TOPOLOGICAL SEMI-CONJUGACY AND CHAOTIC MAPPINGS

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Abstract
We analyze α_m-mappings ($m \geq 2$) in symbol space Σ_2 and prove that the maps are chaotic in Σ_2. We show that there exists semi-conjugacy between $\alpha_m : I \to I$ ($I \subset \Sigma_2$) and corresponding class E_m of mappings in [0,1[. The topological semi-conjugacy and sensitive dependence on initial conditions guarantee that mappings E_m are chaotic.

Key words
Topological semi-conjugacy, symbol space, chaotic mapping, binary expansion.

1 Preliminaries

Our purpose is finding for classes of chaotic mappings in the segment [0,1]. We offer one of such possibilities.

For this aim at first we reveal the class of chaotic mappings α_m ($m \geq 2$) in symbol space Σ_2 and in subset $I \subset \Sigma_2$ too. At second we find corresponding class E_m ($m \geq 2$) of mappings in [0,1[. We show that there exists topological semi-conjugacy $\tau : I \to [0;1[$ between α_m and E_m too. The topological semi-conjugacy and sensitive dependence on initial conditions guarantee that mappings E_m are chaotic.

Definition 1.1([Holmgren, 1996], [Robinson, 1995]). The set of all infinite sequences of symbols 0 and 1 is called *the symbol space of 0 and 1* and is denoted by Σ_2, i.e.,

$$\Sigma_2 = \{s_0 s_1 s_2 ... \,|s_i = 0 \text{ or } s_i = 1, i = 0, 1, 2, ...\}.$$

We will refer to Σ_2 as the space of sequence of two symbols. We introduce a metric structure on Σ_2 by

$$\forall s = s_0 s_1 s_2 ...,\ t = t_0 t_1 t_2 ... \in \Sigma_2 :$$
$$d(s,t) = \sum_{i=0}^{+\infty} \frac{|s_i - t_i|}{2^i}.$$

This indeed is a metric (see, for example, [Holmgren, 1996]) therefore (Σ_2, d) is a metric space. But this metric is not unique. Σ_2 forms a metric space if we replace number 2 with $\lambda > 1$ as well (for example, in [Robinson, 1995] a case with $\lambda = 3$ and $\lambda = 4$ is examined, in [Holmgren, 1996], [Kitchens, 1998] or [Wiggins, 1988] $\lambda = 2$).

We note again that the two sequences are close if they agree on a long initial symbol block in metric space (Σ_2, d) too. The following lemma makes this precise.

Lemma 1.1.([Holmgren, 1996]) Let $s = s_0 s_1 s_2 ...$ and $t = t_0 t_1 t_2 ...$ be sequences of Σ_2. If $s_i = t_i$ for $i \leq n$, then $d(s,t) \leq \frac{1}{2^n}$. On the other hand, if $d(s,t) \leq \frac{1}{2^n}$, then $\forall\, i < n :\ s_i = t_i$.

The space (Σ_2, d) has more specific and interesting properties (see, [Holmgren, 1996] , [Lind, Marcus, 1995] or [Wiggins, 1988]).

The term "chaos" in reference to functions was first used in Li and Yorke's paper "Period three implies chaos" ([Li, Yorke, 1975], 1975). We use the following definition of R. Devaney [Devaney, 1986]. Let (X, ρ) be metric space.

Definition 1.2.([Devaney, 1986]) The function $f : X \to X$ is *chaotic* if
a) the periodic points of f are dense in X,
b) f is topologically transitive,
c) f exhibits sensitive dependence on initial conditions.

At first we note

Definition 1.3. The function $f : X \to X$ is *topologically transitive* on X if

$$\forall x, y \in X\ \forall \varepsilon > 0\ \exists z \in X\ \exists n \in \mathbf{N} :$$
$$\rho(x,z) < \varepsilon \ \&\ \rho(f^n(z), y) < \varepsilon.$$

Definition 1.4. The function $f : X \to X$ *exhibits sensitive dependence on initial conditions* if

$$\exists \delta > 0\ \forall x \in X\ \forall \varepsilon > 0\ \exists y \in X\ \exists n \in \mathbf{N} :$$
$$\rho(x,y) < \varepsilon \ \&\ \rho(f^n(x), f^n(y)) > \delta.$$

Definition 1.5. Let $A, B \subseteq X$ and $A \subseteq B$. Then A is *dense* in B if for each point $x \in B$ and each $\varepsilon > 0$, there exists $y \in A$ such that $d(x, y) < \varepsilon$.

Devaney's definition is not the unique classification of a chaotic map. For example, another definition can be found in [Robinson, 1995]. Also mappings with only one property — sensitive dependence on initial conditions — frequently are considered as chaotic (see, [Gulick, 1992]). Banks, Brooks, Cairns, Davis and Stacey [Banks, Brooks, Cairns, Davis, Stacey, 1992] have demonstrated that for continuous functions, the defining characteristics of chaos are topological transitivity and the density of periodic points. It means that we can not check up exhibits sensitive dependence on initial conditions of continuous mapping. This property follows from others.

The shift map $\sigma : \Sigma \to \Sigma$

$$\forall\ s = s_0 s_1 s_2 ... \in \Sigma_2 :\ \sigma(s) = s_1 s_2 ....$$

is well known example of a chaotic map (see [Holmgren, 1996], [Robinson, 1995], [Lind, Marcus, 1995] and others). But it is not unique chaotic map in space (Σ_2, d).

2 α_m-mappings ($m \geq 2$) in symbol space

Definition 2.1. The α_m*-mapping* ($m = 2, 3, ...$) $\alpha_m : \Sigma_2 \to \Sigma_2$ is defined by

$$\alpha_m(s_0 s_1 s_2 ...) = s_1 s_2 ... s_{m-1} s_{m+1} s_{m+2} ...$$

This mapping is not the kth iteration of the shift map, the α-mapping "forgets" two symbols of the sequence in every iteration. This mapping is simple (similar as shift map) but it is not investigate.

It is possible to prove that the every α_m-mapping ($m \geq 2$) is continuous, the set of periodic points of the α_m-mapping is dense in Σ_2 and the α_m-mapping is topologically transitive on Σ_2 too. By Banks, Brooks, Cairns, Davis and Stacey [Banks, Brooks, Cairns, Davis, Stacey, 1992] follows that the α_m-mapping is chaotic mapping. This proof is not complicated but it is long. If we observe that every α_m-mapping ($m \geq 2$) is increasing mapping, then it is much shorter proof of the fact that α_m-mapping is chaotic.

From now on A will denote a finite *alphabet*, i.e., a finite nonempty set

$$\{a_0, a_1, a_2, ..., a_n\}$$

and the elements are called *symbols*. We assume that A contains at least two symbols. We consider infinite sequences of symbols over a finite set A . *One-sided* infinite sequence over A is any total map $\omega : \mathbf{N} \to A$. The set A^ω contains all infinite sequences.

Let

$$f_\omega(x) = x_{f(0)} x_{f(1)} x_{f(2)} \dots x_{f(i)} \dots, \quad i \in \mathbf{N},\ x \in A^\omega.$$

In this case the function f is called *the generator function* of mapping f_ω.

Definition 4.2. ([Bula, Buls, Rumbeniece, 2006]) A function $f : \mathbf{N} \to \mathbf{N}$ is called *positively increasing function* if

$$0 < f(0) \text{ and } \forall i\ \forall j :\ i < j \Rightarrow f(i) < f(j)\,.$$

The mapping $f_\omega : A^\omega \to A^\omega$ is called *increasing mapping* if its generator function $f : \mathbf{N} \to \mathbf{N}$ is positively increasing.

Theorem 2.1. ([Bula, Buls, Rumbeniece, 2006]) The increasing mapping $f_\omega : A^\omega \to A^\omega$ is chaotic in the set A^ω.

In our case $A^\omega = \Sigma_2$ and α_m-mapping is increasing mapping because its generator function $f : \mathbf{N} \to \mathbf{N}$ is positively increasing:

$$f(x) = \begin{cases} x+1,\ x = 0, 1, 2, ..., m-2, \\ x+2,\ x = m-1, m, m+1, ... \end{cases}$$

Corollary 2.1. The α_m-mapping is chaotic in the symbol space Σ_2, $m = 2, 3, ...$

3 Topological semi-conjugacy

At second we use properties of topological semi-conjugacy and show that there exists for every α_m-mapping corresponding mapping $E_m : [0, 1] \to [0, 1]$ such that it is chaotic in unit segment $[0, 1]$, $m = 2, 3, ...$

Definition 3.1. ([Robinson, 1995]). Let $f : A \to A$ and $g : B \to B$ be functions. A map $h : A \to B$ is called a *topological semi-conjugacy from f to g* provided 1) h is continuous, 2) h is onto, and 3) $h \circ f = g \circ h$. The map h is called a *topological conjugacy* if it is homeomorphism and $h \circ f = g \circ h$.

Essential result for our purpose is following:

Theorem 3.1. ([Peitgen, Juergen, Saupe, 1994]) Let A and B be subsets of the metric spaces, $f : A \to A$, $g : B \to B$, and $\tau : A \to B$ be a topological semi-conjugacy of f to g. If f is chaotic on A, then g is topologically transitive on B and has dense set of periodic points in B. If $\tau : A \to B$ be a topological conjugacy of f and g, then f is chaotic on A if and only if g is chaotic on B.

In [Peitgen, Juergen, Saupe, 1994] is shown that for chaotic shift map corresponding chaotic mapping in unit segment is

$$S(x) = \begin{cases} 2x \bmod 1, & x \in [0, 1[, \\ 1, & x = 1 \end{cases} .$$

This result suggest to find for chaotic α_m-mapping corresponding chaotic mapping in unit segment.

Now we consider binary expansion of numbers from segment $[0, 1]$. Every number x from $[0, 1]$ it is possible to write in form $x = a_0 a_1 a_2 ...$ where $a_k \in \{0, 1\}$

and $x = a_0 2^{-1} + a_1 2^{-2} + a_2 2^{-3} + \ldots$. For example, $\frac{1}{2} = 1000\ldots$ or $\frac{1}{7} = \overline{001}\ldots$ (infinite sequence which periodically repeat after some fixed length will be denoted by the finite length sequence with an overline). But we has one problem: for example, the number $\frac{1}{2}$ has two binary expansions $1\overline{0}$ and $0\overline{1}$. We assume that we consider only first variant of binary expansion. Therefore we consider set $I = \Sigma_2 \setminus J$, where

$$J = \{s_0 s_1 s_2 \ldots \in \Sigma_2 | \exists N \geq 0 \ \forall i \geq N \ s_i = 1\}.$$

Then we has second problem with number 1, its binary expansion $\overline{1} \notin I$. But $\alpha_m(\overline{1}) = \overline{1}$ - this point is fixed point for every mapping α_m, $m = 2, 3, \ldots$ and all iterations are same. Finally we consider set I as binary expansion of numbers from segment $[0, 1[$.

The mapping $\tau : \Sigma_2 \to [0, 1[$ defined by equality

$$\forall s = s_0 s_1 s_2 \ldots \in I \quad \tau(s) = s_0 2^{-1} + s_1 2^{-2} + s_2 2^{-3} + \ldots$$

is onto, continuous (see, for example, [Peitgen, Juergen, Saupe, 1994] and [Kudrjavcev, 1988]) but it is not one-to-one. The mapping $\tau : I \to [0, 1[$ is onto, continuous and one-to-one. Here are more possibilities how the number from segment $[0, 1[$ transforms to binary expansion. We use method from [Peitgen, Juergen, Saupe, 1994]:

$$x \in [0, 1[\quad \tau^{-1}(x) = s_0 s_1 s_2 \ldots, \text{ where}$$

$$s_i = \begin{cases} 0, \ z(x)_i < \frac{1}{2}, \\ 1, \ z(x)_i \geq \frac{1}{2}, \end{cases}$$

$$z(x)_0 = x, \ z(x)_i = 2z(x)_{i-1} \bmod 1, \ i = 1, 2, \ldots.$$

For example, if $x = \frac{1}{7}$, then
$z(x)_0 = x = \frac{1}{7} < \frac{1}{2} \Rightarrow s_0 = 0$,
$z(x)_1 = 2z(x)_0 \bmod 1 = \frac{2}{7} \bmod 1 = \frac{2}{7} < \frac{1}{2} \Rightarrow s_1 = 0$,
$z(x)_2 = \frac{4}{7} \bmod 1 = \frac{4}{7} \geq \frac{1}{2} \Rightarrow s_2 = 1$,
$z(x)_3 = \frac{8}{7} \bmod 1 = \frac{1}{7} < \frac{1}{2} \Rightarrow s_3 = 0, \ldots$, i.e., $\frac{1}{7} = \overline{001}$.

If we consider $\tau : I \to [0, 1[$, then the inverse map τ^{-1} is not continuous. For example, the sequence $x_n = \frac{1}{2} - \frac{1}{2^n}$, $n = 1, 2, \ldots$, converges to $\frac{1}{2}$ but the sequence $\tau^{-1}(x_n)$, $n = 1, 2, \ldots$, converges to $0\overline{1} \notin I$. Therefore $\tau : \Sigma_2 \to [0, 1[$ and $\tau : I \to [0, 1[$ are not homeomorphisms and not topological conjugacy.

In Section 2 we has shown that α_m-mapping is chaotic in symbol space Σ_2. It is chaotic in set I too? Indied, notice that $\alpha_m : J \to J$ and $\alpha_m : I \to I$. The α_m-mapping is increasing mapping in I too. It follows that α_m-mapping is chaotic in subset $I \subset \Sigma_2$.

We assume that for the α_m-mapping exists corresponding chaotic mapping in segment $[0, 1]$. What can we find for α_m-mapping corresponding mapping E_m in segment $[0, 1]$? For this aim we make numerical experiment: at first, we write number x from segment $[0, 1[$ (with step, for example, 0.01) in its binary expansion $s \in I$, at second, we consider $\alpha_m(s)$, at thirdly, we write $\alpha_m(s)$ in its decimal expansion $E_m(x)$ and make graph. Finally for $m = 2$ we find

$$E_2(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{8}, \\ 4x - \frac{1}{2}, & \frac{1}{8} \leq x < \frac{3}{8}, \\ 4x - 1, & \frac{3}{8} \leq x < \frac{4}{8}, \\ 4x - 2, & \frac{4}{8} \leq x < \frac{5}{8}, \\ 4x - \frac{5}{2}, & \frac{5}{8} \leq x < \frac{7}{8}, \\ 4x - 3, & \frac{7}{8} \leq x < 1. \end{cases}$$

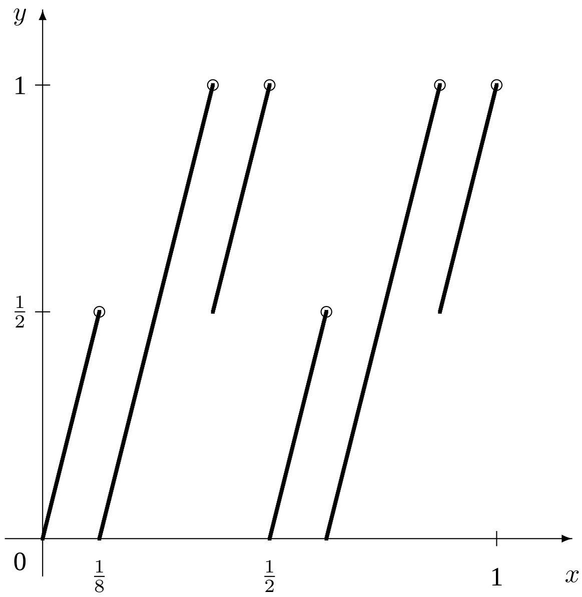

Fig.1. Graph of E_2.

It is necessary to show $\tau \circ \alpha_2 = E_2 \circ \tau$.
Let $s = s_0 s_1 s_2 \ldots \in I$, then

$$\alpha_2(s_0 s_1 s_2 \ldots) = s_1 s_3 s_4 \ldots,$$
$$\tau(\alpha_2(s)) = s_1 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + \ldots.$$

For the right side $E_2(\tau(s))$ we remark that value of

$$\tau(s_0 s_1 s_2 s_3 \ldots) = s_0 2^{-1} + s_1 2^{-2} + s_2 2^{-3} + s_3 2^{-4} + \ldots$$

belongs to one of 8 segments depending of $s_0, s_1, s_2 \in \{0, 1\}$:

1) If $s_0 = s_1 = s_2 = 0$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) = s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + \ldots \in \left[0, \frac{1}{8}\right[.$$

Therefore

$$E_2(\tau(s)) = 4\tau(s) = 2^2\left(s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...\right) =$$
$$= s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

2) If $s_0 = s_1 = 0$, $s_2 = 1$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) = 2^{-3} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ... \in \left[\frac{1}{8}, \frac{2}{8}\right[.$$

Therefore

$$E_2(\tau(s)) = 4\tau(s) - \tfrac{1}{2} =$$
$$= 2^2(2^{-3} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - \tfrac{1}{2} =$$
$$= 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... - \tfrac{1}{2} =$$
$$= s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

3) If $s_0 = s_2 = 0$, $s_1 = 1$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{2}{8}, \frac{3}{8}\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - \tfrac{1}{2} =$$
$$= 2^2(2^{-2} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - \tfrac{1}{2} =$$
$$= 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... - \tfrac{1}{2} =$$
$$= s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

4) If $s_0 = 0$, $s_1 = s_2 = 1$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{3}{8}, \frac{1}{2}\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - 1 =$$
$$= 2^2(2^{-2} + 2^{-3} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - 1 =$$
$$= 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... =$$
$$= 2^{-1} s_1 + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

5) If $s_0 = 1$, $s_1 = s_2 = 0$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{1}{2}, \frac{5}{8}\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - 2 =$$
$$= 2^2(2^{-1} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - 2 =$$
$$= s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

6) If $s_1 = 1$, $s_0 = s_2 = 0$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{5}{8}, \frac{6}{8}\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - \tfrac{5}{2} =$$
$$= 2^2(2^{-1} + 2^{-3} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - \tfrac{5}{2} =$$
$$= s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

7) If $s_2 = 0$, $s_0 = s_1 = 1$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{6}{8}, \frac{7}{8}\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - \tfrac{5}{2} =$$
$$= 2^2(2^{-1} + 2^{-2} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - \tfrac{5}{2} =$$
$$= s_1 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

7) If $s_0 = s_1 = s_2 = 1$ and by assumption all $s_i \neq 1, i > 2$, then

$$\tau(s) \in \left[\frac{7}{8}, 1\right[,$$

therefore

$$E_2(\tau(s)) = 4\tau(s) - 3 =$$
$$= 2^2(2^{-1} + 2^{-2} + 2^{-3} + s_3 2^{-4} + s_4 2^{-5} + s_5 2^{-6} + ...) - 3 =$$
$$= s_1 2^{-1} + s_3 2^{-2} + s_4 2^{-3} + s_5 2^{-4} + ... = \tau(\alpha_2(s)).$$

Similary we find E_3:

$$E_3(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{16}, \\ 4x - \frac{1}{4}, & \frac{1}{16} \le x < \frac{3}{16} \\ 4x - \frac{1}{2}, & \frac{3}{16} \le x < \frac{5}{16}, \\ 4x - \frac{3}{4}, & \frac{5}{16} \le x < \frac{7}{16}, \\ 4x - 1, & \frac{7}{16} \le x < \frac{8}{16}, \\ 4x - 2, & \frac{8}{16} \le x < \frac{9}{16}, \\ 4x - \frac{9}{4}, & \frac{9}{16} \le x < \frac{11}{16}, \\ 4x - \frac{5}{2}, & \frac{11}{16} \le x < \frac{13}{16}, \\ 4x - \frac{11}{4}, & \frac{13}{16} \le x < \frac{15}{16}, \\ 4x - 3, & \frac{15}{16} \le x < 1. \end{cases}$$

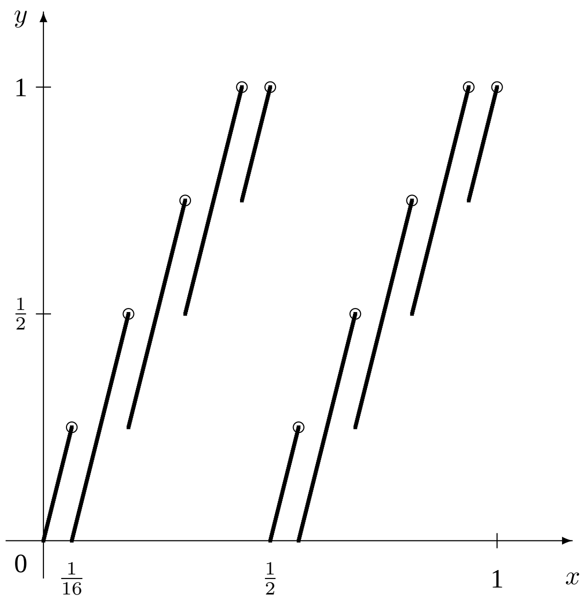

Fig.2. Graph of E_3.

Similarly we can find E_4, E_5,.. and finally we give formula for E_m, $m \geq 2$, in general case.

$$E_m(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{2^{m+1}}, \\ 4x - \frac{1}{2^{m-1}}, & \frac{1}{2^{m+1}} \leq x < \frac{3}{2^{m+1}} \\ 4x - \frac{2}{2^{m-1}}, & \frac{3}{2^{m+1}} \leq x < \frac{5}{2^{m+1}}, \\ \ldots & \ldots, \\ 4x - \frac{i}{2^{m-1}}, & \frac{2i-1}{2^{m+1}} \leq x < \frac{2i+1}{2m+1}, \\ \ldots & \ldots, \\ 4x - 1, & \frac{2^m-1}{2^{m+1}} \leq x < \frac{1}{2} = \frac{2^m}{2^{m+1}}, \\ 4x - 2, & \frac{1}{2} \leq x < \frac{2^m+1}{2^{m+1}}, \\ 4x - \frac{1}{2^{m-1}} - 2, & \frac{2^m+1}{2^{m+1}} \leq x < \frac{2^m+3}{2^{m+1}}, \\ \ldots & \ldots, \\ 4x - 3, & \frac{2^{m+1}-1}{2^{m+1}} \leq x < 1. \end{cases}$$

Obviously similarly as for $m = 2$ we can prove equality $\tau \circ \alpha_m = E_m \circ \tau$, i.e., $\tau : I \to [0, 1[$ is topological semi-conjugacy from α_m to E_m. Similarly as for tent map it is possible to prove that $E_m : [0; 1] \to [0; 1]$ ($E_m(1) = 1$) exibits sensitive dependence on initial conditions. We conclude

Theorem 3.2. Let $E_m(1) = 1$. The every mapping $E_m : [0, 1] \to [0, 1]$, $m = 2, 3, ...$, is chaotic in $[0, 1]$.

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