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BRIDGING CONNECTIONS ENHANCE SYNCHRONIZABILITY OF GROWING NETWORKS

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Abstract
We investigate the effect of growth mechanisms on the synchronizability of dynamical networks. We consider the effect of adding new nodes to an existing synchronized dynamical network using the Barabási-Albert growth algorithm. Our main contribution is to show that an algorithm that combines a version of preferential attachment with a small number of randomly selected bridging connections enhances the synchronizability of the network by providing a more compact eigenvalue spectrum while preserving the general structural characteristics of scale-free networks. We illustrate our results with numerical simulations.

Key words
Complex networks, synchronization, evolving networks

1 Introduction

A network is a set of entities, called nodes, that interact through connections, called edges. In particular, if the pattern of connections between nodes is not trivial the network is called structurally complex, or simply, a complex network [Newman, 2010]. The study of networks has focus mainly on the structural aspects where nodes and edges are assumed to be void of dynamics. In this way, one can use mathematical tools like graph and probability theories to determine key features of the network structure. In this context, it has been observed that real-world networks share different structural features such as the now famous small-world (SW) and scale-free (SF) effects [Wang and Chen, 2003]. Prominent examples of networks that display these structural features are the Internet, metabolic networks, social network, etc. However, it's well-known that for many real-world networks, nodes are not just static entities. In fact, the states of their nodes change over time. The concept of complex dynamical network naturally arises by considering a network with complex structural topology where each node is a dynamical system. In recent years, significant results have been obtain in regards to the stability of collective behaviors in complex dynamical networks [Boccaletti *et al.*, 2006; Wu, 2007; Arenas *et al.*, 2008; Barajas-Ramírez, 2012]. These investigations have establish well-known criteria for network synchronization, like the Master Stability Function (MSF) [Pecora and Caroll, 1998] and the so-called λ_2 criterion [Li, 2005], which highlight the crucial influence of the network structure on the stability of its synchronized dynamics.

Network synchronizability is understood as how easily the synchronized behavior emergences as a stable solution of the network dynamics. In basic terms, the synchronizability of a network is determined by the structural features of the network interconnections [Pecora and Caroll, 1998]. The stability of the synchronized behavior depends on two factors: the node dynamics and the eigenspectrum of the network coupling matrix. Recent investigations have related the structural features of the network, including average distance, degree distribution and betweeness centrality, to the network synchronizability [Chen and Duan, 2008; Zhao *et al.*, 2007]. However, these results do not consider other forms of evolution that are present in real-world networks, like growth. Some efforts have been published considering the addition of one or two extra links or nodes to the current network structure [Chen and Duan, 2008]. A complementary question directly follows: how the network synchronizability is affected by the structural evolution of the network? In this contribution a step is taken in this direction, we investigate how the choice of growth mechanisms can enhance the synchronizability of the network. In previous works, Wang and Chen analyzed the stability of synchronization on small-world and scale-free networks with different number of nodes, showing that larger networks coupled with a small-world connectivity have better synchronizability, in the sense that a smaller coupling strength is required to achieve synchronization, than nearest neighbors networks of similar size. On the other hand, synchronizability in scale-free networks is relatively independent from the number of nodes in the network [Wang and Chen, 2002a; Wang and Chen, 2002b]. In regards to the synchronizability of growing networks, Fan et al. investigated the effects of using alternative versions of preferential attachment which optimize the criteria for synchronization as the network growths, their synchronous preferential attachment mechanism results on networks with improved synchronizability [Fan and Wang, 2005; Fan *et al.*, 2005]. However, as the network retains predominantly scale-free in structure, its synchronizability is basically independent of the network growth. Despite these results, many aspects of the evolution of real-world networks have are not taken into account. In particular, situations such as the addition of multiple nodes and rewiring of the network connection topology have not being considered.

In this contribution, we propose a growth algorithm that inherits the basic mechanism of the BA model, the preferential attachment, to add nodes to an existing network, and complements the network growth algorithm by adding a small number of links uniformly at random between the nodes already in the network. It should be noted that unlike the previous works, where synchronizability is analyzed for networks constructed up to a fixed number of nodes using a given construction algorithm, we investigate the effect of adding nodes and links to an existing network which is already synchronized. This is a subtle but significant difference, instead of changing the size of the network and effectively growing it from zero, we add nodes to an already constructed network, we consider this to be a realistic situation for network growth, where an initially designed network is augmented and improved in order to include new elements or provide services for a larger population. This alternative view of growth as an ongoing process in the network frames our main contribution as we show that a combination of preferential attachment and random bridging enhances the synchronizability of the network compare to the BA model.

The rest of the paper is organized as follows: Some necessary concepts and definitions are given in Section 2. In Section 3, the growth process is described as a transformation operator with two processes. We illustrate the effects of our alternative growth algorithm on synchronizability in Section 4. Finally, in Section 5, the contribution is concluded with some closing remarks.

2 Some Necessary Basic Concepts

Consider a network of N linearly and diffusively coupled identical n-dimensional dynamical systems described by the following equation:

$$\dot{x}_i = f(x_i) + \gamma \sum_{j=1}^{N} c_{ij} x_j, \text{ with } i = 1, 2, ..., N \quad (1)$$

where $x_i = [x_{i1}, x_{i2}, ..., x_{in}]^\top \in \mathbf{R}^n$ are the state variables of the i-th node; $f : \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 function which describes the dynamics of an isolated node; and the constant $\gamma > 0 \in \mathbf{R}$ is the network's coupling strength. The connectivity is describe by the coupling matrix $\mathcal{C} = \{c_{ij}\} \in \mathbf{R}^{N\times N}$, which is constructed as follows: if the i-th and j-th nodes are coupled the entries c_{ij} and c_{ji} are set to one $(j \neq i)$; otherwise $c_{ij} = c_{ji} = 0$; with the diagonal elements given by $c_{ii} = -\sum_{j=1(j\neq i)}^{N} c_{ij} = -\sum_{j=1(j\neq i)}^{N} c_{ji} = -d_i$, where d_i is the degree of the i-th node.

If there are no isolated nodes, the connectivity matrix is irreducible with an eigenvalue spectrum of the form [Wang and Chen, 2003]:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_N, \quad (2)$$

The dynamical behavior of the network is said to be synchronized if the trajectories of every node asymptotically follow the same reference:

$$\lim_{t\to\infty} \|x_i - \bar{x}\| = 0, \text{ for } i = 1, 2, ..., N \quad (3)$$

Then, the stability of the synchronized behavior can be determine from the deviations to the synchronized solution $x_1 = ... = x_N = \bar{x}$. Here $\bar{x}$ is called the synchronized solution. Linearizing the synchronization error, $\xi_i = x_i - \bar{x}$, around $\bar{x}$ the following variational equation is obtained:

$$\dot{\xi}_i = J(\bar{x})\xi_i + \gamma \sum_{j=1}^{N} c_{ij}\xi_j, \text{ for } i = 1, 2, ..., N \quad (4)$$

where $J(\bar{x})$ is the Jacobian matrix of f at $\bar{x}$. Expressing (4) in terms of the eigenvalues of the coupling matrix we have:

$$\dot{\nu}_i = J(x)\nu_i + \gamma\lambda_i\nu_i, \text{ for } i = 1, 2, ..., N \quad (5)$$

where $[\nu_1, ..., \nu_N] = \Psi[\xi_1, ...\xi_N]$ with $\Psi\mathcal{C}\Psi^\top = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_N\}$.

Applying the conventional definition of Lyapunov exponent ($h_i = \lim_{t\to+\infty} \|J(t, x_0)u_i\|$) to the expression in (5), the nN transverse Lyapunov exponents of the network are given by [Barajas-Ramírez and Femat, 2012]:

$$\mu_i(\lambda_k) = h_i + \gamma\lambda_k, \quad (6)$$

for $i = 1, 2, ..., n$ and $k = 1, 2, ..., N$ where h_i represents the Lyapunov exponents of a node in isolation.

Synchronization is achieved if every transverse direction to the synchronized solution is contracting. That is, if $\mu_i(\lambda_k) < 0$ for $i = 1, 2, ...n$ and $k = 2, 3, ..., N$. Letting $h_{max} = h_1$ be the largest Lyapunov exponent for a node in isolation, and from (2) the synchronization condition becomes $\mu_1(\lambda_2) = h_1 + \gamma\lambda_2 < 0$ or equivalently

$$|\lambda_2| > \frac{h_{max}}{\gamma} \tag{7}$$

From the above results a direct relation can be establish between the stability of synchronization and the eigenvalues of the corresponding coupling matrix. Furthermore, we can say that for a network with a fixed coupling strength, larger values of $|\lambda_2|$ indicate an improved tendency towards synchronization, usually called a strong synchronizability [Li, 2005]. The synchronization region of the network S is the set of values of γ such that (7) is satisfied. This region can be unbounded $[-\infty, \alpha)$, or bounded $[\alpha_1, \alpha_2]$ the size of the synchronization region is related to the eigenratio of the coupling matrix $r = \frac{|\lambda_2|}{|\lambda_N|}$ [Pecora and Caroll, 1998; Chen and Duan, 2008].

From the discussion above, the effect of adding nodes and links on the synchronizability of a network can be determine from the change in the value of the largest non-zero eigenvalue and eigenratio from the initial $(\mathcal{C}_{k-1})$ to the resulting $(\mathcal{C}_k)$ coupling matrix

$$\begin{aligned} \Delta\sigma_k &= |\lambda_2(\mathcal{C}_k)| - |\lambda_2(\mathcal{C}_{k-1})| \\ \Delta r_k &= \tfrac{|\lambda_2(\mathcal{C}_k)|}{|\lambda_N(\mathcal{C}_k)|} - \tfrac{|\lambda_2(\mathcal{C}_{k-1})|}{|\lambda_N(\mathcal{C}_{k-1})|} \end{aligned} \tag{8}$$

where positive values of $\Delta\sigma_k$ and Δr_k indicate that as the network growths its synchronizability becomes stronger, while negative values indicate that becomes more difficult. In this sense, even a small numerical value for $\Delta\sigma_k$ and Δr_k is indicative of enhancement due to growth, as such these criteria is a simple extension of the synchronizability measures defined in [Li, 2005; Pecora and Caroll, 1998; Chen and Duan, 2008].

3 Network Growth Mechanism

The growth mechanism is inspired in the Barabási-Albert (BA) model [Barabási and Albert, 1999]. As a first step we consider the generic steps outlined as follows:

• **Growth.** Starting with an initial network with (N_0) nodes, at every iteration k, a small number n_k $(1 \leq n_k \ll N_k)$ nodes with m_k $(m_k \leq N_k)$ edges are added to the network and coupled to m_k different nodes already present in the network.

• **Attachment.** The m_k nodes to which the new node will be connected are chosen at random, with the probability $\Pi(j \to i)$ of coupling a new (j-th) node to the i-th node already in the network given by a preferential attachment rule.

• **Bridging.** With probability $p_{br} \ll 1$ additional bridge edges are added to the resulting network.

In the original BA model the number of nodes and link added in each iteration is fixed ($n_k = 1$ and $m_k = m$, $\forall k$), the preferential attachment is linear and given by

$$\Pi(j \to i) = \frac{d_i}{\sum_q d_q} \tag{9}$$

where $\Pi(j \to i)$ is the probability of connecting the new node j to the already existing node i, and is a function of its node degree d_i and the sum of all the other node degrees. In the original model once the edges are assigned they remain unaltered ($p_{br} = 0$). In that case, after M iterations the network has $N_M = N_0 + M$ nodes and $L_M = mM$ edges. The main distinctive characteristic of the resulting BA network is that its connectivity follows a power-law degree distribution. That is, the number of links per node is not close to the average for the entire network. For this reason the BA model is usually called the scale-free network model [Albert and Barabási, 2002].

It has been argued that the BA model represents a simplified approximation of the evolutionary processes that produced the scale-free nature of the node distributions observed in many real-world systems [Newman, 2010]. However, many aspects of the evolution of real-world networks are not capture by the original BA model. Many variants have been proposed over the last few years designed to improved on the original model by capturing some specific aspect of network evolution (see [Albert and Barabási, 2002], and references therein). In particular, [Fan *et al.*, 2005] proposed a *synchronous preferential attachment mechanism* in which the probability of connecting a new node into the network was chosen to maximize the value of criteria for synchronization (7) according

$$\Pi(j \to i) = \frac{\lambda_{2i}}{\sum_q \lambda_{2q}} \tag{10}$$

where λ_{2i} is the largest nonzero eigenvalue of the coupling matrix $\mathcal{C}$ obtained if the new node is coupled to the i-th node in the network. Alternatively, in order to minimize the value of λ_2 instead of using a probability (like equations (7) or (10)) the new node can be connected only to the first N_0 nodes, resulting on a multicenter network with N_0 hubs, where the $\lambda_2 = -N_0$ independently of the size of the network [Fan and Wang, 2005]. However, in such a connection all remaining nodes $(N_m - N_0)$ will only be connected to the hub nodes. Although, the works referenced above have significantly advance the understanding of the synchronization phenomenon on networks, many aspects of the evolution of real-world networks are yet to be considered.

In the following Sections, we investigate the effects of network growth on the synchronizability of the network. Where growth is understand as adding nodes to an already synchronized network. In this sense the evolution mechanism is interpreted as a transformation that increases the coupling matrix dimension, while the edges are assigned as a combination of preferential attachment and a complementary bridging process that randomly adds a small number of edges to the existing network.

3.1 Growth Algorithm

The growth process can be interpreted as an event driven system where at each iteration k the transformation, Φ_k, is applied to an initial network as follows:

Step 1: Adding nodes. At each iteration, n_k new nodes are coupled into the network.

This step transforms the previous coupling matrix $\mathcal{C}_k$ into $\bar{\mathcal{C}}_{k+1} \in \mathbf{R}^{N_{k+1}\times N_{k+1}}$ with

$$\bar{\mathcal{C}}_{k+1} = \bar{\Phi}_{k+1}(\mathcal{C}_k) = \begin{pmatrix} \mathcal{C}_k, & \alpha_{N_{k+1}} \\ \alpha_{N_{k+1}}, & 0 \end{pmatrix} \quad (11)$$

for $k = 0, 1, 2, \ldots$ where $N_{k+1} = N_k + n_k$, $\alpha_{N_{k+1}} \in \mathbf{R}^{(N_{k+1}-N_k)\times(N_{k+1}-N_k)}$ are zero matrices of appropriate dimensions.

Step 2: Attachment. Each of the new nodes will be coupled to $m_{k+1}(\leq N_k)$ of the nodes already existing in the network according to the linear preferential attachment rule (9).

After this step the matrix $\bar{\mathcal{C}}_{k+1}$ is transform into $\tilde{\mathcal{C}}_{k+1}$ with the same dimension given by

$$\tilde{\mathcal{C}}_{k+1} = \tilde{\Phi}_{k+1}(\bar{\mathcal{C}}_{k+1}) = \begin{pmatrix} \mathcal{C}_k, & \tilde{\alpha}_{N_{k+1}} \\ \tilde{\alpha}_{N_{k+1}}, & 0 \end{pmatrix} \quad (12)$$

where the matrices $\tilde{\alpha}_{N_{k+1}}$ are obtained by changing m_{k+1} entries of $\alpha_{N_{k+1}}$ into ones according to (9).

Step 3: Bridging. With probability $p_{br} \ll 1$ edges are added between the existing nodes in network.

After this step the matrix $\tilde{\mathcal{C}}_{k+1}$ is transform into $\hat{\mathcal{C}}_{k+1}$ with the same dimension given by

$$\hat{\mathcal{C}}_{k+1} = \hat{\Phi}_{k+1}(\tilde{\mathcal{C}}_{k+1}) = \begin{pmatrix} \hat{\mathcal{C}}_k, & \tilde{\alpha}_{N_{k+1}} \\ \tilde{\alpha}_{N_{k+1}}, & 0 \end{pmatrix} \quad (13)$$

where the matrix $\hat{\mathcal{C}}_k$ is obtained by changing $p_{br}\frac{N_k(N_k-1)}{2}$ entries from zero to one as a random process with a uniform probability p_{br}.

Step 4: Diffusive coupling. In order to have a linearly and diffusively network, the matrix $\hat{\mathcal{C}}_{k+1}$ is transformed into $\mathcal{C}_{k+1} = \check{\Phi}_{k+1}(\hat{\mathcal{C}}_{k+1})$ by adjusting the diagonal entries such that

$$c_{ii} = -\sum_{j=1, i\neq j}^{N_{k+1}} \hat{c}_{ij} \quad (14)$$

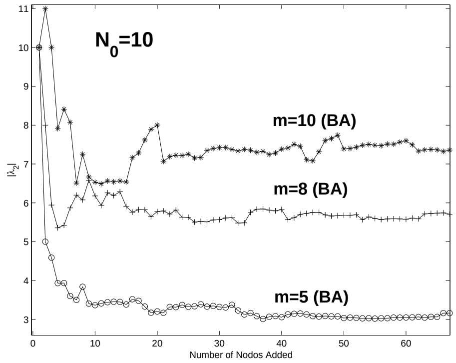

Figure 1. The absolute value of $|\lambda_2(\mathcal{C}_k)|$ of each resulting network as nodes are added one at a time using the BA model

In a compact notation, each iteration of the growth process can be represented by the transformation Φ_{k+1} given, from Step 1 to 4, as:

$$\mathcal{C}_{k+1} = \Phi_{k+1}(C_{(k)}) = (\check{\Phi} \circ \hat{\Phi} \circ \tilde{\Phi} \circ \bar{\Phi})_{k+1}(\mathcal{C}_k) \quad (15)$$

In the following Section, the effect of the proposed growth mechanism in the synchronizability of the network is investigated.

4 Numerical Results

We consider two scenarios: In the first, the network growths by adding nodes one at a time using the BA preferential attachment mechanism. In the second, the network growths following the algorithm described above, for comparison purposes we consider that growth is one node at a time, however, we added a very small number of bridging edges to the resulting network.

Our main concern is to establish if synchronizability is enhanced with our alternative growth mechanism. To this end, our initial network is fully-connected network with ten identical nodes. Is worth noting that different values for ($m_k = m,\ \forall k$) the number of edges added to attache each node to the network is a very sensible parameter, therefore we let it be constant and equal in both scenarios. The bridging probability $p_{br} \ll 1$ is also set constant at small value.

In Figure 1, we have $|\sigma_k|$ plotted as a function of the number of nodes in the network. The results are taken as the average of fifty network realizations. The results show that regardless of the number m_k being used, λ_2 is reduced significantly from the initial values and becomes stationary as the number of nodes increases. Further, the value at which $|\lambda_2(\mathcal{C}_k)|$ becomes stationary depends directly on m_k, with smaller values of m reducing further the final value of $|\lambda_2(\mathcal{C}_k)|$ for larger networks.

In Figure 2, we have $|\sigma_k|$ the resulting λ_2 values for the BA model and our proposed growth algorithm with

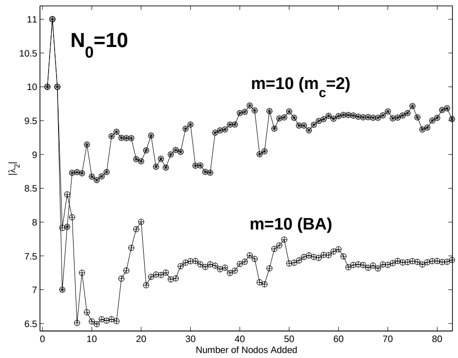

Figure 2. Comparison between $|\lambda_2(\mathcal{C}_k)|$ using the BA method and our proposed growth algorithm with bridging connections.

a probability of additional edges resulting in at most two additional connections ($p_{br} = 0.2$). The results show that the λ_2 with a small number of bridging edges is larger, that is, the synchronizability is enhanced.

5 Conclusion

We investigated the changes in the synchronizability of a dynamical network as its number of nodes and edges growths. The novelty of our analysis cames from two aspects of the analysis: On the one hand, we focus on the effect of the actual growth process, that is, the addition of nodes and links to a network already synchronized. On the other hand, the growth process was described as a transformation which can be design to enhance the stability characteristics of the synchronized behavior. Our results show that adding nodes to an already synchronized network using the preferential attachment mechanism, reduces the value of $|\lambda_2|$ regardless of the choice of m_k, and that this reduction reaches a static value for larger networks. We interpreted the reduction on the value of λ_2 as a lost of synchronizability on the resulting network. Then, we propose an alternative growth model where by adding a small number of bridging edges makes the value of $|\lambda_2|$ larger in the resulting network. In this sense, we proposed a growth mechanism that enhances the synchronizability of the network as it growths.

In this initial investigation, we proposed a formulation of the growth mechanism as a transformation of the coupling matrix of a network. This representation allows for the investigation of the effect of more general situations, such as the addition or elimination of more than one node at a time, and even can help in the formulation of the stability problem for a network with changing dimension. Results related to these different situations on the synchronized behavior of a growing network will be reported elsewhere.

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