Delayed feedback control of discrete systems with transmission delay

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Abstract

In this paper, a new form of delayed feedback control, called *delayed DFC*, is proposed for discrete systems with transmission delay. An inherent limitation of the delayed DFC is found. In order to overcome the limitation, a multi-step recursive delayed DFC is proposed and a necessary and sufficient condition for stability of closed-loop system is obtained.

I. Introduction

Controlling chaos has been widely studied since the well-known OGY method [5] was proposed. One of the important chaos control methods, called delayed feedback control (DFC), was proposed by Pyragas [6] for the first time, in which the control input is the feedback of the difference between the current state and the delayed one. The advantage of DFC is requiring no preliminary calculation of the target periodic orbit or equilibrium point, which is such that controlling chaos becomes simple and convenient. Hence, this method has been paid great attention in chaos control filed [4][8]. However, for chaotic discrete-time systems, Ushio [10] found a limitation, called *odd number limitation*, of DFC. Similar limitation exists in control of chaotic continuous-time systems [3][9]. Actually, *odd number limitation* describes a necessary condition for stabilizability via DFC[11]. Moreover, Tian and Zhu [7][13] obtained some necessary and sufficient conditions for stabilizability of single input discrete systems via DFC, which gave a full characterization on the inherent limitation of DFC.

In order to overcome the *odd number limitation*, some generalized forms of DFC were proposed, for example, observer-based DFC[1], dynamic DFC[11], recursive DFC[12] and so on. However, it should be noted that in the above mentioned DFCs, the current state $\mathrm{x}(k)$ has to be obtained. Actually, in many practical systems such as networked control systems [2], the current state $\mathrm{x}(k)$ can not be transmitted to the controller as soon as possible. Assume the delayed time is m in transmission. Then only the past state $\mathrm{x}(t-m)$ can be obtained. In this case, a natural idea is replacing the original DFC $K[\mathrm{x}(k)-\mathrm{x}(k-1)]$ by a new form $K[\mathrm{x}(k-m)-\mathrm{x}(k-m-1)]$. Then how to describe the limitation of delayed DFC and how to design the stabilizing controller? That is just the main content of this paper.

The present paper proposed a new form of DFC, called delayed DFC, for systems with transmission delay. An inherent limitation of the delayed DFC is revealed. In order to overcome the limitation, a multi-step recursive delayed DFC is proposed. It is obtained that a necessary and sufficient condition of the stabilizability by the multi-step recursive delayed

DFC.

II. NECESSARY CONDITION FOR STABILIZABILITY VIA DELAYED DFC

Consider a nonlinear discrete system described by

$$\mathbf{x}(k+1) = f(\mathbf{x}(k),\ u(k)), \tag{1}$$

where $u(k) \in R^p$ is the control input, $\mathbf{x}(k) \in R^n$ is the state, $f:\ R^n \times R \to R^n$ is a smooth mapping. Assume $\mathbf{x}^*$ is the unstable fixed point of the open-loop system, i.e., $\mathbf{x}^* = f(\mathbf{x}^*, 0)$. Write $A = D_{\mathbf{x}} f(\mathbf{x}^*, 0)$ and $B = D_u f(\mathbf{x}^*, 0)$. Then linearized system of (1) around $\mathbf{x}^*$ is

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where $x(k) = \mathbf{x}(k) - \mathbf{x}^*$. Assume there exists transmission delay in the control system (2) and the number of delayed steps is m. That is the controller can only obtain the delayed state $x(k-m)$ instead of $x(k)$. Consider the new form of DFC

$$u(k) = K[\mathbf{x}(k-m) - \mathbf{x}(k-m-1)], \tag{3}$$

which we call delayed DFC. It is easy to see that (3) is equivalent to

$$u(k) = K[x(k-m) - x(k-m-1)]. \tag{4}$$

With the controller (4), the closed-loop system of (2) is

$$x(k+1) = Ax(k) + BKx(k-m) - BKx(k-m-1), \tag{5}$$

which is an $(m+2)$-order difference equations. Let $y_1(k) = x(k-m-1)$,$y_2(k) = x(k-m)$, $\cdots$, $y_{m+2}(k) = x(k)$, then closed-loop system (5) can be rewritten as

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \\ \vdots \\ y_m(k+1) \\ y_{m+1}(k+1) \\ y_{m+2}(k+1) \end{bmatrix} = \begin{bmatrix} & I & & & & \\ & & I & & & \\ & & & \ddots & & \\ & & & & I & \\ & & & & & I \\ -BK & BK & 0 & \cdots & 0 & A \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_m(k) \\ y_{m+1}(k) \\ y_{m+2}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ B \end{bmatrix} u(k). \tag{6}$$

It is easily seen that the characteristic polynomial is

$$d(s) = \det(s^{m+2}I - s^{m+1}A - sBK + BK). \tag{7}$$

Lemma 1. If $r :=$rank$B < n$, then the characteristic polynomial $d(s)$ has a root $s = 0$ and its multiplicity $q \geq n - r$.

Proof. Denote by A^* the coefficient matrix of (6). It is easy to see

$$\text{rank}A^* = (m+1)n + \text{rank}(BK) \leq (m+1)n + r. \tag{8}$$

By $r < n$, one have $d(0) = \det(BK) = 0$. Then the multiplicity q of $s = 0$ satisfies

$$q \geq (m+2)n - \text{rank}A^* \geq (m+2)n - (m+1)n - r = n - r. \tag{9}$$

The lemma is proved.

Theorem 1. If there exists K such that closed-loop system (5) is asymptotically stable, then

$$0 < \det(I - A) < 2^{(m+1)n+r}, \tag{10}$$

where r =rank$B$.

Proof. By Lemma 1, one can write the characteristic polynomial of closed-loop system (5) as

$$d(s) = s^{n-r} \prod_{i=1}^{(m+1)n-r} (s - \lambda_i), \tag{11}$$

where $|\lambda_i| < 1$ for the asymptotical stability of (5). Thus,

$$0 < d(1) = \prod_{i=1}^{(m+1)n-r} (1 - \lambda_i) \leq \prod_{i=1}^{(m+1)n-r} (1 + |\lambda_i|) < 2^{(m+1)n-r}. \tag{12}$$

Moreover, from (7) one can see

$$d(1) = \det(I - A). \tag{13}$$

Hence, by (12) and (13) the theorem is proved.

Remark 1. If A has an eigenvalue equal to 1 or an odd number of real eigenvalues greater than 1, then det$(I_n - A) \leq 0$. Thus Theorem 2 implies that there exist no delayed DFC (3) such that (5) is asymptotically stable, which is just the *odd number limitation* of the original DFC appeared in [10] or [12].

Remark 2. If $m = 1$ and $r = 1$, then the result is reduced to the necessary and sufficient condition shown in Theorem 2 of [13].

III. Multi-step Recursive Delayed DFC

In order to overcome *odd number limitation* of the original DFC, dynamic DFC [11] and recursive DFC [12] are proposed by Yamamoto *et. al.*. For the same aim, we consider the following multi-step recursive delayed DFC:

$$u(k) = K_0[x(k-m) - x(k-m-1)] + \sum_{i=1}^{m+1} K_i u(k-i). \tag{14}$$

With the controller (14), the closed-loop system is the difference equations

$$x(k+1) = Ax(k) + BK_0[x(k-m) - x(k-m-1)] + \sum_{i=1}^{m+1} BK_i u(k-i) \tag{15}$$

$$u(k) = K_0[x(k-m) - x(k-m-1)] + \sum_{i=1}^{m+1} K_i u(k-i), \tag{16}$$

with the dynamic variables $[x(k),\ u(k-1)]^{\mathrm{T}}$.

Theorem 2. Assume that $(A,\ B)$ is stabilizable, i.e., there is K such that all the characteristic roots of $A+BK$ lie inside the unit circle. Then, there exists a multi-step recursive delayed DFC (14) such that the closed-loop system (15)-(16) is asymptotically stable if and only if $\det(I-A)\neq 0$. Moreover, one of the controller is given by

$$K_0=-K(I-A)^{-1}A^{m+1},\ K_{m+1}=K(I-A)^{-1}A^mB, \tag{17}$$

$$K_i=KA^{i-1}B\quad(i=1,2,\cdots,m). \tag{18}$$

Proof. (Necessity) It is easily seen that the characteristic polynomial of the closed-loop system (15)-(16) is

$$\begin{aligned} H(s) &= \det\begin{bmatrix} Is^{m+2}-As^{m+1}-BK_0s+BK_0 & -\sum\limits_{i=1}^{m+1}BK_is^{m+1-i} \\ -K_0s+K_0 & Is^{m+1}-\sum\limits_{i=1}^{m+1}K_is^{m+1-i}\end{bmatrix} \\ &= \det\begin{bmatrix} Is^{m+2}-As^{m+1} & -Bs^{m+1} \\ -K_0s+K_0 & Is^{m+1}-\sum\limits_{i=1}^{m+1}K_is^{m+1-i}\end{bmatrix}. \end{aligned} \tag{19}$$

Thus,

$$H(1)=\det(I-A)\det(I-\sum_{i=1}^{m+1}K_i). \tag{20}$$

Since the closed-loop system (15)-(16) is asymptotically stable, i.e., all the roots of $H(s)$ lie inside the unit circle, we have $H(1)\neq 0$, which implies $\det(I-A)\neq 0$ by (20).

(Sufficiency) Substituting (17) and (18) into (19), we obtain the characteristic polynomial $H(s)$:

$$\begin{aligned} & s^{(m+1)n}\det\begin{bmatrix} Is-A & -B \\ K(I-A)^{-1}A^{m+1}(sI-I) & Is^{m+1}-K\hat{A}B\end{bmatrix} \\ &= s^{(m+1)n}\det\begin{bmatrix} Is-A & -B \\ \Delta & Is^{m+1}\end{bmatrix}, \end{aligned} \tag{21}$$

where

$$\hat{A} = \sum_{i=1}^{m}A^{i-1}s^{m+1-i}+(I-A)^{-1}A^m \tag{22}$$

$$\begin{aligned} \Delta &= -K\left(\sum_{i=1}^{m}A^{i-1}s^{m+1-i}+(I-A)^{-1}A^m\right)(Is-A) \\ &\quad +K(I-A)^{-1}A^{m+1}(sI-I) \\ &= -K\left(\sum_{i=1}^{m}A^{i-1}s^{m+2-i}-\sum_{i=1}^{m}A^is^{m+1-i}\right) \\ &\quad -K(I-A)^{-1}A^ms+K(I-A)^{-1}A^{m+1} \\ &\quad +K(I-A)^{-1}A^{m+1}s-K(I-A)^{-1}A^{m+1} \\ &= -K\left(s^{m+1}I-A^ms+(I-A)^{-1}A^ms-(I-A)^{-1}A^{m+1}s\right) \\ &= -Ks^{m+1}. \end{aligned} \tag{23}$$

Hence,

$$\begin{aligned}H(s) &= s^{(m+1)n}\det\left[\begin{array}{cc} Is-A & -B \\ -Ks^{m+1} & Is^{m+1}\end{array}\right]\\ &= s^{(m+1)n}\det\left[\begin{array}{cc} Is-(A+BK) & -B \\ 0 & Is^{m+1}\end{array}\right]\\ &= s^{2(m+1)n}\det[sI-(A+BK)].\end{aligned} \tag{24}$$

Since all the characteristic roots of $A+BK$ lie inside the unit circle, $H(s)$ is a stable polynomial, i.e., the closed-loop system (15)-(16) is asymptotically stable.

Remark 3. If $m=0$, then controller (14) reduces to the form of recursive DFC proposed by Yamamoto [12]. Moreover, our proof method and designed controller are different from [12].

Remark 4. If $B=I$, then we can set $K=-A$. By (24), all the characteristic roots are zero, which yields that $x(t)$ and $u(t)$ arrive zero after finite steps. In this case, the feedback gains of the multi-step recursive delayed DFC (14) are

$$K_0=(I-A)^{-1}A^{m+2},\ K_{m+1}=-(I-A)^{-1}A^{m+1}, \tag{25}$$

$$K_i=-A^i \quad (i=1,2,\cdots,m). \tag{26}$$

Remark 5. For the original nonlinear system (1), the feedback gains are dependent on $A=D_{\mathbf{x}}f(\mathbf{x}^*,0)$ and $B=D_uf(\mathbf{x}^*,0)$, which still have relationship with the equilibrium point $\mathbf{x}^*$. To overcome the shortcoming, we replace A, B and K by nonlinear functions $A(\mathbf{x}(k-m))$, $B(\mathbf{x}(k-m))$ and $K(\mathbf{x}(k-m))$ respectively in the feedback gains, where $A(\mathbf{x})=D_{\mathbf{x}}f(\mathbf{x},0)$, $B(\mathbf{x})=D_uf(\mathbf{x},0)$ and $K(\mathbf{x})$ is such that $A(\mathbf{x})+B(\mathbf{x})K(\mathbf{x})$ is a constant stable matrix. With the modified multi-step recursive delayed DFC, the controller only depend on the delayed state $\mathrm{x}(t-m)$ and the linearized closed-loop system around x^* does not be changed.

IV. SIMULATIONS

In this section, we give a numerical example which illustrate the effectiveness of the proposed method. Consider the second order discrete chaotic system [13]: The chaotic discrete-time system of the Henon map is described by

$$\begin{array}{l}\mathbf{x}_1(k+1)=\mathbf{x}_2(k),\\ \mathbf{x}_2(k+1)=0.3\mathbf{x}_1(k)-\mathbf{x}_2^2(k)+b_0+u(k),\end{array} \tag{27}$$

where $b_0\in[1.07,\ 1.44]$ is an uncertain parameter. The chaotic system has two unknown equilibrium points:

$$\begin{aligned}\xi_1^*&=(y_1^*,y_1^*), \qquad y_1^*=\frac{1}{2}\left(-0.7+\sqrt{0.49+4b_0}\right),\\ \xi_2^*&=(y_2^*,y_2^*), \qquad y_2^*=\frac{1}{2}\left(-0.7-\sqrt{0.49+4b_0}\right).\end{aligned} \tag{28}$$

It is easy to see

$$A(\mathrm{x}) = D_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 0.3 & -2\mathbf{x}_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{29}$$

$$\det(I_2 - D_{\mathbf{x}} f(\xi_1^*)) = \sqrt{0.49 + 4b_0} > 0, \tag{30}$$

$$\det(I_2 - D_{\mathbf{x}} f(\xi_2^*)) = -\sqrt{0.49 + 4b_0} < 0. \tag{31}$$

Assume the transmission delay $m = 2$. By Remark 5, we try to design a nonlinear multi-step recursive delayed DFC such that both the equilibrium points are asymptotically stabilized. Let $K(\mathrm{x}) = [\,-0.3 \quad 2\mathrm{x}_2\,]$, then

$$A(\mathrm{x}) + BK(\mathrm{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{32}$$

whose characteristic roots are zero. By (17) and (18), we set

$$\begin{aligned} &K_0(\mathrm{x}) = -K(\mathrm{x})[I_2 - A(\mathrm{x})]^{-1}[A(\mathrm{x})]^3 \\ &= \begin{bmatrix} -2.4\mathrm{x}_2^3 + 0.36\mathrm{x}_2^2 - 0.36\mathrm{x}_2 + 0.027 \\ 16\mathrm{x}_2^4 - 2.4\mathrm{x}_2^3 + 3.6\mathrm{x}_2^2 - 0.36\mathrm{x}_2 + 0.09 \end{bmatrix}^{\mathrm{T}} \frac{1}{2\mathrm{x}_2 + 0.7}; \end{aligned}$$

$$K_3(\mathrm{x}) = K(\mathrm{x})[I_2 - A(\mathrm{x})]^{-1}[A(\mathrm{x})]^2 B = \frac{8\mathrm{x}_2^3 - 1.2\mathrm{x}_2^2 + 1.2\mathrm{x}_2 - 0.09}{2\mathrm{x}_2 + 0.7};$$

$$K_1(\mathrm{x}) = K(\mathrm{x})B = 2\mathrm{x}_2; \quad K_2(\mathrm{x}) = K(\mathrm{x})A(\mathrm{x})B = -4\mathrm{x}_2^2 - 0.3.$$

Then the designed nonlinear multi-step recursive delayed DFC is

$$\begin{aligned} u(k) = {} & K_0(\mathrm{x}(k-2))[\mathrm{x}(k-2) - \mathrm{x}(k-3)] + K_1(\mathrm{x}(k-2))u(k-1) \\ & + K_2(\mathrm{x}(k-2))u(k-2) + K_3(\mathrm{x}(k-2))u(k-3). \end{aligned} \tag{33}$$

Since stabilization is guaranteed only in a neighborhood of the fixed point, we adopt the following small control law proposed by Pyragas in [6]:

$$u_s(k) = \begin{cases} u(k), \ \text{if } u(k) < \varepsilon, \\ 0, \ \text{otherwise}, \end{cases} \tag{34}$$

where ε is a sufficiently small positive number. Shown in Fig. 1 and Fig. 2 are the behavior of the controlled system under the same controller (34) with perturbed parameters $b_0 = 1.152$ and $b_0 = 1.2$ respectively.

V. Conclusion

In conclusion, we proposed a new form of delayed feedback control, called *delayed DFC*, for discrete systems with transmission delay. Theoretical analysis showed that it still has an inherent limitation. In order to overcome the limitation, we designed a multi-step recursive delayed DFC and obtained a necessary and sufficient condition for stability of the closed-loop system.

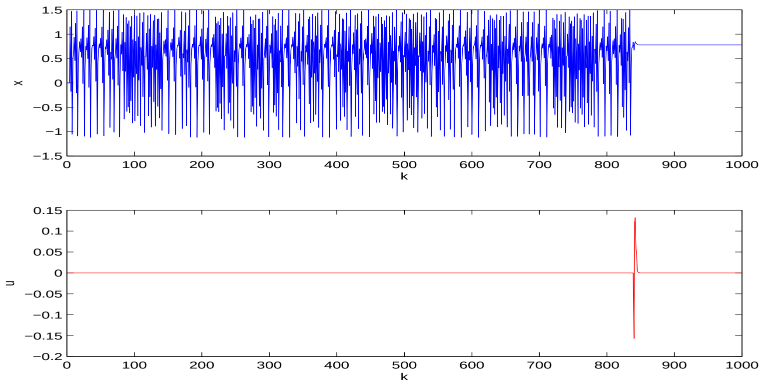

Fig. 1. Controlled behavior of x_1 and control input. $b_0 = 1.152$.

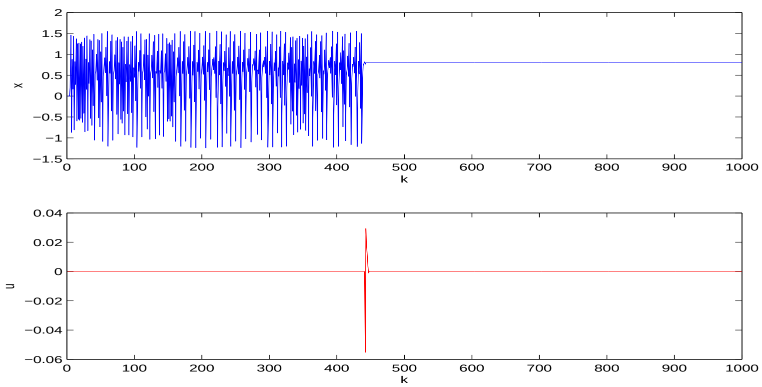

Fig. 2. Controlled behavior of x_1 and control input. $b_0 = 1.2$.

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