SENSITIVITY CONTROL FOR STOCHASTIC INVARIANT MANIFOLDS

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Abstract

We suggest a general approach to investigate a problem of synthesis of spatial stochastic attractors of nonlinear dynamic systems. Our technique is based on stochastic sensitivity function (SSF) and a concept of invariant manifolds. Corresponding notions of accessibility and full controllability are introduced and studied.

Key words

Invariant manifolds; Stochastic system; Sensitivity; Control

1 Introduction

Many nonlinear phenomena of mechanics, electronic generators, lasers, biophysics observed under transition from the order to chaos are frequently connected with a chain of bifurcations: a stationary regime (equilibrium point) - periodic regime (limit cycle) - quasiperiodic regime (torus) - chaotic regime (strange attractor). Each such transition is accompanied by the loss of stability of simple attractor and new more complicated stable attractor birth. Under the random disturbances deterministic attractor is transformed to the stochastic counterpart. Stability analysis of appropriate invariant manifolds is a key for the understanding of the complex behavior of nonlinear stochastic dynamical systems. A synthesis of multi-dimensional attractors with wishful parameters is a challenging problem of the modern control theory. A case of the equilibrium point is well-studied. Now the research interest is moved to a study of limit cycles, tori and another spatial attractors ([Fradkov and Pogromsky, 1998], [Chen and Yu, 2003]). In order to investigate these different cases from a common viewpoint, we use a general approach based on invariant manifolds. The compact invariant manifolds are useful general mathematical models for various important regimes of nonlinear dynamical systems. A study of the compact invariant manifolds attracts an attention of many researchers and leads to the several important mathematical problems in the stability and control theory.

An extension of Lyapunov function technique for the stability analysis of the invariant manifolds of a deterministic system was considered in ([Ryashko and Shnol, 2003]). Stochastic stability analysis for common invariant manifolds was presented in ([Ryashko, 2003]). In this paper, we investigate a control problem for stochastically forced invariant manifolds with the help of stochastic sensitivity function technique. This technique was developed and successfully applied for stochastic limit cycles in ([Bashkirtseva and Ryashko, 2000], [Bashkirtseva and Ryashko, 2004], [Bashkirtseva and Ryashko, 2005]). Here corresponding notions of accessibility and full controllability for stochastic manifolds are introduced and studied.

2 Synthesis of Stochastic invariant manifolds

Consider the deterministic system

$$dx \;=\; f(x,u)\,dt, \qquad x,f\in\mathbf{R}^n,\; u\in\mathbf{R}^1. \tag{1}$$

Here a smooth vector function $f(x,u)$ depends on a control parameter u. Suppose the system (1) for $u=0$ has a smooth compact invariant manifold $\mathcal{M}$. Stability of the manifold $\mathcal{M}$ is not supposed.

Consider a set $\mathcal{F}$ of the admissible feedback controls $u=u(x)$. A function $u(x)\in\mathcal{F}$ if the following condition

$$u(x)|_{\mathcal{M}} \;=\; 0 \tag{2}$$

holds and for a closed system

$$dx \;=\; f(x,u(x))\,dt$$

the manifold $\mathcal{M}$ is exponentially stable in an invariant vicinity U.

The condition (2) means that the manifold $\mathcal{M}$ remains invariant for all admissible controls. Thus, operating the system (1) we do not interfere in the dynamics of its solutions lying on the invariant manifold $\mathcal{M}$.

Consider along with (1) a corresponding stochastic system

$$dx = f(x,u)dt + \varepsilon\sigma(x,u)dw(t). \tag{3}$$

Here $w(t)$ is n-dimensional Wiener process, $\sigma(x,u)$ is a smooth $n \times n$-matrix function characterizing a dependence of disturbances on a state and control, ε is a scalar parameter of the noise intensity.

To describe the dynamics of deviations of the random trajectories of the system (3) from the manifold $\mathcal{M}$ we will use a linear extension system

$$\begin{aligned} dx &= f(x,u)\ dt, \\ dz &= F(x,u)zdt + \varepsilon\sigma(x,u)dw(t), \\ &\quad x \in \mathcal{M},\ z \in \mathbf{R^n}, \end{aligned} \tag{4}$$

where

$$F(x,u) = \frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial u}(x,u)\frac{\partial u}{\partial x}(x).$$

System (4) for $u \in \mathcal{F}$ with the condition (2) can be written as

$$\begin{aligned} dx &= f_0(x)\ dt, \\ dz &= (A(x) + B(x)K(x))zdt + \varepsilon G(x)dw, \\ &\quad x \in \mathcal{M}, z \in \mathbf{R^n}, \end{aligned} \tag{5}$$

where

$$f_0(x) = f(x,0),\quad A(x) = \frac{\partial f}{\partial x}(x,0),$$

$$B(x) = \frac{\partial f}{\partial u}(x,0),\quad G(x) = \sigma(x,0),\quad K(x) = \frac{\partial u}{\partial x}(x).$$

In a character of the dependence of system (5) on a control, we note the following feature. The system (5) dynamics depends on the values of derivatives $\frac{\partial u}{\partial x}$ on the manifold $\mathcal{M}$ only.

For each $x \in \mathcal{M}$, denote by T_x the tangent subspace to $\mathcal{M}$ at x. Denote by N_x the orthogonal complement to T_x in $\mathbf{R^n}$. Let P_x be the operator of the orthogonal projection onto the subspace N_x.

Due to (2), for $K(x) = \frac{\partial u}{\partial x}(x)$ a condition

$$\forall x \in \mathcal{M} \quad K(x)P_x = K(x). \tag{6}$$

holds.

Here, without lost a generality we restrict a consideration by the following control functions

$$u(x) = K(\gamma(x))\Delta(x), \tag{7}$$

where

$$\gamma(x) = argmin_{y\in\mathcal{M}}\|x-y\|, \qquad \Delta(x) = x - \gamma(x),$$

$\|\cdot\|$ is Euclidean norm, $\gamma(x)$ is a point of the manifold $\mathcal{M}$ that is nearest to x, $\Delta(x) = x - \gamma(x)$ is a vector of a deviation of the point x from $\mathcal{M}$.

A necessary and sufficient condition of the exponential stability of the manifold $\mathcal{M}$ for the deterministic system (1) with the control (7) is P-stability of the linear extension system ([Ryashko and Shnol, 2003])

$$\begin{aligned} dx &= f_0(x)\ dt, \\ dz &= (A(x) + B(x)K(x))zdt. \end{aligned} \tag{8}$$

Consider a set $\mathbf{K}$ of $l \times n$-matrix functions $K(x)$ satisfying the condition (6) for which the system (8) is P-stable. Between the set of admissible controls $\mathcal{F}$ and the set $\mathbf{K}$ there are following simple connections.

Lemma 1. In order to $u(x) \in \mathcal{F}$, it is necessary and sufficient that $\frac{\partial u}{\partial x}(x) \in \mathbf{K}$.

Lemma 2. In order to $u(x) = K(\gamma(x))\Delta(x) \in \mathcal{F}$ it is necessary and sufficient that $K(x) \in \mathbf{K}$. We assume that the set $\mathbf{K}$ is not empty. For any $K(x) \in \mathbf{K}$, in the closed stochastic system (3), (7) near the manifold $\mathcal{M}$ the bundle of random trajectories is formed. As shown in ([**?**]) a distribution of the trajectories in this bundle can be described by stochastic sensitivity function defined on $\mathcal{M}$. Values of this function depend on a choice of a feedback coefficient $K(x)$ of the regulator (7). Now our main aim is to study possibilities of the control of this function by a variation of regulator parameters.

Consider a parametrical description of stochastic sensitivity function. For this purpose we take any fixed point $x \in \mathcal{M}$ and consider a solution $x(t) = X(t,x)$ of the equation

$$dx = f_0(x)\ dt$$

with the initial condition $X(0,x) = x$.

Define the following functions

$$A(t) = A(x(t)), \ \ B(t) = B(x(t)),$$

$$G(t) = G(x(t)), \ \ S(t) = G(t)G^\top(t),$$

$$P(t) = P_{x(t)}, \ \ K(t) = K(x(t))$$

and the set

$$\mathbf{K}^x = \{K(t) \mid K(t) = K(x(t)), \ K(x) \in \mathbf{K}\}.$$

Note that due to (6) for all $K(t) \in \mathbf{K}^x$ the following identity

$$\forall t \in \mathbf{R}^1 \quad K(t)P(t) \ = \ K(t) \tag{9}$$

holds.

For the system (3), the stochastic sensitivity function at the points of the trajectory $x(t)$ is determined by a matrix $W(t)$.

This matrix for any $x \in \mathcal{M}$ and $K(t) \in \mathbf{K}^x$ is a unique solution of the equation

$$\begin{aligned} \dot{W} = (A(t) + B(t)K(t))W + \\ +W(A(t) + B(t)K(t))^\top + P(t)S(t)P(t). \end{aligned} \tag{10}$$

The matrix $W(t)$ is connected by the formula

$$\lim_{t\to+\infty}(P(t)V(t)P(t) - W(t)) = 0$$

with a covariation matrix $V(t) = \mathrm{cov}(y(t), y(t))$ of an arbitrary solution $y(t)$ of the linear stochastic system

$$dy = (A(t)y + B(t)v)dt + P(t)G(t)dw \tag{11}$$

with a regulator

$$v = K(t)y. \tag{12}$$

For the small noise, the matrix $W(t)$ gives us a precise description of the distribution of random trajectories near the manifold $\mathcal{M}$. The value $\varepsilon^2 W(t)$ is a first approximation of the covariance matrix for intersection points of random trajectories of the nonlinear stochastic system (3) with a hyperplane that is orthogonal to the manifold $\mathcal{M}$ at the point $x(t) \in \mathcal{M}$.

The stochastic sensitivity function $W(t)$ of the manifold $\mathcal{M}$ depends on a choice of a feedback matrix $K(t)$. Under these circumstances, the following control problem is a main point of this paper interest.

Consider a family of solutions $X(t, x)$ of the deterministic system (1) with initial condition $X(0, x) = x$. For any fixed point $x \in \mathcal{M}$ the solution $x(t) = X(t, x)$ defines a t-parametrization of the all points of the manifold $\mathcal{M}$. Let us introduce a set

$$\mathcal{K}^x \ = \ \{V(t) | V(t) = V(X(t, x)), \ t \in \mathbf{R^1}\},$$

where $V(x)$ is P-positive matrix function. A symmetric matrix function $V(x)$ is called P-positive if the following condition holds

$$\forall x \in \mathcal{M} \ \forall z \in \mathbf{R^n} \ P_x z \neq 0 \ \Rightarrow \ (z, V(x)z) > 0.$$

Consider a set of admissible stochastic sensitivity functions

$$\mathbf{M}^x = \mathcal{K}^x \cap \mathbf{C^1}(\mathbf{R^1}).$$

Main Problem.

Let $\bar{W}(t) \in \mathbf{M}^x$ be a desired stochastic sensitivity matrix. It is required to find such matrix $K \in \mathbf{K}^x$ that a matrix $W_K(t)$ is a solution of the equation (10). It means that for any $t \in \mathbf{R^1}$ the following identity

$$W_K(t) = \bar{W}(t). \tag{13}$$

holds.

This Main Problem is a problem of the synthesis of the required stochastic sensitivity function. The solution of this problem assumes a fitting of a suitable feedback matrix $K(t)$ in the regulator (12).

Let us fix some matrix $\bar{W}(t) \in \mathbf{M}^x$. If to put $W(t) = \bar{W}(t)$ in the (10), we have an equation

$$\begin{aligned} & B(t)K(t)\bar{W}(t) + \bar{W}(t)K^\top(t)B^\top(t) = \\ & = \frac{d}{dt}\left[\bar{W}(t)\right] - A(t)\bar{W}(t) - \bar{W}(t)A^\top(t) - \\ & -P(t)S(t)P(t). \end{aligned} \tag{14}$$

for the unknown matrix $K(t)$.

Thus, the Main Problem of a formation of the desirable stochastic sensitivity function is reduced to the decision of the matrix algebraic equation (14).

Solvability of this equation is connected with properties of the matrix $B(t)$.

Lemma 3. Let for any $t \in \mathbf{R^1}$ the matrix $B(t)$ is quadratic ($n = l$) and nonsingular ($\mathrm{rank}B = n$). Then the system (14) has the solution

$$\bar{K}(t) = B^{-1}(t)\left(\frac{d}{dt}\left[\bar{W}(t)\right]\bar{W}^+(t) +\right.$$

$$+\frac{1}{2}\bar{W}(t)\frac{d}{dt}\left[\bar{W}^{+}(t)\right]-\frac{1}{2}P(t)S(t)\bar{W}^{+}(t)-A(t)\Big),$$

where "+" means a pseudoinversion. Really, for the matrix $\bar{K}(t)$ due to formulas

$$\bar{W}^{+}\bar{W}=\bar{W}\bar{W}^{+}=P,$$

$$\frac{d}{dt}\left[\bar{W}\right]=\frac{d}{dt}\left[\bar{W}\right]P+P\frac{d}{dt}\left[\bar{W}\right]+\bar{W}\frac{d}{dt}\left[\bar{W}^{+}\right]\bar{W}$$

the following relations hold

$$B\bar{K}\bar{W}+\bar{W}\bar{K}^{\top}B^{\top}=\frac{d}{dt}\left[\bar{W}\right]P+$$

$$+\frac{1}{2}\bar{W}(t)\frac{d}{dt}\left[\bar{W}^{+}\right]\bar{W}-\frac{1}{2}PSP-A\bar{W}+$$

$$+P\frac{d}{dt}\left[\bar{W}\right]+\frac{1}{2}\bar{W}\frac{d}{dt}\left[\bar{W}^{+}\right]\bar{W}-\frac{1}{2}PSP-\bar{W}A^{\top}=$$

$$=\frac{d}{dt}\left[\bar{W}\right]-PSP-A\bar{W}-\bar{W}A^{\top}.$$

It means that $\bar{K}$ is a solution.

If $\mathrm{rank}B<n$, the system (14) not always has a solution.

3 Accessibility and Controllability

Let us introduce the concepts of an accessibility and full controllability.

Definition 1. If an element $\bar{W}\in\mathbf{M}^{x}$ for some $K\in\mathbf{K}^{x}$ for all t satisfies the equality $W_{K}(t)=\bar{W}(t)$ then $\bar{W}$ is called *accessible* in the system (3). A set of all accessible elements

$$\mathbf{W}^{x}=\{\bar{W}\in\mathbf{M}^{x}\mid\exists K(t)\in\mathbf{K}^{x}\quad W_{K}(t)=\bar{W}(t)\}$$

is called *a set of accessibility* of the system (3).

Definition 2. The manifold $\mathcal{M}$ is named *fully stochastic controllable* in the system (3), if

$$\forall\,x\in\mathcal{M}\;\forall\,W\in\mathbf{M}^{x}\;\exists\,K\in\mathbf{K}^{x}\quad W_{K}(t)\equiv W(t).$$

In this case, we will speak shortly that the system (3) is *fully stochastic controllable*.

Directly from these definitions the Proposition follows.

Proposition 1. The manifold $\mathcal{M}$ is fully stochastic controllable if and only if

$$\forall\,x\in\mathcal{M}\quad\mathbf{W}^{x}=\mathbf{M}^{x}.$$

Now the sufficient condition of full stochastic controllability on the basis of the Lemma 3 can be written as follows.

Proposition 2. The condition

$$\forall\,x\in\mathcal{M}\qquad\mathrm{rank}\frac{\partial f}{\partial u}(x,0)=n=l$$

is sufficient for a full stochastic controllability of the system (3).

Remark. The constructive application of this general theory for the case of equilibrium points and limit cycles with a physically meaningful examples can be found in ([Bashkirtseva and Ryashko, 2005], [Bashkirtseva and Ryashko, 2008]).

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