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# REDUCED-ORDER MODELING OF STRONGLY NONLINEAR MODAL INTERACTIONS THROUGH SLOW-FAST PARTITION OF THE DYNAMICS AND EMPIRICAL MODE DECOMPOSITION

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**Abstract**

We perform nonlinear system identification of strongly nonlinear transient modal interactions occurring in a damped system possessing essential stiffness nonlinearity. The nonparametric identification is performed, (i) by slow – fast partition of the transient dynamics through a complexification – averaging technique, and (ii) by Empirical Mode Decomposition (EMD) of the nonlinear time series of the responses, applied together with the numerical Wavelet Transform (WT). We show that the dominant intrinsic mode functions (IMFs) resulting from EMD coincide with the slow-flow responses. Based on this observation, we formulate a reduced – order modeling methodology that can find applicability to a broad range of applications involving nonlinear modal interactions.

**Key words**

Applications, Identification, Modeling

## 1 Introduction

The dynamics of passive energy transfer from a damped linear oscillator to an essentially nonlinear end attachment was studied recently [Kerschen et al., 2005]. It was shown that complicated transitions in the damped dynamics can be interpreted based on the topological structure and bifurcations of the periodic solutions of the underlying undamped system. In this work, we study this system, in order to perform nonlinear system identification of the occurring strongly nonlinear transient modal interactions. We show the direct relationship between the slow flow dynamics (studied by a technique combining complexification and averaging) and the intrisic mode functions (IMFs) that resulting from application of empirical mode decomposition (EMD) in combination to Hilbert transform. Based on this relationship we identify the strongly nonlinear modal interactions in the system and formulate a reduced-order modeling methodology, with wide range of applicability.

## 2 Empirical Mode Decomposition (EMD) and the Hilbert Transformation

The Empirical Mode Decomposition (EMD) decomposes a signal (time series) in terms of oscillatory components, termed Intrinsic Mode Functions (IMFs) [Huang et al., 1998, 2003]. The IMFs satisfy three main conditions, which are also used for their computation:

(i) For the duration of the entire time series, the number of extrema and of zero crossings of each IMF should either be equal or differ at most by one.

(ii) At any given time instant, the mean value (moving average) of the local envelopes of the IMFs defined by their local maxima and minima should be zero.

(iii) The linear superposition of all IMFs should reconstruct the time series.

By construction, the lowest-order IMFs contain the oscillatory components of the signal with the highest frequency components. As the order of the IMFs increases, their frequency contents decrease accordingly. Hence, EMD analysis provides a decomposition of the signal over different time scales and, as such, is a useful computational tool for multi – scale system identification of the dynamics. Indeed, the EMD extracts oscillating modulations or modes embedded in the data. The essence of our method is to identify the intrinsic oscillatory modes that compose the data (time series), and to categorize them in terms of their characteristic time scales.

Moreover, by Hilbert – transforming the IMFs one computes the temporal evolutions of their instantaneous amplitudes and phases (frequencies) which, in turn, can be used for the construction of the Hilbert spectrum of the signal [Huang et al., 1998, 2003]. Denoting the Hilbert transform of a time series $c(t)$ by,

$$\hat{c}(t) = \left(\frac{1}{\pi}\right) \int_{-\infty}^{+\infty} \frac{c(\tau)}{t-\tau}\, d\tau \equiv \frac{1}{\pi t} * c(t) \qquad (1)$$

we may produce the following complexified analytical signal:

$$\Psi(t) = c(t) + j\,\hat{c}(t)\,, \quad j = (-1)^{1/2} \qquad (2)$$

From (2) we compute the instantaneous amplitude $A(t)$, and phase, $\varphi(t)$, of the signal by employing the polar representation,

$$\Psi(t) = A(t)\; e^{j\varphi(t)} = A(t)\,\cos\varphi(t) + jA(t)\sin\varphi(t)$$

thus expressing the signal in the form:

$$c(t) = A(t)\,\cos\varphi(t)\,, \quad A(t) = \sqrt{c(t)^2 + \hat{c}(t)^2}$$
$$\varphi(t) = \tan^{-1}\left[\hat{c}(t)/c(t)\right] \qquad (3)$$

The above representations enable one to compute also the instantaneous frequency $f(t)$ of the signal $c(t)$ according to the following definition:

$$f(t) = \frac{\dot{\varphi}(t)}{2\pi} = \frac{c(t)\,\dot{\hat{c}}(t) - \hat{c}(t)\,\dot{c}(t)}{2\pi\left[c(t)^2 + \hat{c}(t)^2\right]} \qquad (4)$$

## 3 Slow – Flow Analysis

Slow – flow analysis of the transient responses of strongly nonlinear systems can be performed by the complexification/averaging technique first introduced by Manevitch [1999]. This technique does not necessarily require weak nonlinearity, although for steady state responses it is similar to the (classical) method of averaging; once the proper *ansatz* regarding the frequency content and the slow-fast partition of the dynamics is included, it was numerically verified that the resulting slow-flows capture accurately the essential (important) dynamics of the original dynamical system, and provide good approximations of the original dynamics.

The technique will be demonstrated by considering the following two-degree-of-freedom (DOF) strongly nonlinear, dissipative system [Kerschen et al., 2005],

$$\ddot{y} + y + \lambda_1 \dot{y} + \lambda_2(\dot{y} - \dot{v}) + C(y-v)^3 = 0$$
$$\varepsilon\ddot{v} + \lambda_2(\dot{v} - \dot{y}) + C(v-y)^3 = 0 \qquad (5)$$

where $0 < \varepsilon << 1$ is a small parameter. This represents a linear oscillator (LO) with a strongly nonlinear, lightweight attachment. We wish to identify the nonlinear modal interactions governing *targeted energy transfer (TET)* in this system, whereby energy from the LO gets irreversibly and passively transferred to the nonlinear attachment, which acts, in essence, as *nonlinear energy sink (NES)* [Tsakirtzis, 2006; Georgiades, 2006].

Specifically, we will study 1:3 subharmonic interactions between the two oscillators in (5) by expressing the damped transient responses in the form,

$$y(t) = y_1(t) + y_{1/3}(t)\,, \quad v(t) = v_1(t) + v_{1/3}(t) \qquad (6)$$

where index 1 denotes terms possessing fast frequency $\omega$, and index $1/3$ those with fast frequency $\omega/3$. We introduce the complex variables,

$$\begin{aligned} \psi_1(t) &= \dot{y}_1(t) + j\omega y_1(t) \equiv \phi_1(t)\,e^{j\omega t} \\ \psi_3(t) &= \dot{y}_{1/3}(t) + j(\omega/3)y_{1/3}(t) \equiv \phi_3(t)\,e^{j(\omega/3)t} \\ \psi_2(t) &= \dot{v}_1(t) + j\omega v_1(t) \equiv \phi_2(t)\,e^{j\omega t} \\ \psi_4(t) &= \dot{v}_{1/3}(t) + j(\omega/3)v_{1/3}(t) \equiv \phi_4(t)\,e^{j(\omega/3)t} \end{aligned} \qquad (7)$$

where $\phi_i(t)$ represent 'slowly' varying (complex) modulations of the 'fast' frequency components $e^{j\omega t}$ and $e^{j(\omega/3)t}$. Expressing the transient responses in terms of the new complex variables,

$$y = \frac{\psi_1 - \psi_1^*}{2j\omega} + \frac{3\left(\psi_3 - \psi_3^*\right)}{2j\omega},\; v = \frac{\psi_2 - \psi_2^*}{2j\omega} + \frac{3\left(\psi_4 - \psi_4^*\right)}{2j\omega} \qquad (8)$$

substituting into (5), and averaging over each of the two fast frequencies $\omega$ and $\omega/3$, we derive the set of modulation equations that govern the slow-flow dynamics of the transient responses of the system in the form:

$$\dot{\underline{\phi}} = g(\underline{\phi};\varepsilon), \quad g: C^4 \to C^4 \qquad (9)$$

The explicit expressions of the slow-flow equations can be found in [Kerschen et al., 2005]. The slow-flow dynamics capture the nonlinear resonance captures that occur in the dynamics of system (5); then the responses of the original system can be estimated by inversing the applied coordinate transformations.

In the next Section we demonstrate the direct relation between the slow-flow dynamics and the results of EMD of the time series of the exact dynamics.

## 4 Relation Between EMD Analysis and Slow-Flow Dynamics and Reduced-order Models

We consider EMD analysis of the transient response of system (5) for initial conditions $v(0) = y(0) = 0$ and $\dot{v}(0) = 0.01499, \dot{y}(0) = -0.059443$ , e.g., with initial energy being confined mainly in the LO. The time series of the LO and the NES are depicted in Figures 1b and 2c, respectively. It turns out that the response of the LO possesses a single dominant IMF, $c_{LO1}(t)$, whereas the response of the NES possesses two, $c_{NES1}(t)$ and $c_{NES2}(t)$. In Figures 1a and 2a,b we depict the instantaneous frequencies of the dominant IMFs resulting from EMD analysis, superimposed to the corresponding WT spectra. In particular, in the time interval $t \in$ [0,100], both $c_{NES1}(t)$ and $c_{NES2}(t)$ are needed to reconstruct the response of the NES, whereas, for $t \in$ [100,1000] only $c_{NES1}(t)$ is needed. Regarding the LO, only $c_{LO1}(t)$ is needed to reconstruct the response over the entire time window considered in the study.

We now show that the dominant IMFs of the LO and NES responses are directly related to the slow-flow responses of system (9). Indeed, integrating the slow-flow with the corresponding initial conditions [Kerschen et al., 2005] $\varphi_1(0) = -0.0577$, $\varphi_2(0) = 0.0016$, $\varphi_3(0) = -0.0017$, $\varphi_4(0) = 0.0134$ and $\omega = 1.0073$, we make the following comparison. We plot the slow-flow modulations $\phi_i$ in the complex plane for varying time, together with the complexified analytical signals $\Psi_{LO1}(t)$, $\Psi_{NES1}(t)$ and $\Psi_{NES2}(t)$ resulting from the Hilbert transform of the dominant IMFs, $c_{LO1}(t)$, $c_{NES1}(t)$ and $c_{NES2}(t)$, respectively. Considering the complex plots of Figure 3, we note that the complexifications of the dominant IMFs of the LO and the NES correspond closely to the complex modulations $\phi_1$ and $3\phi_4$, respectively, of the slow-flow. This is more clearly inferred from the plots of Figure 4, where it is shown that the complexification $\Psi_{LO1}(t)$ matches closely the slow modulation $\phi_1$ of the fast frequency component at frequency $\omega$ over the entire time window of the dynamics (cf. Figure 4a). On the contrary, two complexifications $\Psi_{NES1}(t)$ and $\Psi_{NES2}(t)$ are needed to approximate the slow modulation $3\phi_4$ (cf. Figures 4b,c). *This fact is not a numerical artifact of the numerical EMD algorithm, but is related to the dynamics of the problem.*

Indeed, from the complex plot of Figure 5, where the slow modulation $3\phi_4$ of the NES response corresponding to fast frequency $\omega/3$ is plotted, we note that there exist two different regimes in the dynamics: an initial regime for $0 < t < 100$, and the main regime for $t > 100$. This is consistent with the findings of [Kerschen et al., 2005], where it was shown that in the initial regime of the dynamics there occurs 1:1 transient resonance capture (TRC) between the LO and the NES, during which energy gets transferred from the fundamental component of the response of the LO ($\phi_1$) to the fundamental component of the response of the NES ($\phi_2$). After this initial regime there occurs a 1:3 subharmonic TRC during which there occurs strong TET from the fundamental component of the LO to the subharmonic component of the NES ($\phi_4$). *The EMD analysis captures both regimes*, as indicated by the IMF instantaneous frequency plots of Figure 2, and the results of Figures 3 and 4. It follows that through EMD it is possible to perform system identification of the strongly nonlinear modal interactions that occur between the LO and the NES, over different time windows of the response. Moreover, the results of Figures 3 and 4 indicate that the dominant IMFs of the two subsystems have a direct physical meaning: they are approximately coincident with the slow-flow responses of the system, over different time windows of the dynamics.

Based on the previous system identification of the nonlinear modal interactions in the dynamics, and the knowledge that the dominant IMFs derived by the EMD of the time series coincide approximately with the slow flow dynamics, we can formulate a reduced-order methodology to model the nonlinear resonance interactions between the LO and the NES. This methodology consists of the following steps:

(i) Analysis of the measured nonlinear time series by numerical WT and computation of the corresponding WT spectra.

(ii) Analysis of the measured nonlinear time series by EMD, and derivation of the IMFs of the responses.

(iii) By comparing the instantaneous frequencies of the responses with the corresponding WT spectra, identification of the set of dominant IMFs for each time series; identifications of the fast frequencies at different time windows of the time series.

(iv) By comparing the instantaneous frequencies of the dominant IMFs of the two interacting systems, determination of the transient or sustained resonance captures that occur at different regimes of the transient responses.

(v) In the corresponding time windows where resonance captures occur, due to the correspondence between the complexified IMFs and the slow-flow dynamics, modeling of the resonance modal interactions by slow and fast components. In the example considered above, there exist two different regimes of the motion (where transient resonance interactions occur), during which the transient responses of the LO and the NES are approximated as follows, Regime 1 ( $0 < t < 100$ ):

$$
\begin{aligned}
y(t) &\approx \left|\Psi_{LO1}(t)\right| \sin\left[t + \beta_{LO1}(t)\right], \\
v &\approx \left|\Psi_{NES1}(t)\right| \sin\left[t + \beta_{NES1}(t)\right] + \\
&\quad \left|\Psi_{NES2}(t)\right| \sin\left[(t/3) + \beta_{NES2}(t)\right]
\end{aligned}
\tag{10}
$$

Regime 2 ( $t > 100$ ):

$$
\begin{aligned}
y(t) &\approx \left|\Psi_{LO1}(t)\right| \sin\left[t + \beta_{LO1}(t)\right], \\
v &\approx \left|\Psi_{NES1}(t)\right| \sin\left[(t/3) + \beta_{NES1}(t)\right]
\end{aligned}
\tag{11}
$$

where the phases in the trigonometric functions are the phases of the complexified IMFs in the time windows indicated. These reduced-order representations are consistent with the results depicted in Figures 1 and 2.

(vi) The reduced-order models of step (v) can be regarded as responses of coupled modal oscillators; then, reduced-order models in terms of ordinary differential equations can be derived with responses being equal to (10) and (11), and modeling the previous resonance modal interactions in the two regimes.

## 5 Conclusions

We performed nonlinear system identification of strongly nonlinear transient modal interactions occurring in systems possessing essentially nonlinear attachments. Nonparametric system identification was performed, by performing slow – fast partitions of the transient dynamics through a complexification – averaging technique, and by applying Empirical Mode Decomposition and Wavelet Transforms of the measured nonlinear time series. We showed that the dominant IMFs resulting from EMD have direct physical interpretations, as they approximately coincide with the slow-flow dynamics of the system. This observation enabled us to formulate a general reduced-order methodology, which, since it deals directly with measured time series and does not rely on any parametric models of the system, can find applicability applied to a broad range of problems where identification of nonlinear modal interactions is needed.

## References


Georgiades, F. (2006). *Nonlinear Localization and Targeted Energy Transfer Phenomena in Vibrating Systems with Smooth and Non-smooth Stiffness Nonlinearities*, PhD Thesis, National Technical University of Athens, Athens, Greece.

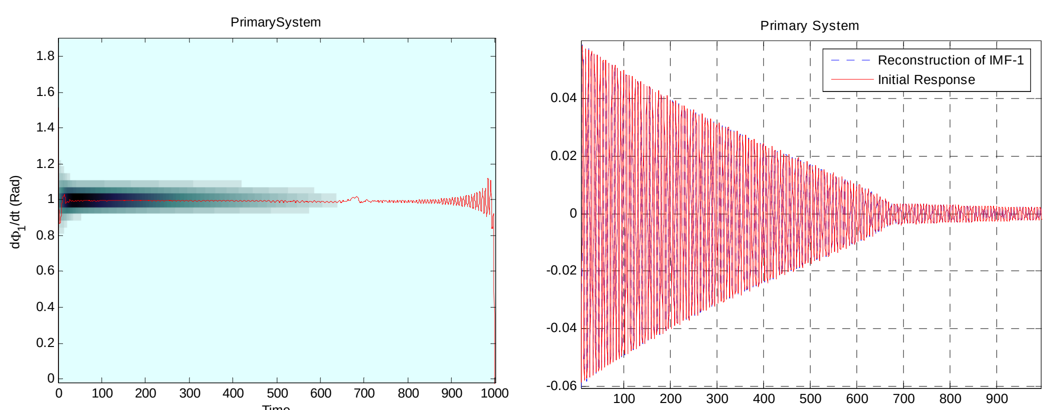
Huang, N., Shen, Z., Long, S., Wu, M., Shih, H., Zheng, Q., Yen, N.-C., Tung, C., and Liu, H. (1998). The empirical mode decompostion and the Hilbert spectrum for nonlinear and nonstationary time series analysis, *Proc. Royal Soc. London, Ser. A*, **454**, pp. 903-995.

Huang, N., Wu, M., Qu W., Long, S., and Shen, S.S.P. (2003). Applications of Hilbert-Huang transform to non-stationary financial time series analysis, *Appl. Stoch. Models Bus. Ind.*, **19**, pp 245-268.

Kerschen, G., Lee, Y.S., Vakakis, A.F., McFarland, D.M., and Bergman, L.A. (2006). Irreversible passive energy transfer in coupled oscillators with essential nonlinearity, *SIAM J. Appl. Math.*, **68**(2), pp. 648-679.

Manevitch, L.I. (1999). Complex representation of dynamics of coupled nonlinear oscillators, in *Mathematical Models of Non-Linear Excitations, Transfer Dynamics and Control in Condensed Systems and Other Media*, Uvarova, L., Arinstein, A. and Latyshev, A. (eds.), Kluwer Academic / Plenum Publishers.

Tsakirtzis S. (2006). *Passive Targeted Energy Transfers From Elastic Continua to Essentially Nonlinear Attachments for Suppressing Dynamical Disturbances*, PhD Thesis, National Technical University of Athens, Athens, Greece.


(a) (b)

Figure 1. Transient response of the LO: (a) instantaneous frequency of the dominant IMF superimposed to the WT spectrum; (b) signal reconstruction using the first dominant IMF.

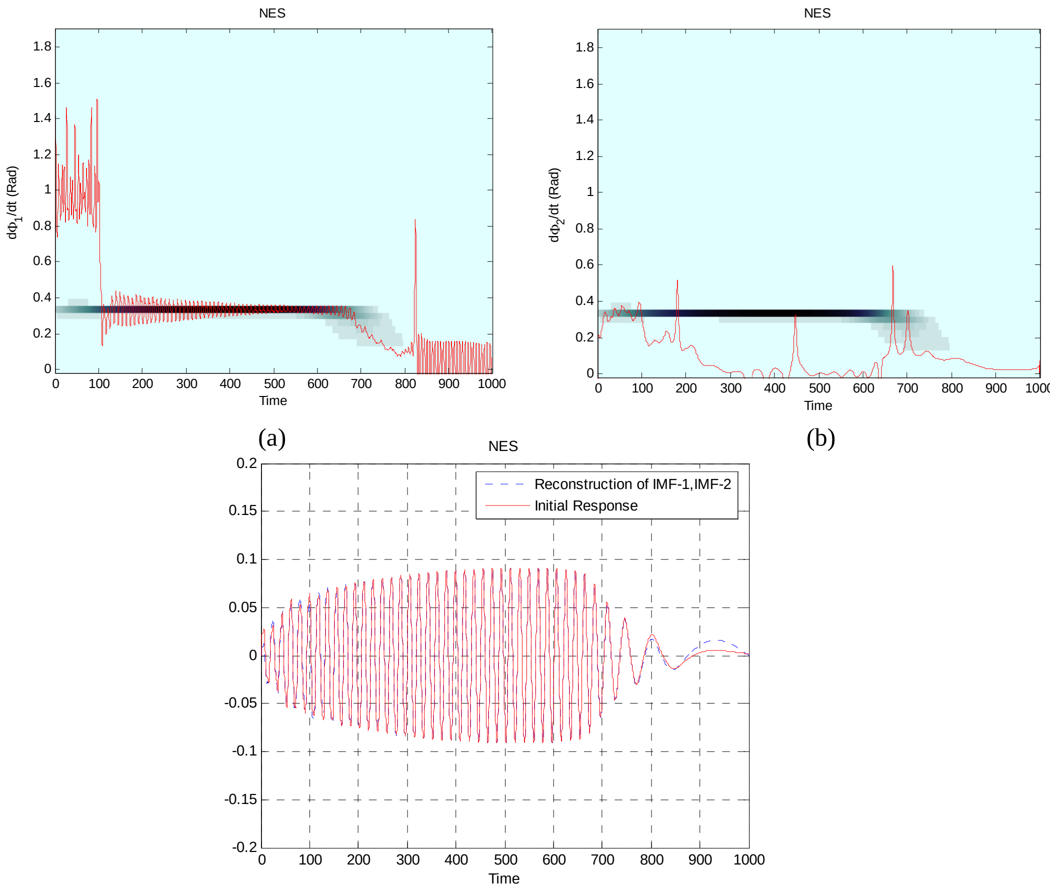



Figure 2. Transient response of the NES: (a) instantaneous frequency of the two dominant IMFs superimposed to their WT spectra; (b) signal reconstruction based on superposition of the two dominant IMFs.

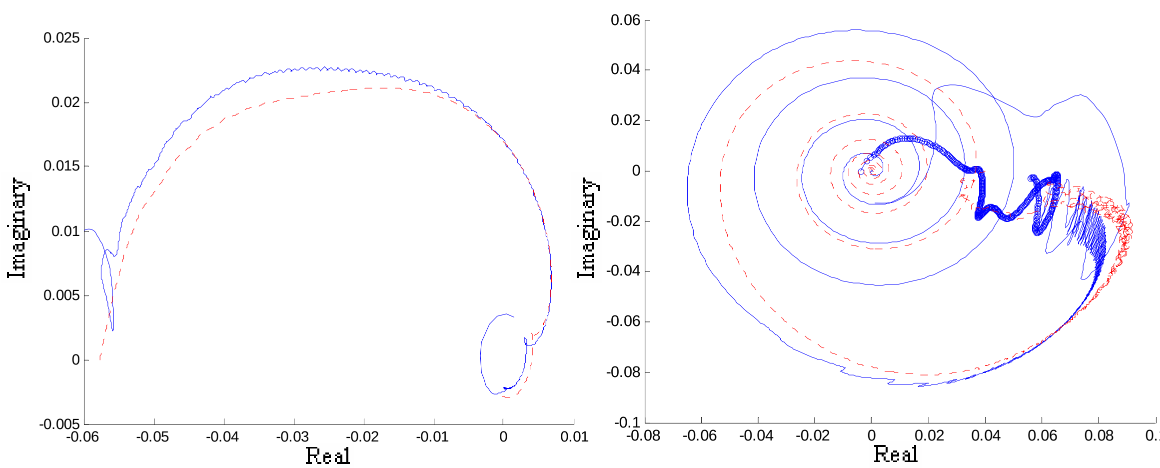



Figure 3. Comparison of IMF complexifications and slow-flow modulations in the complex plane: (a) $\Psi_{LO1}(t)$ (solid line) compared to $\phi_1(t)$ (dashed line), $0 < t < 1000$, (b) $\Psi_{NES1}(t)$ (solid line), $100 < t < 1000$, and $\Psi_{NES2}(t)$ (**ooooooo**), $0 < t < 100$, compared to $3\phi_4(t)$ (dashed line), $0 < t < 1000$.

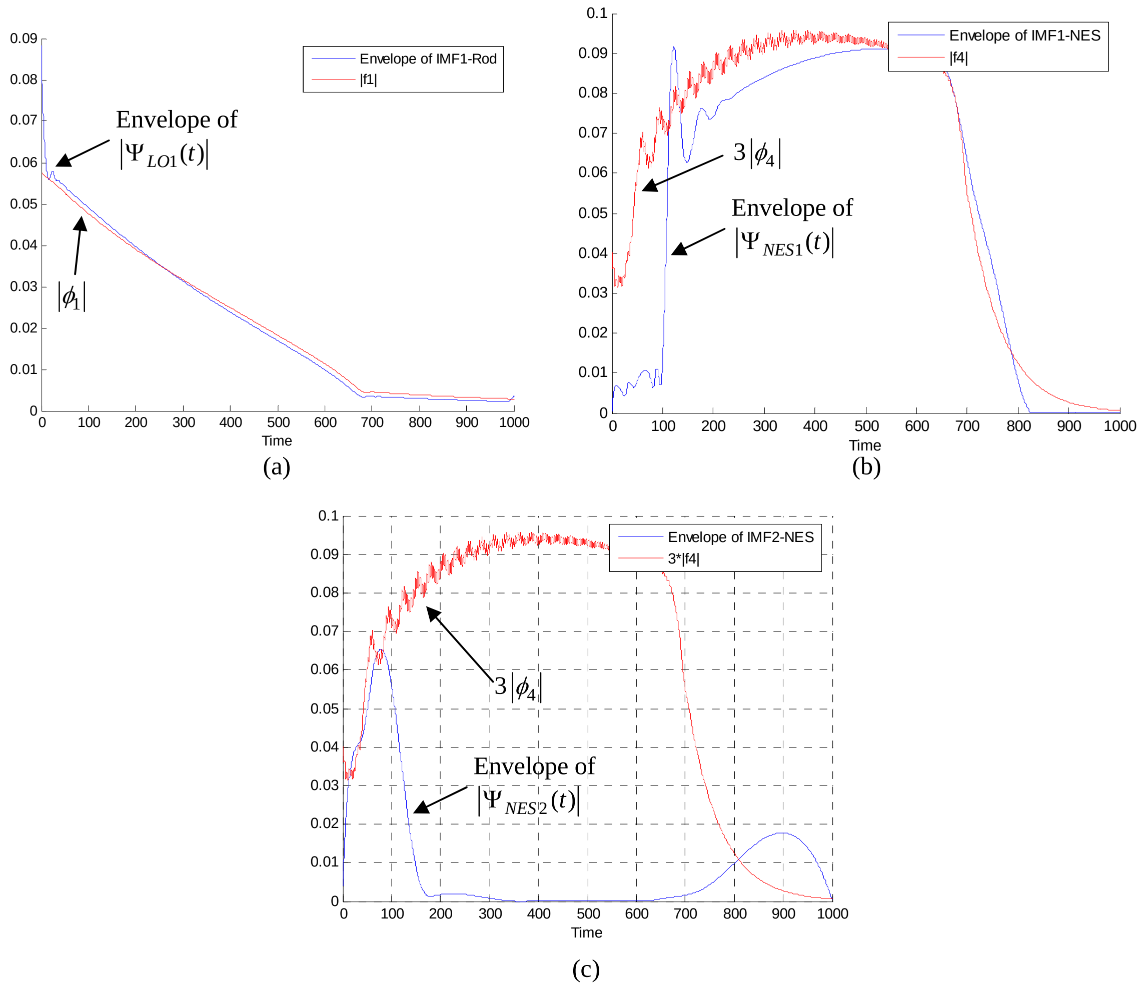



Figure 4. Comparison of envelopes of IMF complexifications and moduli of the slow-flow modulations: (a) envelope of $|\Psi_{LO1}(t)|$ and $|\phi_1|$; (b,c) envelopes of $|\Psi_{NES1}(t)|$ and $|\Psi_{NES2}(t)|$, and modulus $3|\phi_4|$.

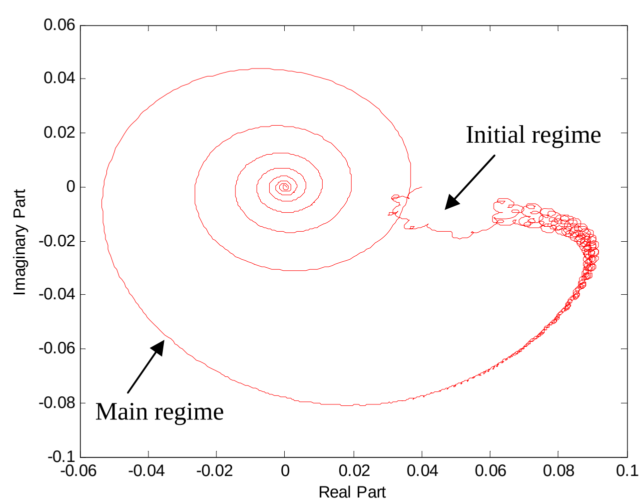



Figure 5. Plot in the complex plane of the subharmonic slow-flow component $3\phi_4$ of the NES; the two regimes of the dynamics can be clearly noted.