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RESONANCE OF PROPER FREQUENCIES 1:2 AS A REASON FOR HARD EXCITATION OF OSCILLATIONS FOR THE PLATE IN ULTRASONIC GAS FLOW

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Abstract

A well know problem of oscillations for plate in ultrasonic gas flow is considered. The chosen mathematical model is the boundary value problem proposed by V.V. Bolotin where aerodynamic forces are accounted for on the basis of flat section law (piston theory). The linear and nonlinear version of the problem is considered with the damping coefficient assumed to be small. It is shown that proper frequency 1:2 oscillation occurs for velocities significantly smaller than the velocity of flutter. In the nonlinear version this situation allows us to show that there always exist unstable periodic solutions in a small neighbourhood about the equilibrium state. The latter comment implies that in the range up to critical velocities there is a possibility of hard excitation oscillations which can be a result in the destruction of the construction. The analysis of the problem in the nonlinear setting is based on the direct application of the normal form method to the nonlinear boundary value problem.

Key words

bifurcation, stability, flutter.

1 Introduction

In the paper we consider the well known problem of elastic stability theory - flutter plate in ultrasonic gas flow in the nonlinear version. It follows from the result of analysis of the problem, that in most cases the linear boundary value problem does not provide answers to the major questions concerning the plate in ultrasonic gas flow. Hard oscillations may arise causing destruction of the construction particularly for velocities determined from the linear problem which are significantly smaller than the velocity of flutter. At the level of physical representation this hypothesis was formulated in V.V. Bolotin's well- known monograph [Bolotin,1961]. In this we propose a concrete mechanism partially explaining this hypothesis from mathematical point of view. It is based on the fact that accounting for such a factor as resonance of self frequency 1:2 very often leads to the appearance of unstable oscillations for those velocities of the flow for which the formal stable equilibrium state results from the linear theory. Research of the nonlinear problem is based on the application of Poincare – Dulac normal form method adapted to the boundary value problems which simulate the phenomenon of nonlinear panel flutter. Evidently, the full analysis in linear version is necessary too. In this case it was carried out without the use of the traditional Galerkin's method. Numerical methods were used at the final analysis stage for research of 2 nonlinear equations.

Let us consider a more detailed problem setting which is taken from the of V.V. Bolotin's monograph and assumes the case of cylindrical curvature. A similar boundary value problem was considered in F.Holmes and J. Marsden's works [Holmes,1977;Holmes,Marsden,1978].

Let us represent the corresponding boundary value problem in nondimensionalised form

$$\begin{aligned} &w_{tt} + gw_t + w_{xxxx} + cw_x + m_1(w_x)^2 + \\ &+m_2(w_x)^3 - \beta w_{xx} \int_0^1 (w_x)^2 dx = 0, \end{aligned} \tag{1}$$

$$w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0. \tag{2}$$

Equation (1) which describes the plate oscillation in ultrasonic gas flow is considered with the boundary condition (2), when the plate is supported. The choice of such a variant is not required and can be replaced by different boundary value conditions. For example,

$$w(t,0) = w(t,1) = w_x(t,0) = w_x(t,1) = 0.$$

Explicit form of positive coefficients g, c, m_1, m_2, β can be found in the monograph [1], and also in other works on the same subject. For example, c – nondimensionalised flow velocity. Here
$c = p_\infty \kappa l^3 U/(c_\infty D)$, where U – flow velocity, l – plate length, κ – exponent of polytrop, D – cylindrical rigidity, p_∞ – in inperturbed gas, c_∞ – velocity of sound in gas. Aerodynamic force are accounted for on the basis of the law of flat sections [Il'ushin,1956;Lighthill,1953] (piston theory), while equation (1) contains only summands of Tailor's decomposition of the corresponding formula [Bolotin,1961].

Let's consider boundary value problem (1),(2) linearized in zero

$$w_{tt} + g w_t + w_{xxxx} + c w_x = 0, \qquad (3)$$

$$w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0. \quad (4)$$

A study of the spectrum of the stability of the boundary value problem (3),(4) implies finding the nontrivial solution of the form $w(t,x) = \exp(\lambda t) v(x)$, where $v(x)$ fulfills the boundary conditions (4). Therefore $v(x)$ is a proper function of linear differential operator

$$A(c)v = v^{(IV)} + cv', \qquad (5)$$

the domain of definition consists of smooth functions $v(x)$, satisfying the boundary conditions

$$v(0) = v(1) = v''(0) = v''(1) = 0. \qquad (6)$$

Let $g \sim 1$. Then for $c = 0$ proper values of operator (5) $\mu_n = \pi^4 n^4$ ($n \in \mathbb{N}$) and, it implies that all the points of the stability spectrum lie on the left plane ($\lambda_n^2 + g\lambda_n + \mu_n = 0$). The smallest $c = c_0$, above which equilibrium state becomes unstable is called flutter velocity. The stability spectrum contains the pair of simple proper values $\pm i\sigma_0$ for $c = c_0$. Nonlinear analysis assumes the extension Andronov - Hopf's bifurcation theory on the corresponding class of boundary value problem. In this direction, the possibility of soft and hard auto excitation has been shown in [Holmes,1977;Holmes,Marsden, 1978; Kulikov,Liberman,1975;Kulikov,1976;Kolesov V.,Kolesov Yu.,Kulikov,Fedik,1978].

A different problem stems out if nondimensionalised damping coefficient g is sufficiently small. In A.A. Movchan's works he introduced the notion of lower critical velocity of flutter i.e. a minimum positive $c = c_1$, above which the points of stability spectrum leave the real axis for the first time. For $c = c_1$ the stability spectrum of the boundary value problem contains a multiple pair of purely imaginary proper values. This problem was considered in [Kulikov,2006;Kulikov,2006]. The obtained results were presented at the IX Russian congress on theoretical and applied mechanics [Kulikov,2006]. In this case it is shown that hard auto oscillations mode is common. It is clear that in this case $c_1 < c_0$, i.e there is a loss of stability of the equilibrium state and oscillations of large amplitude emerges with velocities less than the classical velocity of flutter.

It will be shown below that a similar picture is true if $c \approx c_2$, where for $c = c_2$ proper values $\pm i\sigma, \pm 2i\sigma$ emerges in the stability spectrum. The analysis of the nonlinear problem shows that even in this case hard mode auto oscillations can be realized ($c_2 < c_1 < c_0$). It is useful to mention that the assumption of a small damping coefficient is sufficiently natural. It is true, for example, in case when the coefficient of cilindrical rigidity is large. Remember that cylindrical rigidity coefficient is proportional to the modulus of elasticity E, and for aluminium $E = 7 \times 10^7 H/m^2$.

2 Linear analysis of the problem

Let's consider the boundary value problem (3),(4) for $g = 0$:

$$w_{tt} + w_{xxxx} + c w_x = 0, \qquad (7)$$

$$w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0. \quad (8)$$

Stability spectral points λ are related with the proper values of the differential operator $A(c)$ through equality $\lambda = \pm i\sqrt{\mu}$. Therefore resonance of proper frequency 1:2 occurs if among the proper numbers of the operator $A(c)$ there exist proper values μ_1, μ_2 such as $\mu_1 : \mu_2 = 1 : 4$. We apriori consider only those c, for which the including $c \in (0; c_1)$ is true implying that in this interval of values c all proper values of the operator $A(c)$ are real. The minimum possibility for the realization of this kind of resonance c is represented by c_2. To find this value we need to consider the boundary value problem

$$v^{(IV)} + cv' = \mu v, \qquad (9)$$

where $v(x)$ sytisfies the boundary conditions (8).

The general solution of equation (9)

$$v(x) = a_1 \exp(\gamma_1 x) + a_2 \exp(\gamma_2 x) + \\ + a_3 \exp(\gamma_3 x) + a_4 \exp(\gamma_4 x), \qquad (10)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are roots of the characteristic equation

$$\gamma^4 + c\gamma - \mu = 0. \qquad (11)$$

In [Kulikov,2006;Movchan,1956] it was established that the roots of the equation (11) has the form

$$\gamma_{1,2} = \alpha \mp \beta, \gamma_3 = -\alpha - \Delta, \gamma_4 = -\alpha + \Delta,$$

where $\Delta = \sqrt{\beta^2 - 2\alpha^2}, \alpha, \beta \in \mathbb{R}, \beta > 0$. Evidently,that

$$c = 4\alpha(\beta^2 - \alpha^2),\ \lambda = (\alpha^2 + \beta^2)(\beta^2 - 3\alpha^2). \quad (12)$$

Substitute the solution of (10) in the boundary conditions (8). To find $a_j (j = 1, 2, 3, 4)$ we obtain a system of linear equations

$$\sum_{j=1}^{4} a_j = 0,\ \sum_{j=1}^{4} a_j q_j = 0,\ \sum_{j=1}^{4} a_j \gamma_j^2 = 0,$$

$$\sum_{j=1}^{4} a_j \gamma_j^2 q_j = 0\ (q_j = \exp(\gamma_j)).$$

The condition of the existence of nontrivial solution leads to the characteristic equation

$$P(\alpha, \beta) = 0, \quad (13)$$

where $P(\alpha, \beta) = (3\alpha^2 + \beta^2\Delta^2)\sin\beta sh\Delta + 2\alpha^2\beta\Delta(ch(2\alpha - \cos\beta ch\Delta))$. By using Cardano's formulas, the assumption that $\mu \in \mathbb{R}\ \ \beta > 0$, from equation (12) can be expressed as $\alpha\ \ \beta$

$$\alpha = \alpha(\mu, c) = \sqrt{\Theta(\mu, c)^{\frac{1}{3}} - (\mu/12)\Theta(\mu, c)^{-\frac{1}{3}}},$$

$$\beta = \beta(\mu, c) = \sqrt{\alpha^2(\mu, c) + \sqrt{4\alpha^4(\mu, c) + \mu}}, \quad (14)$$

where $\Theta(\mu, c) = c^2/128 + \sqrt{(\mu/12)^3 + (c^2/128)^2}$. After substituting α, β, expressed using μ and c with the help of equalities (14) for $\mu = \sigma^2$ and $\mu = 4\sigma^2$ correspondingly in the characteristic equation (13) we obtain a system of equation for the determination of c and σ

$$P_1(\sigma^2, c) = P(\alpha(\sigma^2, c), c) = 0,$$
$$P_2(4\sigma^2, c) = P(\alpha(4\sigma^2, c), c) = 0. \quad (15)$$

The solution of the system (15) is determined $c = c_2$, for which resonance 1:2 is realized by its proper frequency. Analysis of this system showed that it has a countable number of solutions $(\sigma_n, c_{2n})\ n \in \mathbb{N}$. Here c_2 is chosen as $min(c_n)$, where $n \in \mathbb{N}$. The given system was solved by Zeidel's method with localization of the roots. Fig. 1 shows two graphs of implicit functions $P_1(\sigma^2, c) = 0$ and $P_2(4\sigma^2, c) = 0$ in the case of realizing the solution which gives $min(c_{2n})\ n \in \mathbb{N}$. It turns out that $c_2 = c_{21} = 225.04379, \sigma^2 = \sigma_1^2 = 369.43038$. It is clear that $c_2 < c_1$, where c_1 – lower critical flutter's velocity ($c_1 = 343.35592$ [Kulikov,2006]). Formulas (14) enable the restore of the value α and β, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and finally having solved the system of algebraic equation for a_1, a_2, a_3, a_4 we found the corresponding proper functions. For $\mu = \sigma^2$ we obtain

$$\gamma_{1,2} = 2.47077 \mp 5.37388i, \gamma_3 = -6.55317,$$

$$\gamma_4 = 1.61163, a_1 = -0.28964 \pm 0.04293i,$$

$$a_3 = -0.42072, a_4 = 1.$$

A given function will henceforth be represented by $e_1(x)$. For $\mu = 4\sigma^2$ let the proper function of the differential operator $A(c)$ be represented by $e_2(x)$. In this case we substitute in (10)

$$\gamma_{1,2} = 1.45477 \mp 6.38669i, \mu_3 = -7.50105,$$

$$\mu_4 = 4.59148, A_1 = -7.6272 \mp 38.02131i,$$

$$A_3 = 14.2544, A_4 = 1.$$

This problem can be studied by Galerkin's method, were as usual we choose the basis function of the form $\sin\pi x, \sin 2\pi x, \ldots$. Having chosen a two term Galerkin's approximation we can find that $c_2 = 200.87$. In the case of a three term approximation $c_2 = 227.05$, by using Galerkin's method when four basic functions are chosen we obtain $c_2 = 224.43$.

In fig.1 the dashed lines represent a graph of the function $P(4\sigma^2, c) = 0$, solid lines represent a graph of the function $P(\sigma^2, c) = 0$. Their intersection gives the desired solution.

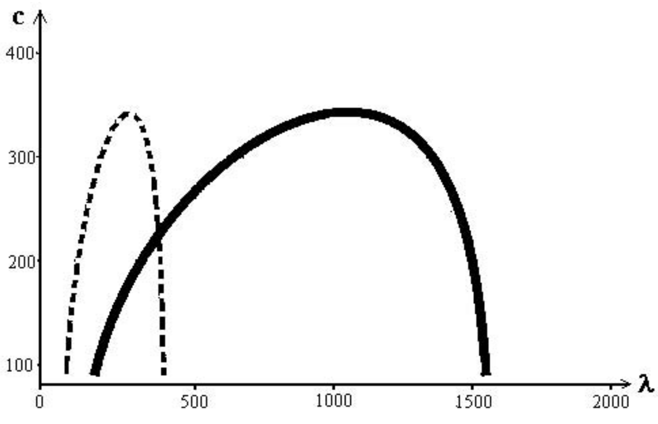

Fig. 1

3 Analysis of the problem in a non linear setting

Let us consider the nonlinear boundary value problem (1), (2) when $c = c_2 + \varepsilon_0 a_0 (a_0 \in \mathbb{R})$. We also consider that

$$g = 2g_0\varepsilon(g_0 > 0).$$

In this case the solution of the boundary value problem (1),(2) can be found in the form of the sum taken according to the power of ε

$$w(t,x,\varepsilon) = \varepsilon w_1(t,s,x)+$$
$$+\varepsilon^2 w_2(t,s,x) + \varepsilon^3 w_3(t,s,x) + \ldots .$$

Here $s = \varepsilon t$, points are used to represent the terms with higher order of smallness on ε. At last, $w_1(t,s,x), w_2(t,s,x), w_2(t,s,x)$ are sufficiently smooth functions for a set of variables. As functions of x , for all t,s they are included in the domain of definition of linear differential operator $A(c)$ and as for variable t for all s and x they are almost periodic functions. Let

$$w_1(t,x,s) = \sum_{k=1}^{\infty}(z_k(s)E_k(t,x) + \overline{z}_k\overline{E}_k(t,x)),$$
$$E_k(t,x) = e_k(x)\exp(i\sigma_k t),$$

where $\sigma_1 = \sigma, \sigma_2 = 2\sigma$, and the rest σ_k are chosen so that σ_k^2 are other proper values of the differential operator $A = A(c)$, different from $\sigma^2, 4\sigma^2$, and $e_k(x)$ are their corresponding proper functions . In this case functions $z_k(s)$ for all s must be chosen so that the function $w_1(t,x,s) \in \overset{\circ}{W}{}_2^4[0;1]$ – Sobolev's function space, which have generalized derivatives by x up to fourth order and which can be integrable square (contains in $L_2(0;1)$) for all $t \in [0;T]$.

”Zero” means that only functions satisfying the boundary condition (2) are considered. The functions $w_2(t,s,x), w_3(t,s,x)$ are to be chosen as the solution of linear nonhomogeneous problem.

So the function $w_2(t,x)$ is a solution of the boundary value problem

$$\frac{\partial^2 w_2}{\partial t^2} + \frac{\partial^4 w_2}{\partial x^4} + c_2\frac{\partial w_2}{\partial x} = F_2(t,s,x),$$

$$w_2(t,s,0) = w_2(t,s,1) = \frac{\partial w_2}{\partial x}|_{x=0} = \frac{\partial w_2}{\partial x}|_{x=1} = 0,$$

where $F_2(t,s,x) = -2(\partial^2 w_1/\partial t\partial s + g_0(\partial w_1/\partial t)) - a_0(\partial w_1/\partial x) - m_2(\partial w_1/\partial x)^2$. We will not write the boundary value problem for $w_3(t,s,x)$ because its analysis results will be needed latter.

From the conditions of the solvability of the boundary value problem in the class of trigonometry polynomes by t it follows that $z_n(s)$ satisfy the system of usual differential equations

$$z_1'(s) = -(g_0 - 0.07a_0 i)z_1(s) + 93.739m_2\overline{z}_1(s)z_2(s),$$
$$z_2'(s) = -(g_0 + 0.013a_0 i)z_2(s) - 0.001m_2 z_1^2(s),$$
$$z_k'(s) = -(g_0 + \alpha_k i)z_k(s),$$

where $k = 3,4,5,\ldots,\alpha_k \in \mathbb{R}$. These constants are computed from the conditions of the solvability of the boundary value problem in the class of the trigonometry polynomes of nonhomogeneous boundary value problems

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c_2\frac{\partial w}{\partial x} = -a_0\frac{\partial}{\partial x}E_k(t,x),$$

$$w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,\pi) = 0.$$

It is clear that for any value α_k the statement $\lim_{s\to\infty} z_k(s) = 0$ is true. Therefore only the first two equations play an important role in the study of the system.

Let

$$z_1(s) = d_1\exp(i\alpha_1 s)u_1(s),$$
$$z_2(s) = d_2\exp(i\alpha_2 s)u_2(s),$$

where

$$d_1^2 = \frac{1}{ab}, d_2 = \frac{1}{a}, a = -93.739m_2,$$
$$b = -0.001m_2, \alpha_2 - 2\alpha_1 = \frac{\pi}{2}.$$

Then $u_1(s), u_2(s)$ satisfy the following system of differential equations

$$u_1'(s) = -(g_0 + i\beta_1)u_1(s) + \overline{u}_1 u_2(s),$$
$$u_2'(s) = -(g_0 + i\beta_2)u_2(s) + u_1^2(s), \quad (16)$$

where $\beta_1 = -0.07a_0, \beta_2 = 0.013a_0$.

Lemma. *The system of differential equation (16) has a periodic solution*

$$u_1(s) = v_1(s) = \rho_1\exp(i\omega s),$$
$$u_2(s) = v_2(s) = \rho_2\exp(i\eta)\exp(2i\omega s), \quad (17)$$

where $\rho_1 = \sqrt{g_0^2 + (\omega+\beta_1)^2}, \rho_2 = \rho_1, \omega = -(\beta_1 + \beta_2)/3, \eta = arccos(g_0/\rho_1)$. *This solution is unstable.*

The proof of the statement consists of two parts. The existence of this solution is checked by substituting (17) in (16). To check the stability let's take

$$u_1(s) = u_{1p}(s)(1+w_1(s)), u_2(s) = u_{2p}(s)(1+w_2(s))$$

and for $w_1(s), w_2(s)$ we obtain an auxiliary system of equations

$$w_1' = (-i\omega - (g_0 + \beta_1 i)w_1 + \rho_2(\overline{w}_1 + w_2)\exp(i\eta),$$

$$w_2' = (-2i\omega - (g_0 + \beta_2 i))w_2 + (2\rho_1^2/\rho_2)w_1 \exp(-i\eta).$$

The zero equilibrium state stability of this linear system is studied in a standard way.

Theorem. *The unstable periodic solution of boundary value problem (1),(2)*

$$w(t,x,\varepsilon) = [\frac{1}{\sqrt{ab}}\exp(i\sigma t + i\alpha_1\varepsilon t)e_1(x) +$$

$$+\frac{1}{a}\exp(2i\sigma t + 2i\alpha_2\varepsilon t)e_2(x) + c.c.] + o(\varepsilon).$$

corresponds to the periodic solution (17) of the system (16) if $c = c + a_0\varepsilon$. The symbol c.. in the brackets represents a complex conjugate function.

The proof of the theorem stems from the results in [Kolesov,Kulikov,2003] (see section main bifurcation theorem).

Finally, we admit that the following conclusion can be drown: zero equilibrium state for $c = c_2 + \varepsilon$ asymptotically is stable from a formal point of view but in a small neighbourhood around it, there exists a nonstable periodic solution which may cause hard excitation oscillation.

Remark: The problem of nonlinear flutter was also considered in [Dowell,1966;Dowell,1967;Thompson,1982].

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