One side invertibility for implicit hyperbolic systems with delays

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Abstract— **This paper deals with left invertibility problem of implicit hyperbolic systems with delays in infinite dimensional Hilbert spaces. From a decomposition procedure, invertibility for this class of systems is shown to be equivalent to the left invertibility of a subsystem without delays.**

I. Introduction

We shall deal here with left invertibility for a class of implicit systems with delays in Hilbert spaces which are either left or right invertible (hereafter called "one side invertible") and whose state is described by

$$E\ddot{z}(t) + \alpha\dot{z}(t) + A_0 z(t) + A_1 z(t-h) = Bu(t) \qquad (1)$$

With the output function

$$y(t) = Cz(t) \qquad (2)$$

where $z(0) = 0$, $\dot{z}(0) = 0$ et $z(t) \equiv 0$, $\forall t \in [-h, 0[$, $(h > 0)$, $\alpha \geq 0$, $\ddot{z}(t)$, $\dot{z}(t)$, $z(t) \in H$ (the state space); $u(t) \in U$ (the input space); $y(t) \in Y$ (the output space). E, A_0, A_1, B and C are linear operators defined as follows: $E : H \longrightarrow H$; $A_0 : H \longrightarrow H$; $A_1 : H \longrightarrow H$; $B : U \longrightarrow H$; $C : H \longrightarrow Y$, where U, H and Y are real Hilbert spaces.

The concept of left invertibility is the problem of determining the conditions under which a zero output corresponding to a zero initial state can only be generated by a zero input. The left invertibility problem has been investigated within its different versions in detail in the finite dimensional case. We cite [1], [3] and [6] whose approaches are generalized in this study. In infinite dimension, this notion has been generalized essentially for non implicit systems with delays and for the particular case of bounded operators (see for instance [4]).

The aim of this paper is to extend this notion in the direction of infinite dimensional linear systems with delays and to use this approach for solving left invertibility for implicit hyperbolic systems with delays. The motivation for considering this class is given by the article of Bonilla [1] that gave a natural left or right inverse for implicit descriptions without delays.

Our first contribution is to give necessary and sufficient conditions for the system to be left invertible. The paper is structured as follows. The first part deals with solvability and invertibility of this class of systems in the frequency domain. The second part deals with systems without delays. A criterion will be given for one side invertibility in this section.

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II. Solvability and one side invertibility

Taking $y = \dot{z}$, $\dot{y} = \ddot{z}$, $w = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}$,

$$\widetilde{A}_0 = \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix},\ \widetilde{A}_1 = \begin{pmatrix} 0 & 0 \\ -A_1 & 0 \end{pmatrix},$$
$$\widetilde{E} = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}, \widetilde{C} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Then the system (1), (2) can be written as:

$$(\widetilde{\Sigma}) \begin{cases} \widetilde{E}\dot{w}(t) = \widetilde{A}_0 w(t) + \widetilde{A}_1 w(t-h) + \widetilde{B}u(t) \\ \widetilde{y}(t) = \widetilde{C}w(t) \end{cases} \qquad (3)$$

As usual, u, w, $\widetilde{y}$ represent respectively the input, state and output of the system $(\widetilde{\Sigma})$. The setting is very general in the sense that $\widetilde{E}$ is not invertible, $\widetilde{A}_0$, $\widetilde{A}_1$ are unbounded operators, $\widetilde{B}$ and $\widetilde{C}$ are restricted to be bounded, uniqueness of the solution is not required, and an explicit solution, will even not be demanded.

By using Laplace transform (this was used in [2]) the system $(\widetilde{\Sigma})$ can be rewritten as follows:

$$(\widehat{\Sigma}) \begin{cases} \left(s\widetilde{E} - \widetilde{A}_0 - \widetilde{A}_1 \exp(-sh)\right) \widehat{w}(s) = \widetilde{B}\widehat{u}(s) \\ \widehat{y}(s) = \widetilde{C}\widehat{w}(s) \end{cases} \qquad (4)$$

where s is the classical Laplace variable.

Definition 1: The system $(\widehat{\Sigma})$ is said to be solvable if $(s\widetilde{E} - \widetilde{A}_0 - \widetilde{A}_1 \exp(-sh))$ is invertible.

Definition 2: The system $(\widetilde{\Sigma})$ is left invertible if the following condition is fulfilled

$$\widetilde{y}(t) \equiv 0 \implies u(t) \equiv 0 \qquad (5)$$

By $U_{-1}(s)$ we shall denote all functions that are strictly proper (see [8] for more detail and information). This notion may also be expressed in terms of transfer function of the system $(\widetilde{\Sigma})$ as follows.

Lemma 1: The system $(\widetilde{\Sigma})$ is left invertible if and only if

$$T(s, \exp(-sh))\ \widehat{u}(s) \equiv 0 \implies \widehat{u}(s) \equiv 0 \qquad (6)$$

where $\widehat{u}(.) \in U_{-1}(s)$ and $T(s, \exp(-sh) = \widetilde{C}(s\widetilde{E}\text{-}\widetilde{A}_0\text{-}\widetilde{A}_1 \exp(-sh))^{-1}\widetilde{B}$ is the transfer function of the system $(\widetilde{\Sigma})$.

Proof: It directly obtained by using the Laplace transform of input-output relation. ■

Proposition 1: Under the condition that

$$\forall s \in \mathbb{C}\ \ker \begin{bmatrix} s\widetilde{E} - \widetilde{A}_0 - \widetilde{A}_1 \exp(-sh) & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix} = \{0\} \qquad (7)$$

The system $(\widetilde{\Sigma})$ is left invertible.

Proof: This can be easily proved using the lemma1. ■

Proposition 2: If the system $(\widetilde{\Sigma})$ is left invertible, then

$$ker\begin{bmatrix} s\widetilde{E}-\widetilde{A}_0-\widetilde{A}_1\exp(-sh) & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}=\{0\} \tag{8}$$

for some s$\in\mathbb{C}$.

Proof: It follows from lemma1. ∎

Note that, in order to simplify the exposition, we just consider here systems having only one delay in the state. Our results may easily be extended to systems with several integer delays in the state, in the input and in the output.

III. Classical systems without delays and one side invertibility

We can associate with the system $(\widetilde{\Sigma})$ the following quadruples of operators $(\widetilde{D}_k,\widetilde{F}_k,\widetilde{G}_k,\widetilde{H}_k)$ representing the family of classical (without delays) implicit systems (see for instance [5], [7]), say $(\widetilde{\Sigma}_k)$:

$$(\widetilde{\Sigma}_k)\begin{cases} \widetilde{D}_k\dot{x_k}(t)=\widetilde{F}_kx_k(t)+\widetilde{G}_kv_k\left(t\right) \\ y_k\left(t\right)=\widetilde{H}_k\ x_k(t) \end{cases} \tag{9}$$

where

$$\widetilde{F}_k=\begin{pmatrix} \widetilde{A}_0 & 0 & . & 0 \\ \widetilde{A}_1 & . & . & . \\ . & . & . & 0 \\ 0 & . & \widetilde{A}_1 & \widetilde{A}_0 \end{pmatrix},\ \widetilde{D}_k=\begin{pmatrix} \widetilde{E} & 0 & . & 0 \\ 0 & \widetilde{E} & . & . \\ . & . & . & 0 \\ 0 & . & . & \widetilde{E} \end{pmatrix},$$

$$\widetilde{H}_k=\begin{pmatrix} \widetilde{C} & 0 & . & 0 \\ 0 & \widetilde{C} & . & . \\ . & . & . & 0 \\ 0 & . & . & \widetilde{C} \end{pmatrix},\ \widetilde{G}_k=\begin{pmatrix} \widetilde{B} & 0 & . & 0 \\ 0 & \widetilde{B} & . & . \\ . & . & . & 0 \\ 0 & . & . & \widetilde{B} \end{pmatrix},$$

$$x_k(t)=\begin{pmatrix} w_0(t) \\ . \\ . \\ w_k(t) \end{pmatrix},\ v_k\left(t\right)=\begin{pmatrix} u_0(t) \\ . \\ . \\ u_k(t) \end{pmatrix},$$

$w_k(t)=w(t+kh)$, $u_k(t)=u(t+kh)$ for all $t\in[0,h]$, $w_k(0)=w_{k-1}(h)$, and $u_k(0)=u_{k-1}(h)$.

Let us denote the transfer function of $(\widetilde{\Sigma}_k)$ as :

$$\Phi_k\left(s\right)=\widetilde{H}_k(s\widetilde{D}_k-\widetilde{F}_k)^{-1}\widetilde{G}_k \tag{10}$$

First we shall start by giving a result for invertibility of the operator $(s\widetilde{D}_k-\widetilde{F}_k)$.

Lemma 2: The operator $(s\widetilde{D}_k-\widetilde{F}_k)$ is invertible if and only if $(s\widetilde{E}-\widetilde{A}_0)$ is invertible.

Proof: i) Suppose that $(s\widetilde{E}-\widetilde{A}_0)$ is invertible. Let $v=(x_0,....,x_k)^T$ be a vector in the Kernel of $(s\widetilde{D}_k-\widetilde{F}_k)$. This implies

$$\begin{cases} (s\widetilde{E}-\widetilde{A}_0)x_0=0 \\ (s\widetilde{E}-\widetilde{A}_0)x_1-\widetilde{A}_1x_0=0 \\ (s\widetilde{E}-\widetilde{A}_0)x_2-\widetilde{A}_1x_1=0 \\ . \\ . \\ (s\widetilde{E}-\widetilde{A}_0)x_k-\widetilde{A}_1x_{k-1}=0 \end{cases}$$

This is equivalent to saying that $x_0=x_1=....=x_k=0$.

ii) Reciprocally, suppose that $(s\widetilde{D}_k-\widetilde{F}_k)$ is invertible and let x_0 be an element such that $(s\widetilde{E}-\widetilde{A}_0)x_0=0$, then we have

$$\begin{pmatrix} s\widetilde{E}-\widetilde{A}_0 & 0 & . & 0 \\ -\widetilde{A}_1 & s\widetilde{E}-\widetilde{A}_0 & . & 0 \\ . & . & . & . \\ 0 & . & -\widetilde{A}_1 & s\widetilde{E}-\widetilde{A}_0 \end{pmatrix}\begin{pmatrix} 0 \\ . \\ . \\ x_0 \end{pmatrix}=0$$

This implies that $x_0=0$. ∎

The next proposition is the central result of this section.

Proposition 3: The system $(\widetilde{\Sigma}_k)$ is left invertible if and only if the subsystem $\left(\widetilde{E},\widetilde{A}_0,\widetilde{B},\widetilde{C}\right)$ is also invertible.

Proof: Let $X_k=(x_0,....,x_k)^T$, $Y_k=(y_0,....,y_k)^T$ be two elements such that $(s\widetilde{D}_k-\widetilde{F}_k)X_k=Y_k$. If the operator $(s\widetilde{E}-\widetilde{A}_0)$ is invertible, then it is not hard to show that

$$\begin{pmatrix} x_0 \\ . \\ . \\ x_k \end{pmatrix}=\begin{pmatrix} R_0(s) & 0 & . & 0 \\ R_1(s) & R_0(s) & . & . \\ . & . & . & . \\ R_k(s) & . & R_1(s) & R_0(s) \end{pmatrix}\begin{pmatrix} y_0 \\ . \\ . \\ y_k \end{pmatrix}$$

where $R_i(s)=[(s\widetilde{E}-\widetilde{A}_0)^{-1}\widetilde{A}_1]^i(s\widetilde{E}-\widetilde{A}_0)^{-1}$.

Furthermore the transfer function of the system $(\widetilde{\Sigma}_k)$ is given by:

$$\Phi_k\left(s\right)=\begin{pmatrix} T_0(s) & 0 & . & 0 \\ T_1(s) & T_0(s) & . & . \\ . & . & . & . \\ T_k(s) & . & T_1(s) & T_0(s) \end{pmatrix}$$

where $T_0(s)=\widetilde{C}R_0(s)\widetilde{B}$ and $T_i(s)=\widetilde{C}R_i(s)\widetilde{B}$. According to the proof of lemma 2, we conclude that the system $(\widetilde{\Sigma}_k)$ is left invertible if and only if the subsystem $\left(\widetilde{E},\widetilde{A}_0,\widetilde{B},\widetilde{C}\right)$. ∎

As an easy corollary of this proposition one has the following result.

Corollary 1: The system $(\widetilde{\Sigma}_k)$ is left invertible if the following condition hold:

$$\forall s\in\mathbb{C}\quad \ker\begin{bmatrix} s\widetilde{E}-\widetilde{A}_0 & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}=\{0\} \tag{11}$$

Proof: It immediately results from proposition 3 and proposition 1. ∎

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