrac{1}{(-M_{12}^2 + M_{11}M_{22})}\big(M_{11}(q_1' + (q_3\omega_{13} + q_4\omega_{14})q_3' + (q_3\omega_{33} + q_4\omega_{34})q_4')^2 + 2M_{12}(q_1' + (q_3\omega_{13} + q_4\omega_{14})q_3' + (q_3\omega_{33} + q_4\omega_{34})q_4') (q_2' + (q_3\omega_{23} + q_4\omega_{24})q_3' + (q_3\omega_{43} + q_4\omega_{44})q_4') + M_{22}(q_2' + (q_3\omega_{23} + q_4\omega_{24})q_3' + (q_3\omega_{43} + q_4\omega_{44})q_4')^2)\Big)$$

(here $M_{11} = \big(b_1 + a_{13}q_3^2 + a_{14}q_4^2\big),\ M_{22} = \big(b_2 + a_{23}q_3^2 + a_{24}q_4^2\big),\ M_{12} = \big(b_4 + a_3q_3^2 + a_4q_4^2\big)$). To this end, it is sufficient to add equations of cyclic integrals to the equations, which define IM for the Routh equations. This can easily be verified by using the definition of IM [Irtegov, Titorenko, 2010].

3 A rigid body with a fixed point in Euler's case

In the capacity of the second problem we shall consider the motion of a rigid body with a fixed point in Euler's case. This system's Lagrangian in terms of Euler's angles θ, ϕ, ψ has the form [Macmillan, 1936]:

$$L == \tfrac{1}{2}(A\cos[\varphi]^2 + B\sin[\varphi]^2)\theta'^2 + \tfrac{1}{2}C\varphi'^2 + \\ ((A-B)\cos[\varphi]\sin[\theta]\sin[\varphi]\theta' + C\cos[\theta]\varphi'[t])\psi' \\ + \tfrac{1}{2}M\psi'^2, \tag{11}$$

where

$$M = C\cos[\theta]^2 + \sin[\theta]^2\left(B\cos[\varphi]^2 + A\sin[\varphi]^2\right).$$

It is known that the differential equations in this case assume the cyclic integral

$$\partial L/\partial\psi' = ((A-B)\cos[\varphi]\sin[\theta]\sin[\varphi]\theta' \\ + C\cos[\theta]\varphi' + M\psi' = p_1. \tag{12}$$

Let us conduct reduction of the system. In this case, the Legendre transformation allows one to obtain the following Routh function

$$R = L - p_1\psi', \tag{13}$$

where ψ' has to be removed with the aid of (12).

By adding the addend $f[\theta]\dot{\theta}$ to R, we obtain the "extended" Routh function

$$\tilde{R} = \big(-p_1^2 + \big(AB\sin[\theta]^2 + C\cos[\theta]^2\big(A\cos[\varphi]^2 + \\ B\sin[\varphi]^2\big)\big)\theta'^2 + C\sin[\theta]^2\big(B\cos[\varphi]^2 + A\sin[\varphi]^2\big)\varphi'^2 \\ + 2C\cos[\theta]p_1\varphi' + (A-B)\sin[\theta]\sin[2\varphi]\theta'(p_1 \\ - C\cos[\theta]\varphi')\big)/2M + f[\theta]\theta'$$

The conditions of stationarity of $\tilde{R}$ with respect to the phase variables $\theta, \varphi, \theta', \varphi'$

$$\left.\begin{array}{l}
\cos[\theta]p_1 + \sin[\theta]\big((B-A)\cos[\theta]\cos[\varphi]\sin[\varphi]\theta' \\
+\sin[\theta]\big(B\cos[\varphi]^2 + A\sin[\varphi]^2\big)\varphi'\big) = 0, \\
M\,f[\theta] + (A-B)\cos[\varphi]\sin[\theta]\sin[\varphi]p_1 \\
+\big(AB\sin[\theta]^2 + C\cos[\theta]^2\big(A\cos[\varphi]^2 + B\sin[\varphi]^2\big)\big)\theta' \\
-(A-B)C\cos[\theta]\cos[\varphi]\sin[\theta]\sin[\varphi]\varphi' = 0, \\
(A-B)\big(2\cos[\varphi]\sin[\theta]p_1 - 2\big(C\cos[\theta]^2 \\
+A\sin[\theta]^2\big)\sin[\varphi]\theta' - C\cos[\varphi]\sin[2\theta]\varphi'\big) \\
\big(\sin[\theta]\sin[\varphi]p_1 + \cos[\varphi]\big(C\cos[\theta]^2 + B\sin[\theta]^2\big)\theta' \\
-C\cos[\theta]\sin[\theta]\sin[\varphi]\varphi'\big) = 0, \\
4\Big(\cos[\theta]\sin[\theta]\big(-C + B\cos[\varphi]^2 + A\sin[\varphi]^2\big)p_1^2 \\
-\tfrac{1}{4}(A-B)^2C\cos[\theta]\sin[\theta]\sin[2\varphi]^2\theta'^2 \\
-C\sin[\theta]\big(\cos[\theta]^2(A+B-C-(A-B)\cos[2\varphi]) \\
+\sin[\theta]^2\big(B\cos[\varphi]^2 + A\sin[\varphi]^2\big)\big)p_1\varphi' \\
+C^2\cos[\theta]\sin[\theta]\big(B(\cos[\varphi])^2 + A\sin[\varphi]^2\big)\varphi'^2 \\
+\theta'\big((A-B)\cos[\theta]\cos[\varphi]\sin[\varphi] \\
\big(C\cos[\theta]^2 + \sin[\theta]^2\big(2C - B\cos[\varphi]^2 - A\sin[\varphi]^2\big)\big)p_1 \\
+\big(C\cos[\theta]^2 + \sin[\theta]^2\big(B\cos[\varphi]^2 + A\sin[\varphi]^2\big)\big)^2 f'[\theta] \\
-(A-B)C\cos[\varphi]\sin[\varphi]\big(C\cos[\theta]^2 \\
-\sin[\theta]^2\big(B\cos[\varphi]^2 + A\sin[\varphi]^2\big)\big)\varphi'\big)\Big) = 0,
\end{array}\right\}$$

(here $f[\theta]$ is considered as unknown) have the following of solutions:

$$\left.\begin{array}{l}
\Upsilon_1 = -cot[\varphi]csc[\theta]p_1 + A\theta' = 0, \\
\Upsilon_2 = cot[\theta]csc[\theta]p_1 + A\varphi' = 0, \\
\Upsilon_3 = f[\theta]\big(C\cos[\theta]^2 + B\cos[\varphi]^2\sin[\theta]^2 \\
+A\sin[\theta]^2\sin[\varphi]^2\big) + (A \\
-B)\cos[\varphi]\sin[\theta]\sin[\varphi]p_1 + \big(AB\sin[\theta]^2 \\
+C(\cos[\theta])^2\big(A\cos[\varphi]^2 + B\sin[\varphi]^2\big)\big)\theta' \\
-(A-B)C\cos[\theta]\cos[\varphi]\sin[\theta]\sin[\varphi]\varphi' = 0
\end{array}\right\} \tag{14}$$

where $f[\theta]$ is determined by the equation

$$\cot[\theta](\csc[\theta])^2p_1^2 == f[\theta]f'[\theta].$$

Two families of the solutions $f[\theta]$::

$$f[\theta] = \pm\sqrt{2}\sqrt{C[1] - \tfrac{1}{2}\cot[\theta]^2p_1^2}, \tag{15}$$

here $C[1] > 0$ are constant of integration.

From (14) we find

$$\cos[\varphi] = -\frac{f[\theta]}{\sqrt{f[\theta]^2 + \csc[\theta]^2p_1^2}} \tag{16}$$

It can readily be verified that three equations (14) define the family of IMs for the Routh equations.

Example. Define one of the subfamilies of this family. Choose the second solution in (15)

$$f[\theta] = -\sqrt{2}\sqrt{C[1] - \frac{1}{2}\cot[\theta]^2 p_1^2},$$

hence from (14) we obtain the following differential equation for θ

$$\theta' = \sqrt{2}\sqrt{C[1] - 1/2\cot[\theta]^2 p_1^2}/A. \quad (17)$$

Equation (17) is integrated to obtain elementary functions:

$$\theta = \mp ArcSec[\frac{2\sqrt{2}\zeta\sqrt{2C[1]+p_1^2}}{1+4\zeta^2 C[1]}],$$

here

$$\zeta = \exp(\left(\mathbf{i}t + \sqrt{2}AC[2]\right)\sqrt{2C[1]+p_1^2}/A),$$
$$C[2] = const.$$

If

$$C[2] = \frac{\log\left[1/(4C[1])\right]}{2\sqrt{2}\sqrt{2C[1]+p_1^2}},$$

then (17) has the real solution (family)

$$\cos[\theta] == \frac{\sqrt{2}\sqrt{C[1]}\cos[t\omega]}{\sqrt{2C[1]+p_1^2}},\ \omega = \frac{\sqrt{2C[1]+p_1^2}}{A}. \quad (18)$$

The solution (18) we shall substitute in (16):

$$\cos[\varphi] == \sqrt{2}\sqrt{\frac{C[1]}{2C[1]+\csc[t\omega]^2 p_1^2}} \quad (19)$$

The family of solutions (18) and (19) lies on IM (14) and satisfies Routh equations containing function (13).

These IMs may be "brought up" into the phase space of the system with the Lagrangian (11). To this end it is sufficient to add relation (12) (the cyclic integral) to (14):

$$\Upsilon_4 = (A-B)\cos[\varphi]\sin[\theta]\sin[\varphi]\theta' + C\cos[\theta]\varphi'$$
$$+\left(C\cos[\theta]^2 + \sin[\theta]^2\left(B\cos[\varphi]^2\right.\right.$$
$$\left.\left.+A\sin[\varphi]^2\right)\right)\psi' - p_1 = 0$$

All the computational difficulties, which are bound up with a large volume of trigonometric transformations, have been overcome with the aid of CAS MATHEMATICA.

4 Conclusion

So, the problem of obtaining the stationary invariant manifolds with the aid of "extended" characteristic functions is reduced to finding the solutions of some additional system of equations. Additional equations respect to functions defining invariant manifolds can be algebraic equations (as in task 1) or differential equations (as in task 2). The stationarity property of such invariant manifolds allows us to apply the 2nd Lyapunov method in their qualitative analysis.

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