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FRACTIONAL PI^λ CONTROLLER FOR FRACTIONAL ORDER LINEAR SYSTEMS WITH INPUT HYSTERESIS

Necati Özdemir
Department of Mathematics,
Balikesir University
Balikesir, TURKEY
nozdemir@balikesir.edu.tr

Beyza Billur İskender
Department of Mathematics,
Balikesir University
Balikesir, TURKEY
beyzabillur@hotmail.com

Abstract

In this work, single-input, single-output fractional order linear time invariant systems subject to input hysteresis, Relay and Duhem, are presented. It is assumed that the plant is stable and fractional PI^λ controller are designed with negative unity feedback to overcome the undesirable effects of hysteresis. Tuning of the PI^λ controller is demonstrated numerically by using Grünwald-Letnikov approach. It is shown that PI^λ controller are more effective than PI controller for the system subject to input Duhem hysteresis.

Key words

Fractional order systems, fractional order controller, hysteresis.

1 Introduction

Fractional calculus which is extension of non integer order derivative and integral has great attention in the last few decades, because of its ability to model systems more accurately than integer orders. Therefore, the fractional calculus is an important tool for various areas of science and technology. Basis of the fractional calculus could be found in [Oldham and Spanier, 1974; Miller and Ross, 1993; Podlubny, 1999].

Recently, fractional calculus has been applied to control theory. Fractional order controllers have been developed for fractional order systems. Firstly, Oustaloup [Oustaloup, 1995] has proposed fractional order controller CRONE which is abbreviation of Commande Robuste d'Ordre Non Entier. He demonstrated that performance of the CRONE method more superior than classic *PID* controller. Later, Podlubny [Podlubny, 1994; Podlubny, Dorcak and Kostial, 1997; Machado, 1997] proposed fractional $PI^\lambda D^\mu$ controller with integrator order λ and differentiator order μ and showed that $PI^\lambda D^\mu$ controller had better performance when used for fractional dynamical systems.

In lately, control of systems with hysteresis is an important task of engineering. Because, hysteresis effects on systems as loss of stability, limit cycles and steady state error etc. Hysteresis has different meaning for different fields but in general, it is a special type of memory based relation between input and output. Although it has quite importance in terms of its applications, the mathematical basis of this phenomenon base on last thirty years, for example, [Krasnosel'skii and Pokrovskii, 1989; Mayergoyz, 1991; Macki, Nistri and Zecca, 1993; Visintin, 1994]. Control of hysteresis could be seen in [Tao and Kokotovic, 1994; Sain, Sain and Spencer, 1997; Logemann and Mawby, 1998] where the systems are integer order. In addition to this, there are only a few works on fractional order systems with hysteresis [Bagley and Torvik, 1986; Padovan and Sawicki, 1997; Darwish and El-Bary, 2006; Schafer and Kruger, 2006; Deng and Lü, 2007] which do not deal with control problem.

In this work, single-input and single-output fractional order linear time invariant systems subject to input hysteresis are presented. It is assumed that the plant of the system is stable and hysteresis is compensated by PI^λ controller with negative unity feedback. In section 2, necessary definitions and properties of fractional calculus are given and fractional dynamical system is described. In section 3, fractional order PI^λ controller are shown. In section 4, hysteresis phenomenon is defined with some of hysteresis operators. In section 5, PI^λ controller of the fractional order systems with input hysteresis is mainly investigated. Section 6 shows the numerical examples for Relay and Duhem hysteresis. The numerical solutions are obtained by using Grünwald-Letnikov approach and compared with analytical solutions. Finally, conclusions are deduced in Section 7.

2 Preliminaries For Fractional Calculus

2.1 Fundamental Definitions

There are different fractional derivative definitions, which include Riemann-Liouville (R-L), Grünwald-Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivative in [Oldham and Spanier, 1974; Miller and Ross, 1993; Podlubny, 1999]. Here, the R-L definition is used.

Definition 1. *The R-L derivative of a function $x(t)$ is*

$$ {}_aD_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} x(\tau)\, d\tau \quad (1) $$

where $n-1 \le \alpha < n$, n is an integer and $\Gamma(.)$ is Euler's gamma function.

Definition 2. *The R-L integral of a function $x(t)$ is*

$$ I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau. \quad (2) $$

Definition 3. *The Laplace transform of R-L derivative is the form of*

$$ L\left[{}_aD_t^{\alpha} x(t)\right] = s^{\alpha} X(s) - \sum_{k=0}^{n-1} s^k \left[D^{\alpha-k-1} x(t)\right]_{t=t_0}. \quad (3) $$

Definition 4. *The Mittag-Leffer function is defined by*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha>0,\ \beta>0). \quad (4) $$

2.2 Fractional Order Dynamical Systems

A differential equation which contains fractional derivative is called fractional differential equation. Fractional order dynamical system is a system that is modeled by fractional order differential equation as

$$ {}_{t_0}D_t^{\alpha} x(t) = f(x,u) \quad (5) $$

where $x(t)$ is a state vector, $u(t)$ is an input function and f is a function of x and u. A fractional order linear time invariant system is in the following form

$$ {}_{t_0}D_t^{\alpha} x(t) = Ax(t) + Bu(t) \quad (6) $$

where A and B are suitable coefficient matrix. General solution of the fractional order linear system is obtained by using Laplace transform. In the s-domain, solution of the system is

$$ X(s) = (s^{\alpha} I - A)^{-1} x_0 + (s^{\alpha} I - A)^{-1} BU(s) \quad (7) $$

with initial condition $x_0 = \left[{}_{t_0}D_t^{\alpha-1} x(t)\right]_{t=t_0}$, and in the time domain

$$ x(t) = E_{\alpha,1}(At^{\alpha}) x_0 + \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) Bu(\tau)\, d\tau. \quad (8) $$

Stability of the fractional order systems has been studied by [Matignon, 1996; Matignon, 1998]. The stability condition of the fractional order systems is defined by the following definition.

Definition 5. *The fractional order linear system (6) is stable if for all $i = 1, 2, ..., m$*

$$ |\arg(\Lambda_i(A))| > \alpha \frac{\pi}{2} \quad (9) $$

where $\Lambda_i(A)$ denotes the i-th eigenvalue of A and α is the fractional order of the system.

3 Fractional PI^{λ} Controller

The proportional integral (PI) control which is described by fractional order integral is called as fractional proportional integral controller (PI^{λ}). It is given by

$$ u(t) = k_p e(t) + k_i I^{\lambda} e(t) \quad (10) $$

where $e(t) = r - y(t)$ is the error function, k_p and k_i are the gains of proportional and integral controllers, respectively, and I^{λ} is fractional order integral which is defined by R-L integral. The fractional PI^{λ} controller is more flexible than the integer one because it allows one to tune λ, in addition, k_p and k_i, and it does not effect changes of systems parameters [Podlubny, 1999].

4 Hysteresis Operators

Hysteresis is a nonlinear relation between input and output functions which can be mathematically represented by causal and rate independent operator Φ. There are different kinds of hysteresis operators, which are Relay, Stop, Play, Duhem, Preisach and Prandtl. In this work Relay and Duhem operators are considered.

Relay operator [Macki, Nistri and Zecca, 1993]:
In relay, the relation between input $u(t)$ and output $\omega(t)$ is determined by two threshold values γ and σ, $(\gamma < \sigma)$. The output moves on one of the given function $f_1 : (-\infty, \sigma] \to \mathbb{R}$ and $f_2 : [\gamma, \infty) \to \mathbb{R}$. In this work, it is assumed that f_1 and f_2 functions are

coincided at γ and σ. The relay operator is defined by

$$\omega(t) = \begin{cases} f_1(u), & u(t) \leq \gamma \\ f_2(u), & u(t) \geq \sigma \\ f_1(u), \gamma \leq u(t) \leq \sigma \text{ and } u(\tau(t)) = \gamma \\ f_2(u), \gamma \leq u(t) \leq \sigma \text{ and } u(\tau(t)) = \sigma \end{cases} \tag{11}$$

where $\tau(t) = \{s : s \leq t,\ u(s) = \sigma \text{ or } u(s) = \gamma\}$, i.e. $\tau(t)$ is the last time value that the input function $u(t)$ reached either of the two threshold values. Figure 4 shows the Relay operator for the functions:

$$\left.\begin{aligned} f_1(u) &= 1.7654 - \sqrt{1.1 - u(t)} \\ f_2(u) &= \sqrt{1.1 - u(t)}. \end{aligned}\right\} \tag{12}$$

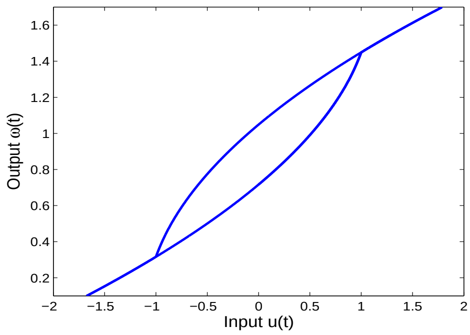

Figure 1. Relay hysteresis

Duhem Operator [Visintin, 1994]

The Duhem operator is based on the property that the output of the operator changes its character when the input changes its direction. This operator is defined by

$$\dot{\omega}(t) = \begin{cases} f_1(u, \omega)\,\dot{u}, & \dot{u}(t) \geq 0 \\ f_2(u, \omega)\,\dot{u}, & \dot{u}(t) \leq 0 \end{cases} \tag{13}$$

where f_1 and f_2 are continuous functions. Figure 2 shows the Duhem operator with the initial condition $\omega(0) = 0$, for the functions:

$$\left.\begin{aligned} f_1(u, \omega) &= \rho[bu - \omega] + c \\ f_2(u, \omega) &= -\rho[bu - \omega] + c \end{aligned}\right\} \tag{14}$$

where ρ, b and c are constants chosen as $\rho = 1$, $b = 3.1635$ and $c = 0.345$ in [Su, Stepanenko, Svoboda and Leung, 2000].

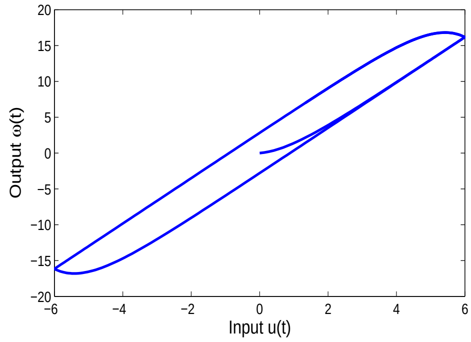

Figure 2. Duhem hysteresis

5 Fractional Order Systems With Input Hysteresis

Consider a fractional order single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$) linear time invariant system with input hysteresis which is the following form

$$\left.\begin{aligned} {}_{t_0}D_t^{\alpha} x(t) &= Ax(t) + B\Phi(u(t)) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{15}$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^m$ and $C \in \mathbb{R}^{1 \times m}$ are coefficient matrix, $x \in \mathbb{R}^m$ $(m \in \mathbb{N})$ is the state variable and Φ is the hysteresis operator, see Figure 3. The General solution of the system is obtained by using Laplace transform with the initial condition x_0 as follows

$$\begin{aligned} x(t) = {} & E_{\alpha,1}(At^{\alpha})x_0 \\ & + \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) B\Phi(u(\tau))\, d\tau, \end{aligned} \tag{16}$$

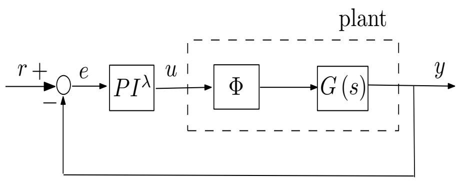

Figure 3. Fractional PI^{λ} control of fractional order linear system with input hysteresis

6 Numerical Example

(Controlled fractional diffusion process) Consider a fractional diffusion process on the one-dimensional spatial domain $[0, 1]$, with diffusion coefficient β and Dirichlet boundary condition. The process is called as subdiffusion when $0 < \alpha < 1$, and diffusion when

$\alpha = 1$. It is assumed that the process has point actuation at $x_b \in (0,1)$. The single-input, single-output system is defined by

$$\left.\begin{aligned} \frac{\partial^\alpha z(t,x)}{\partial t^\alpha} &= \beta \frac{\partial^2 z(t,x)}{\partial x^2} + \delta(x - x_b)\Phi(u(t)) \\ y(t) &= z(t, x_c) \end{aligned}\right\} \quad (17)$$

with boundary and initial conditions, respectively,

$$z(t,0) = z(t,1) = 0 \text{ and } z(0,x) = 0. \quad (18)$$

These equations model the problem of heating a rod of unit length whose ends are kept at zero temperature and which is initially zero temperature across its length. The temperature at x_c is raised along its length, to value r by applying heat at a point x_b along its length. The function $z(t,.)$ is the temperature profile along the length of the rod at time $t \in \mathbb{R}^+$.

Equation (17) is solved by using separation of variables, for this purpose let the partial differential equation has a solution of the form:

$$z(t,x) = T(t)X(x). \quad (19)$$

Firstly, homogen part of equation (17) is considered. Substituting (19) in (17) gives

$$\frac{1}{T}\frac{d^\alpha T}{dt^\alpha} = \beta \frac{1}{X}\frac{d^2 X}{dt^2}. \quad (20)$$

Right hand side of (20) holds if it equals a constant which is called separation constant can be chosen as the following

$$\frac{1}{T}\frac{d^\alpha T}{dt^\alpha} = \beta \frac{1}{X}\frac{d^2 X}{dt^2} = -\beta\lambda^2. \quad (21)$$

Using (18), the solution of the second part of (21) is

$$X(x) = \sin(j\pi x), \quad (j = 1,2,...),$$

which is called eigenfunctions and then the general solution of (17) is

$$z(t,x) = \sum_{k=1}^{\infty} q_k(t)\sin(k\pi x). \quad (22)$$

Since the higher order terms do not contribute much, it could be interest only finite number of terms which is denoted by m. Substituting (22) into (17) gives

$$\begin{aligned} \sum_{k=1}^{m} \frac{d^\alpha q_k(t)}{dt^\alpha}\sin(k\pi x) = -\beta \sum_{k=1}^{m} q_k(t)\sin(k\pi x) \\ + \delta(x - x_b)\Phi(u), \end{aligned} \quad (23)$$

and multiplying both side of (23) by $\sin(k\pi x)$ and then integrated from 0 to 1 gives

$$\frac{d^\alpha q_k(t)}{dt^\alpha} = -\beta k^2\pi^2 q_k(t) + 2\sin(k\pi x_b)\Phi(u) \quad (24)$$

$$q_k(0) = 0, \ k = 1,2,...,m, \quad (25)$$

where (25) are the initial conditions which are calculated from (18). (24) is presented by the state space form

$$\left.\begin{aligned} {}_0D_t^\alpha q(t) &= Aq(t) + B\Phi(u(t)) \\ y(t) &= Cq(t) \end{aligned}\right\} \quad (26)$$

where $q(t) = \left[q_1(t) \ q_2(t) \ ... \ q_m(t) \right]^T$ is state variable, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^m$ and $C \in \mathbb{R}^{1\times m}$ are matrices given by

$$\begin{aligned} &A = diagonal\left[-\beta k^2\pi^2\right], \\ &B = \left[b_1 \ b_2 \ ... \ b_m \right]^T, \ b_k = 2\sin(k\pi x_b) \\ &C = \left[c_1 \ c_2 \ ... \ c_m \right], \quad c_k = \sin(k\pi x_c), \ k = 1,2,...,m. \end{aligned}$$

According to Definition 5, system (26) is stable for all α values because the eigenvalues of the matrix A are on the negative real axis, and this system is solved by using Grünwald Letnikov approximation. For this, let the time interval $[0,T]$ is divided N equal parts and each parts is the size of h. So, it is approximated at node M, $0 \le M \le N$ such that

$$\begin{aligned} {}_0D_t^\alpha q(hM) &= \frac{1}{h^\alpha}\sum_{j=0}^{N} w_j^{(\alpha)} q(hM - jh) \\ &= Aq(hM) + B\Phi(u(hM)) \end{aligned} \quad (27)$$

where for $j = 1,2,...,N$

$$w_0^{(\alpha)} = 1; \ w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}.$$

From this equation, the response of the system is obtained at node M as follows:

$$q(hM) = \left(\frac{1}{h^\alpha} w_0^{(\alpha)} I - A\right)^{-1} \left(B\Phi(u(hM)) - \frac{1}{h^\alpha}\sum_{j=1}^{N} w_j^{(\alpha)} q(hM - jh)\right).$$

Control design of the system (26) is shown by the following simulations at the points $x_b = 0.25$ and $x_c = 0.375$.The system is solved by using Grünwald-Letnikov approximation choosing $h = 0.1$.

In Figure 4, contribution of number of the eigenvalues m to the system output $y(t)$ is demonstrated. For this purpose, the system subject to Relay hysteresis is considered and solved for different values of m with input function $u(t) = 4.5\sin(2.3t)$, and $\alpha = 1$. It can be seen from the figure that after $m = 10$ variation of $y(t)$ is very small between two consecutive values. It is also valid for the system with Duhem hysteresis. Therefore finite number of eigenvalues are sufficient for numerical calculations although the system is infinite dimensional. So $m = 15$ is chosen for the other simulation results.

In Figure 5, accuracy of the algorithm is shown by comparing analytical and numerical solutions of the system (26) subject to both Relay and Duhem hysteresis with input function $u(t) = 4.5\sin(2.3t)$. It is clearly seen that these two solutions are very closed which means the algorithm runs accurately.

Rest of the figures shows the control design of the fractional order systems subject to input hysteresis nonlinearity with $\alpha = 0.8$. Control purposes of the systems are tracking the reference value $r = 1$.

Figure 6 and 7 shows integer and fractional order control design for fractional order system subject to input Relay hysteresis, respectively. In Figure 6, integer order PI controller are designed by adjustment of k_p and k_i. It can be seen from this figure that settling time and overshoots are reduced when k_p and k_i are both increased. In Figure 7, fractional order PI^λ controller are adjusted by $\lambda = 0.7, 1.1$ for fixed $k_p = 1.5$ and $k_i = 1.3$. It can be concluded from the figure that fractional order controller do not have more advantageous with respect to integer order controller.

Figure 8 demonstrates the integer order control of the system subject to input Duhem hysteresis. It is shown that when k_p and k_i are increased, the settling time is reduced but overshoot becomes.

Figure 9 shows the fractional order controller of the same system in Figure 8. It is obvious that decreasing λ values lead to reach best settling time without overshoot. It can be concluded by comparing the last two figures that fractional order control of the system subject to input Duhem hysteresis is twofold advantageous.

7 Conclusions

In this work, single-input, single-output fractional order linear time invariant systems subject to input Relay and Duhem hysteresis were considered. Convergency of the system to the reference signal was achieved by tuning of the PI^λ controller. To obtain solution of the system, Grünwald-Letnikov approach was used. Hence, despite of the fact that PI^λ controller were more effective for Duhem hysteresis, it was not for Relay hysteresis.

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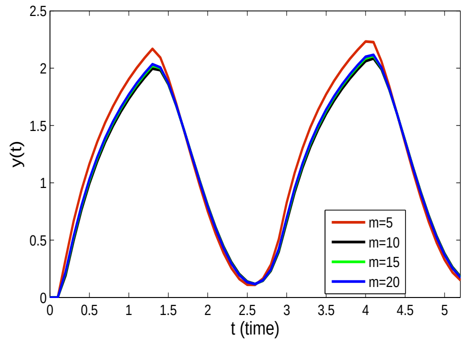

Figure 4. The solution of system (26) with input Relay hysteresis for $\alpha = 1, u(t) = 4.5sin(2.3t)$

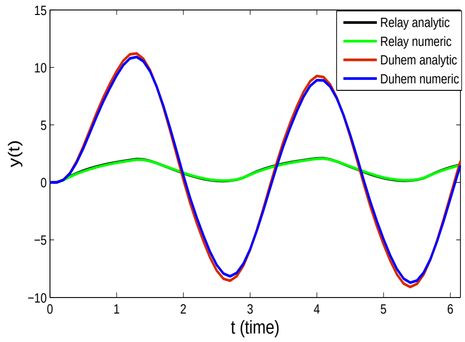

Figure 5. Comparison of analytical and numerical solution of the systems (26) with input Relay and Duhem hysteresis for $\alpha = 1$

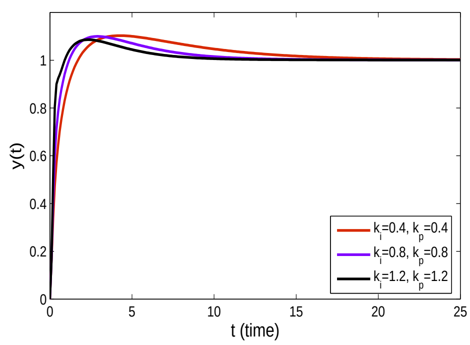

Figure 6. Integer order control of system (26) with input Relay hysteresis for $\alpha = 0.8$

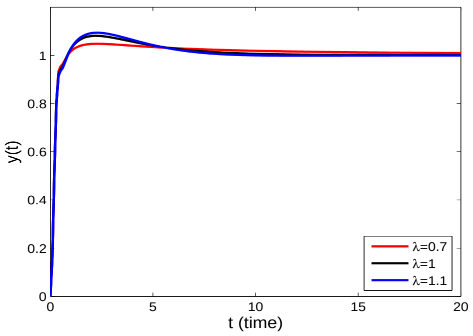

Figure 7. Fractional order control of system (26) with input Relay hysteresis for $\alpha = 0.8$, $k_p = 1.5$ and $k_i = 1.3$

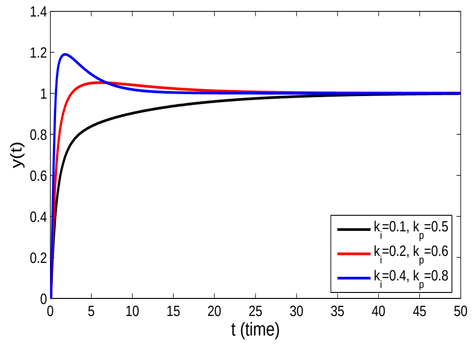

Figure 8. Integer order control of system (26) with input Duhem hysteresis for $\alpha = 0.8$

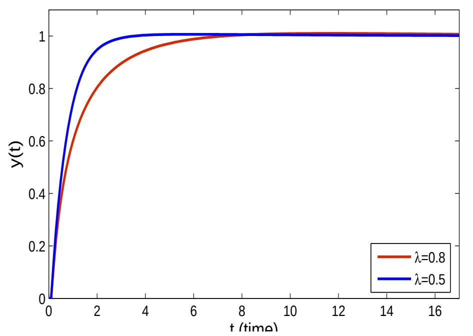

Figure 9. Fractional order control of system (26) with input Duhem hysteresis for $\alpha = 0.8$, $k_p = 0.47$ and $k_i = 0.35$

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