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ADAPTIVE SYNCHRONIZATION OF NETWORKS WITH BOUNDED DISTURBANCES OR DELAYS UNDER INCOMPLETENESS OF MEASUREMENT AND CONTROL

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Abstract

An adaptive master-slave output feedback synchronization problem is investigated firstly for a network of interconnected nonlinear dynamical systems with delayed couplings and then for a network of systems with bounded disturbance. The proposed structure of decentralized controller and adaptation algorithms in both cases is based on the speed-gradient method and passivity. Conditions of synchronization for systems with delayed couplings are established. The problem of convergence with preliminarily specified accuracy is studied for the networks of dynamical systems with disturbances. The effectiveness of the obtained results is demonstrated on the network of Chua systems.

Key words

Networks, Adaptive Controller, Delays, Bounded Disturbances

1 Introduction

During the last years works on networks control occupy more and more essential place in the literature on control theory and practice. Motivating fields of applications are cooperative control of moving objects: robots, flying and underwater subjects, control of industrial and electro energy networks, etc. Although problems of decentralized control for complex dynamical networks of coupled objects were studied before [Šiljak, 1991], new problem statements dictate necessity of taking into account uncertainties, switching structure of the bond graph, partial decomposition of a control, influence of bounded disturbances, nonlinear dynamics of the local subsystem (agent) and uncertainties in measurement of their states.

For the synchronization of networks with delayed couplings some results have already been achieved in [Yao et al., 2006; Wang et al., 2010; Nuno et al., 2010; Liu et al., 2009; Mastellone et al., 2006; Chopra and Spong, 2006]. However, adaptive control laws were derived only for a narrow class of networks, such as fully-controlled and fully-measured networks (e.g. [Yao et al., 2006; Liu et al., 2009]), or the algorithm is not decentralized [Yao et al., 2006]. Some of these works deal with systems with non-switching topology or provide non-adaptive control.

In the current work we overcome these restrictions and propose an adaptive decentralized algorithm for synchronization of networks of dynamic systems with delayed couplings and bounded disturbances. This result is based primarily on the passivity approach and passification lemma.

In contrast to the disturbance-free case, convergence of trajectories of the leader and the agents with bounded disturbances is not possible. To avoid instability of the overall system adaptation algorithms are regularized by means of negative parametric feedback, similarly to [Fradkov et al., 2010].

In this paper unlike the previous ones the problem of convergence with preliminarily specified accuracy is studied. The conditions ensuring achievements of the control goal are given and proven. The results are illustrated by a network of chaotic Chua circuits.

2 Problem statement

Consider a dynamic system that consists of N interconnected subsystems described by:

$$\dot{x}_i = Ax_i + \varphi_0(x_i,t) + Bu_i + \sum_{j=1}^{N} \alpha_{ij}(t)\varphi_{ij}(x_j - x_i) + \\ + \sum_{j=1}^{N} \beta_{ij}(t)x_j(t-\tau) + f_i(t), \\ y_i = Cx_i, \quad i = 1,\dots,N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is a state vector, $u_i \in \mathbb{R}$ is a control input, $y_i \in \mathbb{R}^l$ is a measurable output. Here A, B, C are constant matrices of appropriate dimentions, $\tau > 0$ is the time delay. Nonlinear part is presented by continuously differentiable function $\varphi_0(x,t)\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, functions $\varphi_{ij}(\cdot)$ are used to describe communication among the subsystems, $\alpha_{ij}(t), \beta_{ij}(t)\colon [0,\infty) \to \mathbb{R}_{\geq 0}$ are piecewise continuous functions that describe a topology of the network, $f_i(t)$ is a bounded disturbance of the i-th node:

$$||f_i(t)|| \leqslant d_{f_i} \quad \forall t \in [0,\infty). \tag{2}$$

Suppose that for all $t \in [0,\infty), i = 1,\dots,N$ $\varphi_{ii}(0) = 0$, $\alpha_{ii}(t) = -\sum_{j=1}^N \alpha_{ij}(t)$, $\beta_{ii}(t) = -\sum_{j=1}^N \beta_{ij}(t)$.

Along with the system (1) we will consider an isolated *leader* system:

$$\begin{aligned} \dot{\overline{x}} &= A\overline{x} + B\overline{u} + \varphi_0(\overline{x},t), \\ \overline{y} &= C\overline{x}, \end{aligned} \tag{3}$$

where the control signal $\overline{u}$ is given.

3 Passification lemma

In order to formulate the passification lemma we need to introduce several definitions.

Definition 1. *A linear system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with the transfer matrix $W(\lambda) = C(\lambda I - A)^{-1}B$, where $u(t), y(t) \in \mathbb{R}^l$ and $\lambda \in \mathbb{C}$ is called minimum-phase if the polynomial $\varphi(\lambda) = \det(\lambda I - A)\det W(\lambda)$ is Hurwitz. The system is called hyper-minimum-phase if it is minimum-phase and the matrix $CB = lim_{\lambda\to\infty}\lambda W(\lambda)$ is symmetric and positive definite.*

We will need the passification lemma in the following form [Fradkov, 1976; **?**].

Lemma 1 (Passification lemma). *Let the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $g \in \mathbb{R}^{l\times m}$ be given and the full-rank condition $rank(B) = m$ holds. Then for existence of a positive-definite $n \times n$-matrix $P = P^T > 0$ and $l \times m$-matrix θ_* such that*

$$PA_* + A_*^T P < 0, PB = C^T g, A_* = A - B\theta_*^T C \tag{4}$$

it is necessary and sufficient, that the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= g^T Cx(t) \end{aligned} \tag{5}$$

is hyper-minimum-phase.

4 Networks with delayed couplings

We will begin with a case of disturbance-free linearly connected subsystems, that is, $f_i(t) \equiv 0$ for all $i = 1,\dots,N$ and $\varphi_{ij}(x_j - x_i) \equiv x_j - x_i$ for all $i,j = 1,\dots,N$. As soon as $\alpha_{ii}(t) = -\sum_{j=1}^N \alpha_{ij}(t)$ the equation (1) can be rewriten in the form:

$$\begin{aligned} \dot{x}_i &= Ax_i + \varphi_0(x_i,t) + Bu_i + \sum_{j=1}^N \alpha_{ij}(t)x_j + \\ &\sum_{j=1}^N \beta_{ij}(t)x_j(t-\tau), \\ y_i(t) &= Cx_i(t), \qquad i = 1,\dots,N. \end{aligned} \tag{6}$$

Here we treat a question of synchronization, i. e. the aim of control is to make the trajectories of all the subsystems converge to the trajectory of the leader system:

$$\lim_{t\to\infty}(x_i(t) - \bar{x}(t)) = 0, \quad i = 1,\dots,N. \tag{7}$$

And the problem is to find control functions $u_i = U_i(y_i,t) \quad i = 1,\dots,N$ to ensure achievement of the control goal (7). This problem will be solved under the following assumption:

Assumption 1. *Suppose there exists $g \in \mathbb{R}^l$ such that the transfer function $g^T C(sI_n - A)^{-1}B$ is hyper-minimum-phase.*

4.1 Control synthesis

Denote $e_i(t) = x_i(t) - \overline{x}(t)$. An error equation can be written as follows:

$$\begin{aligned} \dot{e}_i &= Ae_i + \varphi_0(e_i,t) - \varphi_0(\overline{x},t) + \sum_{j=1}^N \alpha_{ij}(t)e_j + \\ &+ \sum_{j=1}^N \beta_{ij}(t)e_j(t-\tau) + B(u_i - \overline{u}), \\ y_i(t) - \overline{y}(t) &= Ce_i(t), \quad i = 1,\dots,N. \end{aligned} \tag{8}$$

Under Assumption 1 the conditions of Lemma 1 hold and, therefore, $PB = C^T g$. Then applying the speed gradient algorithm [Fradkov, 1979] we obtain the following control law:

$$\begin{aligned} u_i(t) &= -\theta_i^T(t)(y_i(t) - \bar{y}(t)) + \bar{u}(t), \\ \dot{\theta}_i(t) &= g^T(y_i(t) - \bar{y}(t))\Gamma_i(y_i(t) - \bar{y}(t)), \end{aligned} \tag{9}$$

where Γ_i is $l \times l$ positive-definite matrix.

4.2 Lipschitz-type nonlinearities

Let us introduce the following additional values:

$$\mu = \sup_{t\in[0,\infty)} \max_{i\in\{1,\dots,N\}} \sum_{\substack{j=1\\ j\neq i}}^{N} (\alpha_{ji}(t) - \alpha_{ij}(t));$$

$$\nu = \sup_{t\in[0,\infty)} \max_{i\in\{1,\dots,N\}} \sum_{j=1}^{N} (|\beta_{ij}(t)| + |\beta_{ji}(t)|).$$

The value μ has the meaning of maximum asymmetricity of the matrix $\alpha(t)$. Thus if the matrix $\alpha(t)$ is symmetric at any time $t \geq 0$, then $\mu = 0$.

As soon as Assumption 1 holds, it follows from Lemma 1 that there exists $\varepsilon > 0$ such that

$$PA_* + A_*^T P < -\varepsilon P, PB = C^T g, A_* = A - B\theta_*^T C \tag{10}$$

This ε has a crucial meaning in the next theorem.

Theorem 1. *Suppose the Assumption 1 holds and $\varphi_0(x,t)$ is Lipschitz with respect to x with a Lipschitz constant η. Then, if*

$$2\eta\frac{\lambda_{max}(P)}{\lambda_{min}(P)} + \mu + \nu < \varepsilon$$

the control algorithm (9) ensures the achievement of the goal (7). Moreover, all tunable parameters $\theta_i(t)$ will stay bounded on the time interval $[0,\infty)$ for all $i = 1,\dots,N$.

Proof. It follows from Assumption 1 that for some $P > 0$, θ_* and $\varepsilon > 0$ the conditions (10) hold.

Consider the following function

$$\begin{aligned} V(e_i) = \sum_{i=1}^{N}\Big[& e_i^T(t)Pe_i(t) + \\ & + (\theta_i(t) - \theta_*)^T\Gamma_i^{-1}(\theta_i(t) - \theta_*) + \\ & + \int_{t-\tau}^{t} e_i^T(s)H_ie_i(s)ds\Big] \geq 0, \end{aligned} \tag{11}$$

where $H_i = \sum_{j=1}^{N}|\beta_{ji}(t)|P \geq 0$. By taking a derivative of V along the trajectories of the system (8), (9), we obtain:

$$\begin{aligned} \dot{V} = & \sum_{i=1}^{N} e_i^T(t)\left[A_*^TP + PA_*\right]e_i(t) + \\ & 2\sum_{i=1}^{N} e_i^T(t)P\left[\varphi_0(x_i,t) - \varphi_0(\bar{x},t)\right] + \\ & 2\sum_{i=1,j=1}^{N} \alpha_{ij}(t)e_i^T(t)Pe_j(t) + \\ & 2\sum_{i=1,j=1}^{N} \beta_{ij}(t)e_i^T(t)Pe_j(t-\tau) + \\ & \sum_{i=1}^{N}\left[e_i^T(t)H_ie_i(t) - e_i^T(t-\tau)H_ie_i(t-\tau)\right], \end{aligned} \tag{12}$$

By bounding sums with coefficients $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ by means of inequality $2x^Ty \leq x^TQx + y^TQ^{-1}y$ and using the Lipschitz condition for $\varphi_0(x,t)$, we may bound (12) as follows:

$$\begin{aligned} \dot{V} \leq & \sum_{i=1}^{N} e_i^T(t)\left[A_*^TP + PA_*\right]e_i(t) + \\ & 2\eta\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \\ & \mu\sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \\ & \sum_{i=1}^{N} e_i^T(t)\left[\sum_{j=1}^{N}|\beta_{ij}(t)|P + H_i\right]e_i(t) + \\ & \sum_{i=1}^{N} e_i^T(t-\tau)\left[\sum_{j=1}^{N}|\beta_{ji}(t)|P - H_i\right]e_i(t-\tau) \end{aligned} \tag{13}$$

Now substituting $H_i = \sum_{j=1}^{N}|\beta_{ji}(t)|P$ and using the first inequality from (10) we obtain:

$$\begin{aligned} \dot{V} \leq & \left(2\eta\frac{\lambda_{max}(P)}{\lambda_{min}(P)} + \mu + \nu - \varepsilon\right) \times \\ & \sum_{i=1}^{N} e_i^T(t)Pe_i(t) \leq 0, \end{aligned} \tag{14}$$

at the same time if $\exists i \in \{1,\dots,N\} : e_i \neq 0$, then $\dot{V} < 0$. Thus, it was shown that the function $V(e)$ is a Lyapunov function. That is $e = 0$ is asymptotically stable solution which means that $x_i(t) - \bar{x}(t) \to 0$ while $t \to \infty$ for $i = 1,\dots,N$.

It is obvious that if $\exists i \in \{1,\dots,N\} : \theta_i(t) \to \infty$ while $t \to \infty$, then $V \to \infty$, which is not possible because V is a bounded function. This proofs a uniform boundedness of $\theta_i(t)$ and ends the proof of the Theorem 1.□

4.3 Coordinated nonlinearity

Under coordinated nonlinearities we will mean nonlinearities of the form $\varphi_0(x,t) = Bh_0(Cx,t)$, where $h_0\colon \mathbb{R}\times[0,\infty)\to\mathbb{R}$ is some function.

Let us introduce the following definition:

Definition 2. *For given vector $G\in\mathbb{R}^l$ a function $f\colon\mathbb{R}^l\to\mathbb{R}$ is called G-monotonically decreasing, if for any $x,y\in\mathbb{R}^l$ the following inequality holds: $(x-y)^TG(f(x)-f(y))\le 0$.*

In case of $l=1$ and $G>0$ this condition repeats the classical definition of a monotonically decreasing function.

Theorem 2. *Suppose the Assumption 1 holds and $\varphi_0(x,t) = Bh_0(Cx,t)$, where $h_0(Cx,t)$ is a g-monotonically decreasing function for any $t\in[0,\infty)$. Then, if*

$$\mu+\nu<\varepsilon,$$

the control algorithm (9) insures the achievement of the goal (7). Moreover, all tunable parameters $\theta_i(t)$ will stay bounded on the time interval $[0,\infty)$ for all $i=1,\dots,N$.

Proof. Consider function (11). By taking a derivative along the trajectories of the system (8), (9) and using estimations from the proof of the Theorem 1, we derive

$$\begin{aligned}&\dot V\le(\mu+\nu-\varepsilon)\sum_{i=1}^N e_i^T(t)Pe_i(t)+\\&2\sum_{i=1}^N e_i^T(t)P\left[\varphi_0(x_i,t)-\varphi_0(\bar x,t)\right]=\\&(\sigma\mu+\sigma\nu-\varepsilon)\sum_{i=1}^N e_i^T(t)Pe_i(t)+\\&2\sum_{i=1}^N(y_i(t)-\bar y(t))^Tg\left[h_0(y_i((t),t)-h_0(\bar y(t),t)\right].\end{aligned}\tag{15}$$

By conditions of the Theorem 2 function h_0 is g-monotonically decreasing, therefore

$$\dot V\le(\sigma\mu+\sigma\nu-\varepsilon)\sum_{i=1}^N e_i^T(t)Pe_i(t)\le 0.\tag{16}$$

Similarly to the proof of the Theorem 1, we conclude $x_i(t)-\bar x(t)\to 0$ while $t\to\infty$ and $i=1,\dots,N$ and $\theta_i(t)$ are uniformly bounded. That ends the proof of the Theorem 2.□

5 Networks with disturbances

Now consider a case when the system has regular topology, doesn't have delays in couplings, but has some bounded disturbances, i.e. $\alpha_{ij}(t)\equiv\alpha_{ij}$, $\beta_{ij}(t)\equiv 0$. In this case we intend to achieve attraction of trajectories of all subsystems to some neighborhood of the trajectory of the leading subsystem. So, the control goal in this case is:

$$\overline{\lim_{t\to\infty}}\,|x_i(t)-\overline{x}(t)|\leqslant\Delta_i.\tag{17}$$

5.1 Control synthesis

Denote $e_i=x_i-\overline{x}$, $\tilde u_i=u_i-\overline{u}$, and then dynamics of e_i can be described as follows:

$$\begin{aligned}&\dot e_i=Ae_i+B\tilde u_i+\varphi_0(x_i)-\varphi_0(\overline{x})+f_i(t)+\\&\quad+\sum_{j=1}^N\alpha_{ij}\varphi_{ij}(x_j-x_i),\\&\tilde y_i=Ce_i,\quad i=1,\dots,d.\end{aligned}\tag{18}$$

As before, we take linear control of the slave subsystem in the following form:

$$\tilde u_i=-\theta_i^T(t)\tilde y_i,\ \theta_i(t)\in\mathbb{R}^l,\ i=1,\dots,d,\tag{19}$$

where $\theta_i(t)$ are adjustable parameters. By applying speed gradient algorithm to the goal function

$$Q(e_i)=\frac{1}{2}e_i^THe_i,\ H=H^T>0.\tag{20}$$

we derive an adaptation algorithm of the form:

$$\dot\theta_i(t)=g^T\tilde y_i(t)\Gamma_i\tilde y_i(t).$$

But this algorithm doesn't take into account the disturbance, and is designed to achieve full convergence to the leading system. Therefore, we suppose the following adaptation algorithm [Druzhinina and Fradkov, 1999]:

$$\dot\theta_i(t)=\begin{cases}g^T\tilde y_i(t)\Gamma_i\tilde y_i(t),\ Q_i(x_i(t),t)>\Delta_i\\ 0\ ,\ Q_i(x_i(t),t)\leqslant\Delta_i.\end{cases}\tag{21}$$

5.2 Lipschitz-type nonlinearities

Further narration requires several assumptions.

Assumption 2. *Suppose that for some $g\in\mathbb{R}^l$ function $g^TW(s-L)$ is hyper-minimum-phase, where $W(s)=C(sI_n-A)^{-1}B$.*

Under Assumption 2 from Lemma 1 it follows that there exist $H=H^*>0$ and θ_*, such that $HA_*+A_*^TH<-\rho H$, $HB=C^Tg$, where $A_*=(A+LI_n)+B\theta_*^TC$.

Assumption 3. *Suppose that $\varphi_0(\cdot)$ and $\varphi_{ij}(\cdot)$ are globally Lipschitz functions with respect to x:*

$$||\varphi_0(x) - \varphi_0(x')|| \leqslant L||x - x'||,\ L > 0,$$

$$||\varphi_{ij}(x) - \varphi_{ij}(x')|| \leqslant L_{ij}||x - x'||,\ L_{ij} > 0.$$

Denote $\lambda_* = \lambda_{max}(H)/\lambda_{min}(H)$ condition number of matrix H.

Theorem 3. *Let assumptions 2 and 3 hold. Denote $\delta_i = \frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} - \sum_{j=1}^{N}|\alpha_{ij}L_{ij}|$. If for any $i = 1, \dots, N$ the following condition is fulfilled:*

$$\delta_i > 0, \tag{22}$$

then adaptive controler (19), (21) *provides achievement of the control goal* (17) *for all Δ_i, that satisfy:*

$$\Delta_i > \frac{d_{f_i}^2 \lambda_{max}(H)}{2\rho\delta_i}, \quad i = 1, \dots, N \tag{23}$$

meanwhile the vector of adjustable parameters θ is bounded for all solutions of the closed-loop system (1), (3), (19), (21).

Proof. To prove this theorem we need an auxiliary lemma. This lemma is a modification of the theorem 2.19 from [Druzhinina and Fradkov, 1999].

Lemma 2. *Consider system that consists of N interconnected subsystems, where each one is described as:*

$$\dot{x}_i = F_i(x_i, \theta_i, t) + h_i(x, \theta, t),\ i = 1, \dots, K, \tag{24}$$

$$\dot{\theta}_i(t) = \begin{cases} -\Gamma_i \nabla_{\theta_i}\omega_i(x_i, \theta_i, t),\ Q_i(x_i(t), t) > \Delta_i \\ 0\ ,\ Q_i(x_i(t), t) \leqslant \Delta_i. \end{cases} \tag{25}$$

where $x_i \in \mathbb{R}^{n_i}, \theta_i \in \mathbb{R}^{m_i}$,

$$\omega_i(x_i, \theta_i, t) = \frac{\partial Q_i}{\partial t} + \nabla Q_i(x_i, t)^T F_i(x_i, \theta_i, t),$$

here $Q_i(\cdot)$ - is some objective function, $N = \sum n_i$, $m = \sum m_i$, $x = col(x_1, \dots, x_l) \in \mathbb{R}^N$. Assume that for (24) *the following groups of conditions hold:*

1. *Functions $F_i(\cdot)$ are continuous with respect to x_i and t_i, are continuously differentiable with respect to θ_i and locally bounded in time $t > 0$; functions $\omega_i(x_i, \theta_i, t)$ are convex by θ_i; there exist vectors $\theta_i^* \in \mathbb{R}^{m_i}$ and scalar continuous growing functions $k_i(Q)$, $\rho_i(Q)$ such that $k_i(0) = \rho_i(0) = 0$, $k_i(Q) \to +\infty$ and $\rho_i(Q) \to \infty$ when $Q \to +\infty$.*

$$\omega_i(x_i, \theta_i^*, t) \leqslant -\rho_i(Q_i(x_i, t)) \tag{26}$$

and

$$Q_i(x_i, t) \geqslant k_i(||x_i - x_i^*(t)||),$$

where $x_i^* = argmin_{x_i}(Q_i(x_i, t))$ and $Q_i(x_i^*(t), t) \equiv 0$

2. *Functions $h_i(x, \theta, t)$ are continuous and the following inequalities hold*

$$|\nabla_{x_i}Q_i(x_i, t)^T h_i(x, \theta, t)| \leqslant \sum_{j=1}^{l}\mu_{ij}\rho_j(Q_j(x_j, t)) + d_i \tag{27}$$

where $M - I$ is Hurwitz matrix, $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, I is identity matrix, $d_i > 0$, and Δ_i in (25) *satisfy the inequalities:*

$$\rho_i(\Delta_i) > r_i, \tag{28}$$

where $r = (I - M)^{-1}d, r = col(r_1, \dots, r_l), d = col(d_1, \dots, d_l)$.

Then all trajectories of the system (24), (25) *are bounded and the control goal*

$$\overline{\lim_{t\to\infty}}\, Q_i(x_i(t), t) \leqslant \Delta_i,\ i = 1, \dots, l$$

is met.

Consider the first group of conditions of Lemma 2. Local boundedness for $t > 0$ is met, because for any $i = 1, \dots, N$ right-hand side of the system (18) and $Q(e_i)$ are continuous functions, not depending from t, and $f(t)$ is bounded. Convexity condition is satisfied because right-hand side of $\dot{Q}_i$ is linear by Q_i. Let's take function $Q \to \rho \cdot Q$ as $\rho_i(\cdot)$, $i = 1, \dots, N$, from Lemma 2. It can be shown that existence of $\theta_* \in \mathbb{R}^l$ and ρ, such that $\omega_i(e_i, \theta_*) \leqslant -\rho Q(e_i)$, is provided by hyper-minimum-phase restriction for function $g^T W(s)$. Indeed, according to the Lemma 1 if $g^T W(s)$ is hyper-minimum-phase then exist $H = H^T > 0$ and θ_* such that.

$$HA_* + A_*^T H < 0,\ HB = C^T g,$$

where $A_* = (A + LI_n) + B\theta_*^T C$.

$$F_i = Ae_i + B\tilde{u}_i + \varphi_0(x_i) - \varphi_0(\overline{x})$$

Taking derivative of Q_i due to error equation for the ith node (18), it can be shown that:

$$\dot{Q}_i = \omega_i(e_i, \theta_*) \leqslant e_i^T H(A + B\theta_i^T C)e_i + \\ + ||e_i^T|| \cdot ||H|| \cdot L \cdot ||e_i|| \leqslant e_i^T H(A + B\theta_i^T C)e_i + \\ + L\frac{\lambda_{max}(H)}{\lambda_{min}(H)} e_i^T H e_i = \frac{1}{2} e_i^T [HA_* + A_*^T H]e_i, \tag{29}$$

where $A_* = (A + LI_n) + B\theta_*^T C$.

Negativeness of $HA_* + A_*^T H$ implies existence of $\rho > 0$, such that $HA_* + A_*^T H \leqslant -\rho H$, and therefore the condition:

$$\omega_i(e_i, \theta_*) \leqslant -\rho Q(e_i),\ i = 1, \ldots, d$$

is met.

Consider the conditions on connections between the systems (second group of conditions in Lemma 2). In our particular case they can be written down as:

$$|e_i^T H[\sum_{j=1}^{N} \alpha_{ij}\varphi_{ij}(e_i - e_j) + f_i(t)]| \leqslant \\ \leqslant \frac{\rho}{2} \sum_{j=1}^{N} \mu_{ij} e_j^T H e_j + d_i,\ i = 1, \ldots, N \tag{30}$$

where $M - I$ is Hurwitz matrix, $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, I is identity matrix.

Let us estimate the left-hand side of (30):

$$\left| e_i^T H[\sum_{j=1}^{N} \alpha_{ij}\varphi_{ij}(e_i - e_j) + f_i(t)] \right| \leqslant \\ \leqslant \sum_{j=1}^{N} |\alpha_{ij} L_{ij}| \cdot \lambda_{max}(H) \cdot (||e_i||^2 + ||e_i|| \cdot ||e_j||) + \\ + |e_i^T H f_i(t)| \leqslant \\ \leqslant \sum_{j=1}^{N} |\alpha_{ij} L_{ij}| \cdot \lambda_{max}(H) \cdot (||e_i||^2 + ||e_i|| \cdot ||e_j||) + \\ + \frac{1}{2}\sigma_i ||e_i||^2 \lambda_{max}(H) + \frac{d_{f_i}^2}{2\sigma_i}\lambda_{max}(H),\ i = 1, \ldots, N,$$

where $\sigma_i > 0$, $i = 1, \ldots, d$, are some numbers.

It can be shown that the lower bound of the right-hand side of (30) is:

$$\frac{\rho}{2} \sum_{j=1}^{N} \mu_{ij} e_j^T H e_j + d_i \geqslant \\ \geqslant \frac{\rho}{2} \sum_{j=1}^{N} \mu_{ij} \lambda_{min}(H) ||e_j||^2 + d_i,\ i = 1, \ldots, d.$$

Thereby, it is sufficient to demand fulfilment of the following inequalities:

$$\begin{cases} \frac{d_{f_i}^2}{2\sigma_i}\lambda_{max}(H) \leqslant d_i \\ \sum_{j=1}^{N} |\alpha_{ij} L_{ij}| \cdot (||e_i||^2 + ||e_i|| \cdot ||e_j||) + \\ + \frac{1}{2}\sigma_i ||e_i||^2 \leqslant \frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} \sum_{j=1}^{N} \mu_{ij} ||e_j||^2, \end{cases} \tag{31}$$

where $i = 1, \ldots, N$.

Consider the following notations: $\mathbf{z} = col(||z_1||, ||z_2||, \ldots, ||z_3||)$, $\nu_i^{(1)}$, $\nu_i^{(2)}$, η_i are described as follows:

$$\nu_i^{(1)} = \begin{pmatrix} 0 \ldots & 0 & \ldots 0 \\ \vdots \ \ddots & \vdots & \vdots \\ 0 \ldots & \sum_{j=1}^{N} |\alpha_{ij} L_{ij}| + \frac{1}{2}\sigma_i & \ldots 0 \\ \vdots & \vdots & \ddots \ \vdots \\ 0 \ldots & 0 & \ldots 0 \end{pmatrix}, \tag{32}$$

here the non-null element is on the main diagonal in the i-th row. Assuming that $\alpha_{ii} = 0$ for all $i = 1, \ldots, N$, we denote $\nu_i^{(2)}$ as:

$$\nu_i^{(2)} = \begin{pmatrix} 0 & \ldots 0 \ldots & 0 \\ \vdots & \ddots \ \vdots & \vdots \\ |\alpha_{i1} L_{i1}| & \ldots 0 \ldots & |\alpha_{iN} L_{iN}| \\ \vdots & \vdots \ \ddots & \vdots \\ 0 & \ldots 0 \ldots & 0 \end{pmatrix}.$$

Where non-null elements are in the i-th row, and the null element is on the main diagonal. As η_i let us take:

$$\eta_i = \frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} \begin{pmatrix} \mu_{i1} & 0 & \ldots & 0 \\ 0 & \mu_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_{iN} \end{pmatrix}.$$

Using this notations we can state that for fulfilment of (30) it is sufficient that for any $i = 1, \ldots, N$ the following inequality is true:

$$\mathbf{z}^T(\nu_i^{(1)} + \nu_i^{(2)}) \leqslant \mathbf{z}^T \eta_i \mathbf{z}, \tag{33}$$

i.e. matrix $\eta_i - \nu_i^{(1)} - \nu_i^{(2)}$ for any $i = 1, \ldots, N$ should be positively defined.

Let us take the following diagonal matrix as $M = \{\mu_{ij}\}$:

$$0 < \mu_{ii} < 1, \mu_{ij} = 0, i \neq j, i = 1, \ldots, N, j = 1, \ldots, N.$$

Then $M - I$ will be Hurwitz matrix.

Non-negative determinacy of $\eta_i - \nu_i^{(1)} - \nu_i^{(2)}$ implies that:

$$\mu_{ii}\frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} - \sum_{j=1}^{N}|\alpha_{ij}L_{ij}| - \frac{1}{2}\sigma_i \geqslant 0,\ i = 1, \ldots, N,$$

or

$$\mu_{ii} \geqslant \left(\frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)}\right)^{-1}\left(\sum_{j=1}^{N}|\alpha_{ij}L_{ij}| + \frac{1}{2}\sigma_i\right),\ i = 1, \ldots, N. \quad (34)$$

returning to the fact that $\mu_{ii} < 1$, we get to the condition (22).

Now, consider (28), together with (34). Assume $\delta_i = \frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} - \sum_{j=1}^{N}|\alpha_{ij}L_{ij}|$. Then (31), (34) can be rewritten as:

$$\begin{cases} \frac{d_{f_i}^2\lambda_{max}(H)}{2\delta_i} \leqslant d_i < \rho\Delta_i, \\ \delta_i > 0, \end{cases} \quad (35)$$

which matches the conditions of the Theorem. Thereby, we can apply Lemma 2, which proves the Theorem.□

Condition (23) of the Theorem 3 says that the higher the disturbance level is, the bigger is the attraction area, which is essential.

Remark 1. *Let ρ_* denote degree of stability of nominator of function $g^TW(s - L)$. Using results of [Fradkov, 2003] it can be shown that if function $g^TW(s-L)$ is hyper-minimum-phase, then as θ_* and ρ we can take $\theta_* = \kappa g$ and any $\rho : 0 < \rho < \rho_*$, where $\kappa > 0$ is sufficiently big number. Thereby, inequality (22) can be replaced with*

$$\sum_{j=1}^{d}|\alpha_{ij}L_{ij}| < \gamma, \quad (36)$$

where $\gamma = \frac{\rho_*}{2\lambda_*}$.

5.3 Coordinated nonlinearity

Let's study a case when $\varphi_0(x_i) = B\psi_0(y_i)$, $\psi_0 : \mathbb{R}^l \to \mathbb{R}$. Then the subsystem can be rewritten as

$$\begin{aligned} \dot{x_i} &= Ax_i + B(u_i + \psi_0(y_i)) + f_i(t) + \\ &+ \sum_{j=1}^{d}\alpha_{ij}\varphi_{ij}(x_i - x_j), \\ y_i &= Cx_i, \end{aligned} \quad (37)$$

and the leading system can be rewritten in the following manner:

$$\dot{\overline{x}} = A\overline{x} + B(\overline{u} + \psi_0(\overline{y})),\ \overline{y} = C\overline{x}, \quad (38)$$

where $\overline{u} \in \mathbb{R}$ - is the given control which is assumed to be known.

Assumption 4. *Suppose thatφ_{ij} are globally Lipschitz functions, with constants $L_{ij} > 0$, and $\psi_0(\cdot)$ is such, that existence and uniqueness of solutions of all the subsystems.*

We choose (19), (21) again as the control input.

Consider real matrices $H = H^T > 0$, g, θ_* of sizes $n \times n$, $l \times 1$, $l \times 1$ respectively, and number $\rho > 0$ such that:

$$HA_* + A_*^TH < -\rho H,\ HB = C^Tg,\ A_* = A + B\theta_*^TC. \quad (39)$$

Theorem 4. *Suppose that Assumptions 1 and 4 hold. Then there exist $H = H^T > 0$, θ_* of orders $n \times n$, $l \times 1$ respectively and $\rho > 0$, such that* (39) *holds. Also assume that function $\psi_0(\cdot)$ is g-monotonically decreasing. Denote $\delta_i = \frac{\rho}{2}\frac{\lambda_{min}(H)}{\lambda_{max}(H)} - \sum_{j=1}^{N}|\alpha_{ij}L_{ij}|$. If for all $i = 1, \ldots, N$ the following condition holds:*

$$\sum_{j=1}^{d}|\alpha_{ij}L_{ij}| < \gamma, \quad (40)$$

where $\gamma = \rho_*/(2\lambda_*)$, λ_* - condition number of matrix H, ρ_* - degree if stability of nominator of $g^TW(s)$. Then for all $i = 1, \ldots, N$ adaptive control (19), (21) *provides fulfilment of the control goal:*

$$\overline{\lim_{t\to\infty}}\,|x_i(t) - \overline{x}(t)| \leqslant \Delta_i, \quad (41)$$

for all Δ_i, that satisfy the following inequality:

$$\Delta_i > \frac{d_{f_i}^2\lambda_{max}(H)}{2\rho\delta_i},$$

meanwhile the vector of adjustable parameters θ_i remains bounded on $[0, \infty)$ for all solutions of the closed loop system (19), (21), (37), (38).

Proof. The proof of this theorem is similar to the proof of Theorem 3. To prove it one should apply lemma 2 with $L = 0$.

Let us prove that $\omega(e_i, \theta_*) \leqslant -\rho Q(e_i)$ for $i = 1, \ldots, N$.

Using previously made assumptions it can be shown that:

$$\begin{aligned} \omega_i(e_i, \theta_*) &= e_i^TH[Ae_i + B(\tilde{u}_i + \psi_0(y_i) - \psi_0(\overline{y}))] = \\ &= e_i^TH(A + B\theta_i^TC)e_i + (y_i - \overline{y})^Tg(\psi_0(y_i) - \psi_0(\overline{y})) \leqslant \\ &\leqslant e_i^TH(A + B\theta_i^TC)e_i, \end{aligned} \quad (42)$$

The latter inequality is fulfilled because $\psi(\cdot)$ is g-monotonically decreasing.

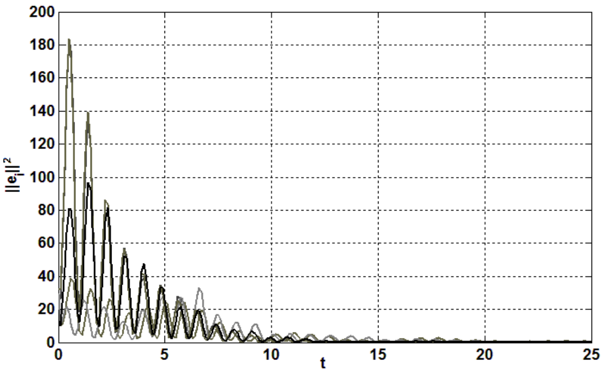

Figure 1. Evolution of $e_i(t)$

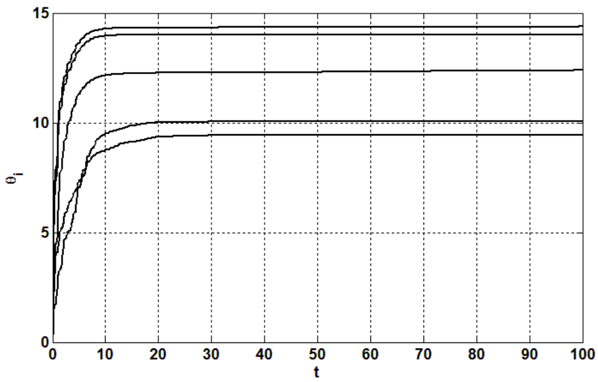

Figure 2. Evolution of $\theta_i(t)$

Then,

$$\omega_i(e_i, \theta_*) \leqslant e_i^T H[A + B\theta_*^T C]e_i = \\ = \frac{1}{2} e_i^T [HA_* + A_*^T H]e_i, i = 1, \ldots, N.$$

Here $A_* = A + B\theta_*^T C$. Since $HA_* + A_*^T H$ is negatively determined, exists such $\rho > 0$, that $HA_* + A_*^T H \leqslant -\rho H$, and that provides fulfilment of the following inequality

$$\omega_i(e_i, \theta_*) \leqslant -\rho Q(e_i), \ i = 1, \ldots, N.$$

Repeating further the proof of the Theorem 3 and taking into account Remark 1, we will prove this theorem. □.

6 Examples

In this section we give an example to demonstrate the effectiveness of the proposed algorithm. We will consider a network consisting of Chua circuits [Chai and Chua, 1995]. The state equation of Chua's circuit is given by

$$\begin{cases} \dot{x}_1 = \alpha_c(x_2 - x_1 - h_0(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\beta_c x_2, \end{cases} \tag{43}$$

where $h_0(x_1) = b_c x_1 + \frac{1}{2}(a_c - b_c)(|x_1 + 1| - |x_1 - 1|)$ and $\alpha_c > 0$, $\beta_c > 0$, $a_c < b_c < 0$ are system parameters. It is easy to check that for any $g > 0$ the function h_0 is g-monotonically decreasing. For simplicity let us take $g = 1$. Suppose that we can control and observe the first component of the state vector of each subsystem (that will ensure hyper-minimum-phaseness), i.e.

$$A = \begin{pmatrix} -\alpha_c & \alpha_c & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_c & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \tag{44}$$

6.1 Network with delays

Suppose the system has five subsystems and a switching topology with the following matrices

$$\alpha(t) = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0{,}5 & -1 & 0{,}5 & 0 & 0 \\ 0{,}5 & 0{,}5 & -1 & 0 & 0 \\ 0 & 0 & 0{,}5 & -1 & 0{,}5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \times \sigma \operatorname{sign}(\sin t),$$
$$\beta(t) = \begin{pmatrix} -1 & 0{,}3 & 0{,}2 & 0 & 0{,}5 \\ 0 & -0{,}3 & 0 & 0{,}2 & 0{,}1 \\ 0{,}4 & 0 & -0{,}7 & 0{,}1 & 0{,}2 \\ 0{,}6 & 0 & 0 & -0{,}6 & 0 \\ 0 & 0{,}3 & 0 & 0{,}4 & -0{,}7 \end{pmatrix} \times \sigma \operatorname{sign}(\cos t), \tag{45}$$

where $\sigma = 0.1$. Note that results are delay-independent, therefore we can take any constant τ. We ran simulations for different values of time delay. Here we present plots for $\tau = 10$. Results for other values of τ do not differ a lot. Let us take the following values of system parameters $\alpha_c = 10$, $\beta_c = 14.87$, $a_c = -1.27$ and $b_c = -0.68$.

It was numerically found that we can take $\varepsilon = 0{,}9167$. For this value of ε all conditions of Theorem 1 are satisfied.

In Fig. 1 the errors evolution is presented. It is easy to see that all e_i tend to zero while $t \to 0$. In Fig. 2 one can see that all θ_i are bounded, moreover θ_i tend to some constant values while $t \to 0$.

6.2 Network with disturbances

Now consider a network consisting of six nodes with disturbances, that is the dynamic of each node is described as follows

$$\dot{x}_i = Ax_i + B(u_i + h_0(y_i)) + f_i + \sum_{j=1}^{5} \alpha_{ij}\varphi_{ij}(x_j - x_i),$$
$$y_i = Cx_i, \quad i = 1, \ldots, 6,$$

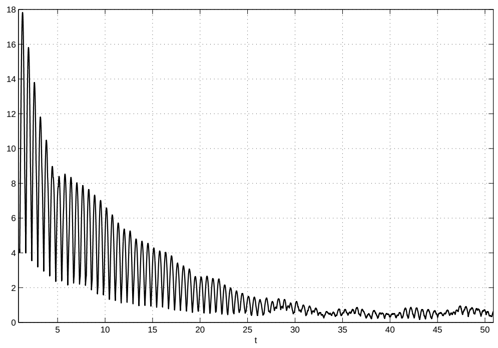

Figure 3. Evolution of $\max_i ||e_i(t)||$

with the same matrices A, B, C and the same function h_0.

We take $a_c = -8/7$, $b_c = -5/7$, $\alpha_c = 15.6$, $\beta_c = 25.58$, $d_{f_i} = 2$, $i = 1, \ldots, 6$.

As the control input we take stepping action with amplitude $1/2$, and period $T = 5$, disturbance f_i we simulate with uniformly-distributed random variable at $[-d_{f_i}, d_{f_i}]$.

Let us denote

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{16} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{26} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{61} & \alpha_{62} & \ldots & \alpha_{66} \end{pmatrix}$$

$$\hat{\alpha} = \begin{pmatrix} 0 & 0.0051 & 0.1395 & 0 & 0.1676 & 0 \\ 0.0662 & 0 & 0.0921 & 0.0065 & 0 & 0.0926 \\ 0.2013 & 0 & 0 & 0.2271 & 0.1430 & 0 \\ 0.0907 & 0 & 0.0675 & 0 & 0 & 0 \\ 0.0663 & 0 & 0 & 0.02773 & 0 & 0.1472 \\ 0.0662 & 0 & 0 & 0.0065 & 0 & 0 \end{pmatrix}$$

50-second simulation with $\alpha = \hat{\alpha}$ shows that the goal is achieved: $||e_i|| < \Delta_i$, for some Δ_i, but does not tend to zero. Evaluation of $\max_i ||e_i||$ is shown in the Fig. 3

7 Conclusion

In this paper unlike the previous ones we have achieved solution of the problem of convergence with pre-specified accuracy and obtained synchronization conditions for delayed coupling networks with switching topology consisting of nonlinear systems with incomplete measurement, incomplete control, incomplete information about system parameters. The design of the control algorithm providing synchronization property is based on the speed-gradient method, while derivation of synchronizability conditions is based on the passification lemma.

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