CYBERNETICS AND PHYSICS, VOL. 2, NO. 1, 2013 , 31–36

MIXED TRACKING AND PROJECTIVE SYNCHRONIZATION OF 5D HYPERCHAOTIC SYSTEM USING ACTIVE CONTROL

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Abstract

This paper examines mixed tracking control and hybrid synchronization of two identical 5-D hyperchaotic Lorenz systems via active control technique. The designed control functions for the mixed tracking enable each of the system state variables to stabilize at different chosen positions as well as control each state variables of the system to track different desired smooth function of time. Also, the active control technique is used to design control functions which achieve projective synchronization between the slave state variables and the master state variables. We also show that the coupling strength is inversely proportional to the synchronization time. Numerical simulations are carried out to validate the effectiveness of the analytical technique.

Key words

Hyperchaos, synchronization, tracking, active control.

1 Introduction

Since the discovery of the first chaotic system by Lorentz in 1963 many new chaotic systems have been successively developed [Chen and Ueta, 1999; Chua and Lin, 1990; Qi *et al*, 2005; Rössler, 1976]. Chaos has gradually moved from simply being a scientific curiosity to a promising subject with practical significance and applications in different fields such as communication[Mengue and Essimbi, 2012], biological systems[Shi, 2012], economics and other fields.

During the beginning of the last decade, one of the most fascinating discoveries that transformed research in the field of nonlinear dynamics and chaos theory is the fact that two or more chaotic systems evolving from different initial conditions can be made to synchronize, either by coupling the systems (locally or globally) or by forcing them. Synchronization means that the state of a response system eventually approaches that of a driving system. This was first demonstrated by Pecora and Carroll [Pecora and Carroll, 1990]. Unrelenting research in chaotic systems has given rise to different types of synchronization including complete synchronization (CS) [Pecora and Carroll, 1990], generalized synchronization (GS) [Kacarev and Parlitz, 1996], projective synchronization (PS) [Mainieri and Rehacek, 1999], function projective synchronization (FPS)[An and Chen, 2008] amongst others.

Chaotic behaviours could be beneficial feature in some cases, but can be undesirable in some engineering, biological and other physical applications; and therefore it is often desired that chaos should be controlled, so as to improve the system performance. Thus, it is of considerable interest and potential utility, to devise control techniques capable of forcing a system to maintain a desired dynamical behaviour even when intrinsically chaotic. The control of chaos and bifurcation is concerned with using some designed control input(s) to modify the characteristics of a parameterized nonlinear system. There might be need for different components of a chaotic system will be required to follow different trajectories when controlled, therefore, the need for mixed tracking or control. A number of methods such as OGY closed-loop feedback method [Ott, Grebogi and Yorke, 1990], active control [Bai and Lonngren, 1997], active backstepping [Zhang, Ma, Li, and Zou, 2005] and recursive active control [Vincent, Laoye, and Odunaike, 2009] exist for the control of chaos in systems. Chaos control is considered as a special case of chaos synchronization. Despite the numerous advantages of mixed tracking, no research work has been done in this regard to the best of our understanding.

The active control method introduced by [Bai and Lonngren, 1997] is efficient technique for the synchronization of chaotic systems because it can be used to synchronize non-identical systems. The active control scheme has received considerable attention during the last decade. Applications to various systems abound some of which include Rossler and Chen system the electronic circuits which model a third-order "jerk" equation Lorenz, Chen and Lu system geophysical model nuclear magnetic resonance (NMR) modeled by the nonlinear Bloch equations, RCL-shunted Josephson junction, inertial ratchets and most recently in extended Bonhoffer-Van der Pol oscillator.

A chaotic system has been defined as one with sensitive dependence on initial condition and possess at least one positive Lyapunov exponent. An hyperchaotic system is one with more than one positive Lyapunov exponent[Li, 2012]. Notable hyperchaotic systems include [Kapitaniak and Chua, 1994; Ning and Haken, 1990; Rössler, 1979]

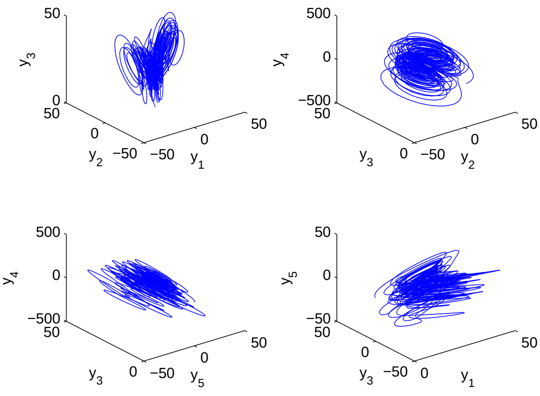

Figure 1. Attractors of 5-D hyperchaotic Lorenz system.

2 System Description

The Lorenz system [Lorenz, 1963] was the first chaotic system to be modeled and one of the most widely studied. The original formula was modified into a 4D hyperchaotic system while Hu [Hu, 2009] constructed a 5-D hyperchaotic Lorenz system by introducing two state feedback to the classical 3-D Lorenz system. The new system, which can generate hyperchaotic attractors with three positive Lyapunov Exponents (LEs) is described by (1)

$$\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 + x_4\\
\dot{x}_2 &= r x_1 - x_2 + x_1 x_3 - x_5\\
\dot{x}_3 &= -\beta x_3 + x_1 x_2\\
\dot{x}_4 &= -x_1 x_3 + k_{1d} x_4\\
\dot{x}_5 &= k_{2d} x_2
\end{aligned} \tag{1}$$

where $\sigma,\ \beta,\ r$ are the parameters of the system and k_{1d}, k_{2d} are the bifurcation parameters. When parameters are set by default as $\sigma = 10,\ \beta = \frac{8}{3},\ r = 28,\ k_{1d} = 2,\ k_{2d}\epsilon(2, 12)$, the system (1) behaves hyperchaotically with three positive LEs. The hyperchaotic attractor of the system is shown in figure (1).

3 Tracking Control of 5D Hyperchaotic Systems

3.1 Design of controllers

We aim to design controllers that will enable (1) to be controlled to a predefined rule. To make it more flexible and adaptable, we preset the controls on each component of the 5D hyperchaotic system to different functions. The system (1) with the control parameters added are given as

$$\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2 + x_4 + u_1\\
\dot{x}_2 &= r x_1 - x_2 + x_1 x_3 - x_5 + u_2\\
\dot{x}_3 &= -\beta x_3 + x_1 x_2 + u_3\\
\dot{x}_4 &= -x_1 x_3 + k_{1d} x_4 + u_4\\
\dot{x}_5 &= k_{2d} x_2 + u_5
\end{aligned} \tag{2}$$

The error function is defined as

$$\begin{aligned}
e_1 &= x_1 - f_1\\
e_2 &= x_2 - f_2\\
e_3 &= x_3 - f_3\\
e_4 &= x_4 - f_4\\
e_5 &= x_5 - f_5
\end{aligned} \tag{3}$$

where f_i are functions to be determined. Differentiating equation (3), we have

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 - \dot{f}_1\\
\dot{e}_2 &= \dot{x}_2 - \dot{f}_2\\
\dot{e}_3 &= \dot{x}_3 - \dot{f}_3\\
\dot{e}_4 &= \dot{x}_4 - \dot{f}_4\\
\dot{e}_5 &= \dot{x}_5 - \dot{f}_5
\end{aligned} \tag{4}$$

substituting (3) and (4) into (1), we have

$$\begin{aligned}
\dot{e}_1 &= -\sigma(e_1 + f_1) + \sigma(e_2 + f_2) + (e_4 + f_4) + u_1 - \dot{f}_1\\
\dot{e}_2 &= r(e_1 + f_1) - (e_2 + f_2) - x_1 x_3 - (e_5 + f_5) + u_2 - \dot{f}_2\\
\dot{e}_3 &= -\beta(e_3 + f_3) + x_1 x_2 + u_3 - \dot{f}_3\\
\dot{e}_4 &= -x_1 x_3 + k_{1d}(e_4 + f_4) + u_4 - \dot{f}_4\\
\dot{e}_5 &= k_{2d}(e_2 + f_2) + u_5 - \dot{f}_5
\end{aligned} \tag{5}$$

eliminating terms which cannot be expressed as linear terms in e_1, e_2, e_3, e_4, e_5 and solving for $u(t)$,

$$\begin{aligned}
u_1 &= \sigma f_1 - \sigma f_2 - f_4 + \dot{f}_1 + v_1(t)\\
u_2 &= -r f_1 + f_2 + x_1 x_3 + f_5 + \dot{f}_2 + v_2 t\\
u_3 &= \beta f_3 - x_1 x_2 + \dot{f}_3 + v_3(t)\\
u_4 &= x_1 x_3 - k_{1d} f_4 + \dot{f}_4 + v_4(t)\\
u_5 &= -k_{2d} f_2 + \dot{f}_5 + v_5(t)
\end{aligned} \tag{6}$$

the parameter v_i will be obtained later. Substituting (6) into (5), the differential of the error becomes

$$\begin{aligned}
\dot{e}_1 &= -\sigma e_1 + \sigma e_2 + e_4 + v_1(t)\\
\dot{e}_2 &= re_1 - e_2 - e_5 + v_2(t)\\
\dot{e}_3 &= -\beta e_3 + v_3(t)\\
\dot{e}_4 &= k_{1d}e_4 + v_4(t)\\
\dot{e}_5 &= k_{2d}e_2 + v_5(t)
\end{aligned} \tag{7}$$

Using the active control method, a constant matrix **A** is chosen which will control the error dynamics (7) such that the feedback matrix is

$$\begin{pmatrix} v_1(t)\\ v_2(t)\\ v_3(t)\\ v_4(t)\\ v_5(t)\end{pmatrix} = A \begin{pmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5\end{pmatrix} \tag{8}$$

Thus, the matrix **A** is chosen to be of the form

$$A = \begin{pmatrix} (\lambda_1+\sigma) & -\sigma & 0 & -1 & 0\\ -r & (\lambda_2+1) & 0 & 0 & 1\\ 0 & 0 & (\lambda_3+\beta) & 0 & 0\\ 0 & 0 & 0 & (\lambda_4+k_{1d}) & 0\\ 0 & -k_{2d} & 0 & 0 & \lambda_5 \end{pmatrix} \tag{9}$$

The eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4, \lambda_5$ are chosen to be negative.

3.2 Numerical simulation

In the simulation, the fourth-order Runge-Kutta integration method is used to solve the differential equation with time step size equal to 0.0001 and the following initial conditions $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 0, 1, 0)$. The system parameters are chosen as $\sigma = 10,\ \beta = \frac{8}{3},\ r = 28,\ k_{1d} = 2,\ k_{2d}\epsilon(2, 12)$, so the system behaves hyperchaotically. We set $f_1 = b\sin(t)$, $f_2 = at^2$,$f_3 = k$, $f_4 = c + d\sin(t)$ and $f_5 = at^2$. Where $b = 5,\ a = 0.1,\ c = 2,\ d = 50$. The results are presented in figure (2). The effectiveness of the control can be seen as various components of the system converges to the preset functions when the controls are applied at time $t > 0$. Before the activation of the controls, the system behaves chaotically while the trajectory was changed to the present function on the activation of the control.

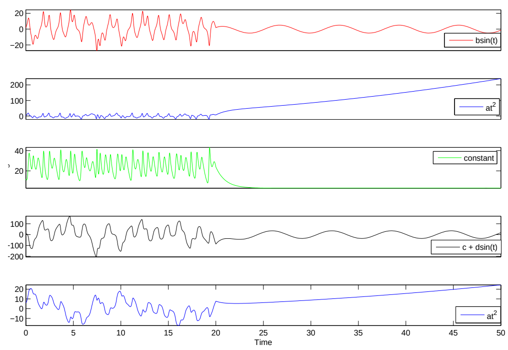

Figure 2. Components of 5D hyperchaotic Lorenz system when controlled to different with different functions when the controls are applied at time $t = 20$. The control functions are (a) tracking of $f_1 = b\sin(t)$ by x_1, (b) tracking of $f_2 = at^2$ by x_2,(c) tracking of $f_3 = k$ by x_3, (d) tracking of $f_4 = c + d\sin(t)$ by x_4 and (e) tracking of $f_5 = at^2$ by x_5.

4 Projective Synchronization of 5D System

We define the master system as equation (1) and the slave system as

$$\begin{aligned}
\dot{y}_1 &= -\sigma y_1 + \sigma y_2 + y_4 + u_1(t)\\
\dot{y}_2 &= ry_1 - y_2 + y_1y_3 - y_3 + u_2(t)\\
\dot{y}_3 &= -\beta y_3 + y_1y_2 + u_3(t)\\
\dot{y}_4 &= -y_1y_3 + k_{1d}y_4 + u_4(t)\\
\dot{y}_5 &= k_{2d}y_2 + u_5(t)
\end{aligned} \tag{10}$$

The goal of this synchronization is to determine the control function $(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))$. The error dynamics of the projective synchronization between (1) and (10) is defined as

$$e_i = y_i - \alpha x_i \tag{11}$$

where $i = 1, 2, \ldots, 5$ and α is the scaling factor.

4.1 Design of Control Functions

By subtracting (1) from (10) and using the notation in (11), we obtain

$$\begin{aligned}
\dot{e}_1 &= -\sigma e_1 + \sigma e_2 + e_4 + u_1(t)\\
\dot{e}_2 &= re_1 - e_2 - e_3 - y_1y_3 + \alpha x_1x_3 + u_2(t)\\
\dot{e}_3 &= -\beta e_3 + y_1y_2 - \alpha x_1x_2 + u_3(t)\\
\dot{e}_4 &= k_{1d}e_4 - y_1y_3 + \alpha x_1x_3 + u_4(t)\\
\dot{e}_5 &= k_{2d}e_2 + u_5(t)
\end{aligned} \tag{12}$$

To achieve asymptotic stability of system (12), we eliminate terms which cannot be expressed as linear terms in e_1, e_2, e_3, e_4, e_5 as follows:

$$\begin{aligned} u_1(t) &= v_1(t) \\ u_2(t) &= -\alpha x_1 x_3 + y_1 y_3 + v_2(t) \\ u_3(t) &= \alpha x_1 x_2 - y_1 y_2 + v_3(t) \\ u_4(t) &= -\alpha x_1 x_3 + y_1 y_3 + v_4(t) \\ u_5(t) &= v_5(t) \end{aligned} \tag{13}$$

substituting (13) into (12)

$$\begin{aligned} \dot{e}_1 &= -\sigma e_1 + \sigma e_2 + e_4 + v_1(t) \\ \dot{e}_2 &= r e_1 - e_2 - e_3 + v_2(t) \\ \dot{e}_3 &= -\beta e_3 + v_3(t) \\ \dot{e}_4 &= k_{1d} e_4 + v_4(t) \\ \dot{e}_5 &= k_{2d} e_2 + v_5(t) \end{aligned} \tag{14}$$

Using the active control method, a constant matrix **A** is chosen which will control the error dynamics (12) such that the feedback matrix is

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \\ v_5(t) \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} \tag{15}$$

with

$$A = \begin{pmatrix} (\lambda_1 + \sigma) & -\sigma & 0 & -1 & 0 \\ -\sigma & (\lambda_2 + 1) & 0 & 0 & 1 \\ 0 & 0 & (\lambda_3 + \beta) & 0 & 0 \\ 0 & 0 & 0 & (\lambda_4 - k_{1d}) & 0 \\ 0 & -k_{2d} & 0 & 0 & \lambda_5 \end{pmatrix} \tag{16}$$

In (16) the five eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are chosen to be negative in order to achieve a stable projective synchronization between two identical 5D hyperchaotic system.

4.2 Numerical simulation

Numerical solutions were carried out using fourth order Runge-Kutta integration scheme to solve systems (1) and (10) with the following initial conditions $(x_1, x_2, x_3, x_4, x_5) = (0, 1, 0, 1, 0)$ and $(y_1, y_2, y_3, y_4, y_5) = (2, 2, 2, 2, 2)$. The system parameters are chosen as $\sigma = 10$, $\beta = \frac{8}{3}$, $r = 28$, $k_{1d} = 2$, $k_{2d} \epsilon (2, 12)$, so the system behaves hyperchaotically.

The error dynamics of the system when the controls are activated at time $t > 20$ is shown in 3. The the synchronization errors between the two system is seen to converge to zero. Figures 4 - 8 shows the dynamics of the state variables (x and y) of the system when compared after activation of control at time $t > 0$ and value of $\alpha = 2.0$. The trajectory of the master is seen to be twice that of the slave as expected. A quantity called synchronization time which gives a value of when the error between the two synchronization approaches zero was also computed. Figure (9) depicts the time it takes for synchronization to occur as the coupling strength is increased. From the graph, an exponential decrease is seen. This synchronization time-coupling strength graph can be used as a measure of the speed of synchronization. In effect, for the system under consideration the synchronization time is seen to decrease with increasing coupling strength.

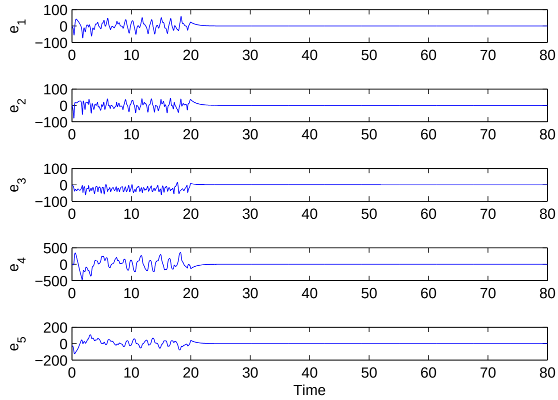

Figure 3. Error dynamics of the state variables when the control functions are activated for $t \geq 20$.

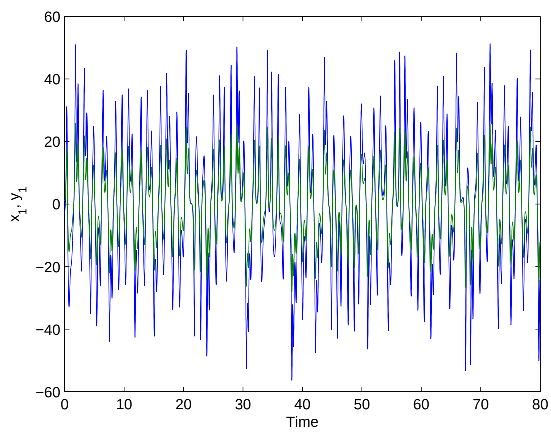

Figure 4. Dynamics of the state variable x_1 and y_1 when controls are activated at $\alpha = 2.0$ and $t \geq 0$.

5 Conclusion

We have designed controllers for the control of 5D-hyperchaotic systems using active control and synchronization of identical 5D hyperchaotic Lorenz systems. From the results obtained, the tracking control was efficient and give practical results as each component of the system were designed to track different functions. Also, from the error dynamics, synchronization using

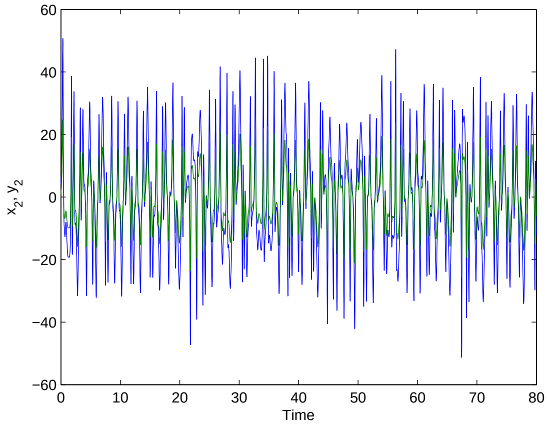

Figure 5. Dynamics of the state variable x_2 and y_2 when controls are activated at $\alpha = 2.0$ and $t \geq 0$.

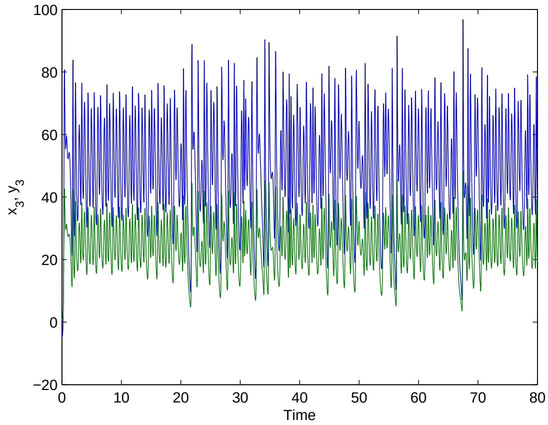

Figure 6. Dynamics of the state variable x_3 and y_3 when controls are activated at $\alpha = 2.0$ and $t \geq 0$.

active control yield good results. Furthermore, the effectiveness of the control obtained was tested using the synchronization time - coupling strength graph. As the coupling strength increases, the synchronization time reduces. The synchronization times obtained are considered good for practical purposes.

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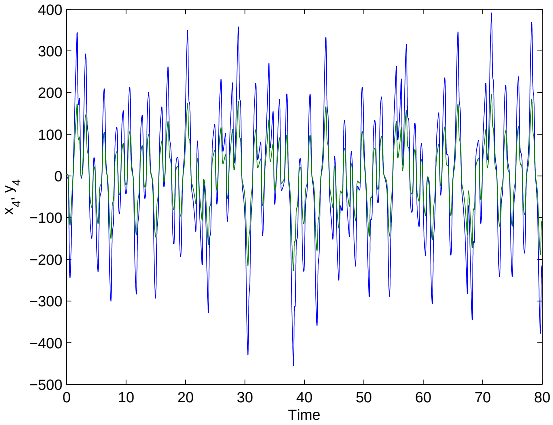

Figure 7. Dynamics of the state variable x_4 and y_4 when controls are activated at $\alpha = 2.0$ and $t \geq 0$.

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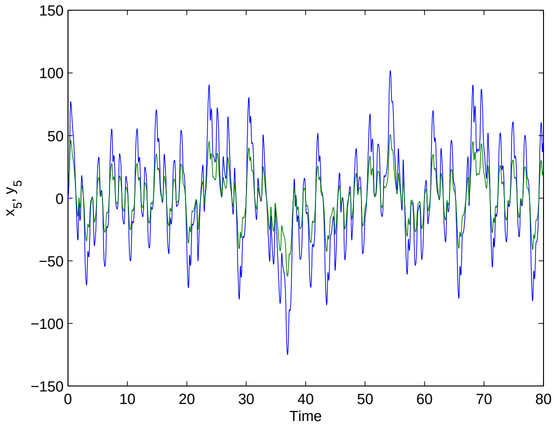

Figure 8. Dynamics of the state variable x_5 and y_5 when controls are activated at $\alpha = 2.0$ and $t \geq 0$.

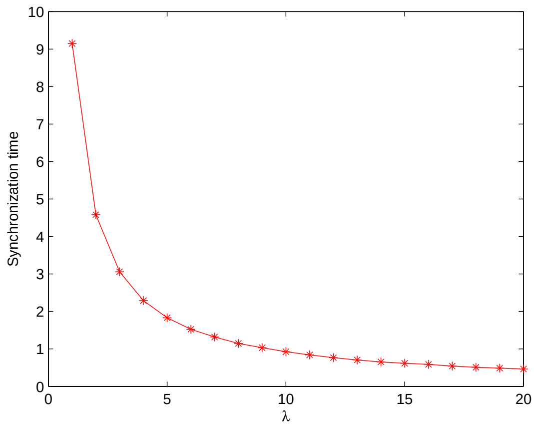

Figure 9. Dependence of the synchronization time on the coupling parameter λ.