Phase-locking phenomena and excitation of damped and driven nonlinear oscillator

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PACS numbers:

I. INTRODUCTION

The phase-locking phenomena for a nonlinear pendulum with a small periodic driving was first studied in Refs.[1, 2] as a model for acceleration of relativistic particles. The main idea of the approach is to drive the oscillator by a periodic driving with a slowly chirping frequency in the vicinity of resonance. If the amplitude of the drive exceeds some critical value, the phase of oscillations can be locked in by the driver which allow to effectivelly excite and control the pendulum. Now the resonant phase-locking phenomena (in another terms – "autoresonance") are widely applied to various physical problems which are associated with nonlinear oscillators. A lot of applications, icluding plasmas and planetary dynamics are referenced in the paper [3]. Now the approach is extended also on infinite dimensional systems, such as vortex dynamics [4] and nonlinear waves [5].

Up to now the most studying of the autoresonance dealt with the problems without dissipation. The analysis was based usually on versions of the hamiltonian variational approach Ref.[6]. Nevertheless, it was supposed [7], that a small linear dissipation can preserve the main features of the autoresonance. In this paper we focus on studying of the phase-locking phenomena for damped and driven systems. For example, some applications on plasma physics [8, 10] induce studying the effect for the forced van der Pol oscillator. The close problem without driving, but with chirping of the main frequency have been investigated in Ref.[9]. One notes that the phase-locking by external forcing can be used as an effective tool not only for excitation, but aslo for control of high amplitude oscillations. This approach may be alternative to intensivelly studying now methods of control of dissipative oscillators (see, e.g. [11]).

We will study in this paper the van der Pol-Duffing oscillator

$$\ddot{u} + u + \mu u^3 - \gamma(1 - \sigma u^2)\dot{u} = \epsilon \cos\Psi \qquad (1)$$

driven by the small periodic forcing with amplitude $\epsilon \ll 1$ and slowly varying frequency

$$\dot{\Psi} = 1 + \Lambda(t). \qquad (2)$$

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We will assume the linear chirp of the frequency $\Lambda(t) = \alpha t$. Then, the slow variation meens $d\Lambda/dt = \alpha \ll 1$. We suppose also that the parameters μ, γ and $\gamma\sigma$ in Eq.(1) will be small too.

II. AVERAGING EQUATIONS

Let us introduce new variabls

$$u(t) = A(t)\cos(t + \phi(t)), \quad \dot{u}(t) = -A(t)\sin(t + \phi(t)), \qquad (3)$$

where the amplitude A and the phase ϕ are supposed to be slow functions of time in accordance with assumption of smallness of parameters in the main equations (1), (2). The standard averaging method [12] gives equations for the new variables:

$$\dot{A} = gA(4 - \sigma A^2) - e\sin\Phi, \qquad (4)$$

$$\dot{\Phi} = -\Lambda(t) + mA^2 - \frac{e}{A}\cos\Phi. \qquad (5)$$

We introduced here the new parameters

$$g = \gamma/8, \; m = 3\mu/8, \; e = \epsilon/2 \; . \qquad (6)$$

The phase $\Phi(t)$ in Eqs.(4),(5) is the difference between phases of the solution (3) and the forcing:

$$\Phi = \phi(t) - \int^t \Lambda(t)dt \qquad (7)$$

Let the frequency shift Λ is a constant. Then, Eqs.(4),(5) have stationary points $(A_0(\Lambda), \Phi_0(\Lambda))$ defined by the equations

$$gA_0(4 - \sigma A_0^2) - e\sin\Phi_0 = 0, \qquad (8)$$

$$-\Lambda + mA_0^2 - \frac{e}{A_0}\cos\Phi_0 = 0. \qquad (9)$$

These points give steady cycles of the original equation (1). Not all cycles are stable. We will study the stability in the framework of the equations (4),(5). Introducing small perturbations of a stationary point $A = A_0 + \delta A$,

$\Phi=\Phi_0+\delta\Phi$ and supposing the dependence $\delta A, \delta\Phi \sim e^{\lambda t}$, we obtain from (4),(5) the quadratic dispersion equation

$$\lambda^2 - \left[\frac{e}{A_0}\sin\Phi_0 + g(4-3\sigma A_0^2)\right]\lambda + g(4-3\sigma A_0^2) \times\frac{e}{A_0}\sin\Phi_0 + e\cos\Phi_0\left[2mA_0 + \frac{e}{A_0^2}\cos\Phi_0\right] = 0. \quad (10)$$

The steady solution (A_0, Φ_0) will be stable if $\mathrm{Re}\lambda \le 0$.

The typical curves $A_0(\Lambda)$ are shown in Fig.1 for three range of parameters which will be studied in the paper:
(a) dissipationless case, $g=0$,
(b) simplest linear dissipation, $g<0$ and $\sigma=0$,
(c) van der Pol-Duffing case, $g>0$ and $\sigma>0$.
We will suppose also that in all cases $\alpha \ge 0$, $m \ge 0$ and $e \ge 0$. Solid parts of the curves in Fig.1 represent stable solutions ($\mathrm{Re}\lambda \le 0$) and dashed lines correspond to unstable solutions ($\mathrm{Re}\lambda > 0$).

In the dissipationless case (a), the main goal of excitation is starting from zero ($A \approx 0$) to attain high amplitude oscillations in the vicinity of line BC in Fig.1*a*. It is obvious that such result impossible to obtain by excitation with constant frequency $\Lambda \ge \Lambda_*$ because in this case one can attain only a low amplitude near the line DE. Appropriate controlling path has to use slow varying frequency (2) [3]. It starts from a large negative initial time t_0, crosses of the resonance $\dot\Psi = 1$ at $t=0$ and, then, forces a solution to move in the vicinity of stable curve BC to excite a high amplitude oscillations. It is important, that such path can be realized only when α is less than some critical value $\alpha_{cr} \sim \epsilon^{4/3}$ [3]. The physical mechanizm of the phenomenon is the phase-locking of exciting oscillations by forcing. In the next section we study new aspects of this phenomena, which allow us to extend the phase-locking approach to dissipative systems of types (b) and (c).

The amplitude curves for dissipative cases are shown in Fig.1*b*,*c*. The main difference with the previous case (when $\mathrm{Re}\lambda = 0$) is that now $\mathrm{Re}\lambda < 0$ and stable parts of the curves associate with atractive focuses of the system (4), (5). For any fixed Λ, a zero initial solution tends to corresponding point $A_0(\Lambda)$ in the curve. But again, the high amplitude solutions in the curve BC are not attainable. One notes that in van der Pol-Duffing case (c) the small amplitude oscillations are unstable themselves and the range of stable limit cycles are restricted in a finite interval of Λ where $A_0^2 \ge 2/\sigma$. It will be shown in following that the phase-locking in dissipative cases are also possible for some restrictions on parameter α and controlling paths.

It is important that in dissipative cases the phase-locking approach have meaning only if the curves have a singular structure like the line BCD. It takes place for a sufficiently small dissipation. In the case (b), the correct limitation is

$$|g| < \frac{1}{4}\left(\frac{me^2}{2}\right)^{1/3}. \quad (11)$$

In case (c), the limitation is more sophisticated, but the simplified sufficient condition reads

$$g \ll \frac{m}{\sigma}. \quad (12)$$

For larger g the curves are not singular and any Λ corresponds to unique value of A_0. In this case no specific control paths is needed to excite high amplitude stable cycles. The simplest way with constant Λ is sufficient to attain any admissible amplitudes.

Let us return to the dynamical equations (4), (5). Differentiating Eq.(5) and excluding $\dot A$ by Eq.(4) we find the equation

$$\ddot\Phi = -\frac{\partial U}{\partial \Phi} + \Gamma\dot\Phi, \quad (13)$$

which describes motion of a "quasiparticle" with coordinate Φ located in the effective potential

$$U(A,\Phi) = [\alpha - 2mgA^2(4-\sigma A^2)]\Phi -2meA\cos\Phi - \frac{eg}{A}(4-\sigma A^2)\sin\Phi - \frac{1}{4}\frac{e^2}{A^2}\cos 2\Phi \quad (14)$$

and with dissipation coefficient

$$\Gamma(A) = -\frac{1}{A}\frac{dA}{dt} + g(4-\sigma A^2). \quad (15)$$

The above equations contain the amplitude A as a parameter. One can see that the potential U may have minima in some range of A if the forcing has a sufficiently great amplitude e. In this case a quasiparticle can be trapped in a potential well which means, in accordance with the definition (7), that the phase difference between forcing and exciting oscillations is bounded, i.e. oscillations are phase-locked. Thus, existence of minima in the effective potential (14) is the necessary condition for phase-locking. It becames sufficient only for appropriate initial conditions in (4), (5) which constrain the system to be phase-locked.

III. OSCILLATOR WITHOUT DISSIPATION

When $g=0$, the minima of the effective potential (14) are located in points where

$$\frac{dU}{d\Phi} = \alpha + \left(2mA + \frac{e}{A^2}\cos\Phi\right)e\sin\Phi = 0. \quad (16)$$

We have found the condition for existence of the minima for any A:

$$\alpha < F(A) = -e\left(2mA + \frac{e}{A^2}\cos\Phi_+\right)\sin\Phi_+, \quad (17)$$

where

$$\cos\Phi_+ = -\frac{m}{2e}A^3 + \sqrt{\frac{m^2}{4e^2}A^6 + \frac{1}{2}}. \quad (18)$$

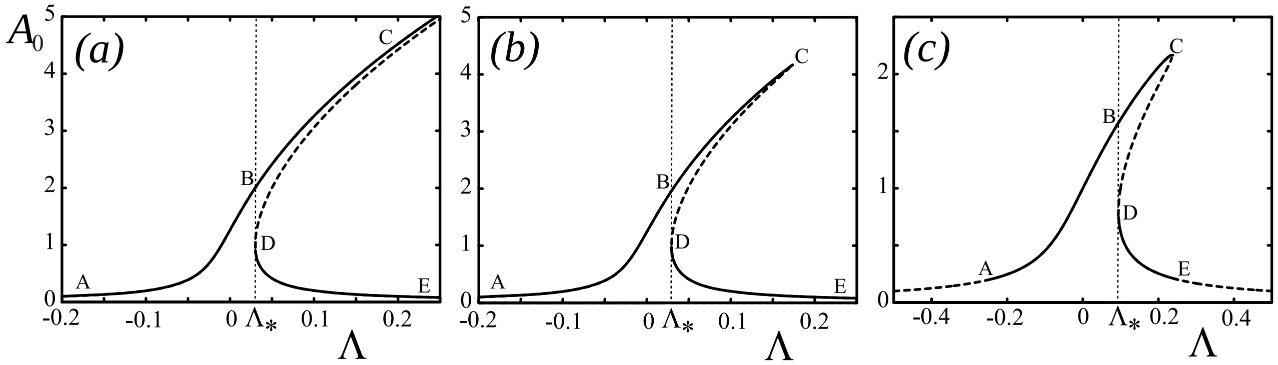

FIG. 1: The amplitude A_0 of stationary solutions of the averaging equations as functions of the frequency Λ. (a) $m = 0.01$, $g = 0$, $e = 0.02$; (b) $m = 0.01$, $g = -0.0012$, $\sigma = 0$, $e = 0.02$; (c) $m = 0.05$, $g = 0.0001$, $\sigma = 50$, $e = 0.05$.

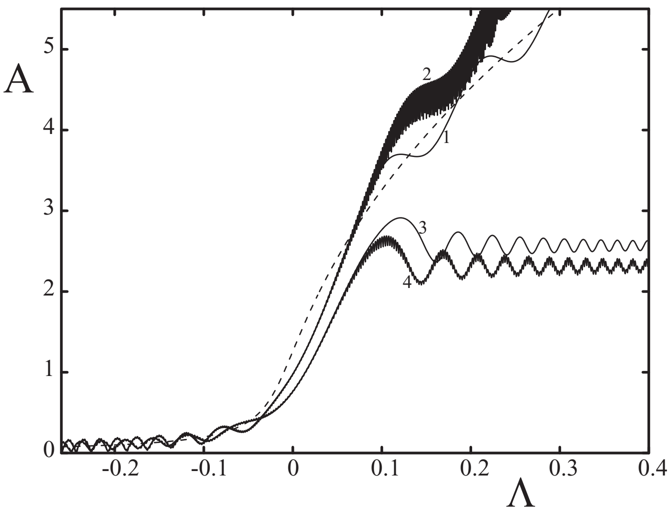

FIG. 2: Dynamics of the system (4), (5) (lines 1 and 3) and Eq.(1) (lines 2 and 4) for linear dependence of the forcing frequency $\Lambda(t) = \alpha t$ at $t \geq t_0 = -400$. $m = 0.01$, $g = 0$, $e = 0.02$. Lines 1 and 2 – $\varkappa = 2.580$ ($\alpha = 0.00065$); lines 3 and 4 – $\varkappa = 3.373$ ($\alpha = 0.00085$). Dashed line is the line ABC from Fig.1*a*.

The function $F(A)$ tends to infinity at $A \to 0$ and $A \to \infty$ and have a minima at $A^3 = e/2m$. Then, we have the sufficient condition for which the effective potential (14) has minima for all $A > 0$:

$$\alpha < \alpha_{cr} = F([e/2m]^{1/3}) = \frac{3^{3/2}}{2^{4/3}} m^{2/3} e^{4/3} \approx 2.062 \, m^{2/3} e^{4/3}. \qquad (19)$$

In opposite case, when the threshold condition (19) is violated, $\alpha > \alpha_{cr}$, there is a finite gap $[A_1, A_2]$ where potential has no minima.

The example of dynamics of the system is shown in Fig.2 . We use the linear dependence $\Lambda(t) = \alpha t$ and started at $t_0 = -400$. This case is associated with the control path used in Ref.[3]. Lines 1 and 2 show dynamics of the systems (4), (5) and (1) accordingly when α is less enough. The amplitude grows infinitly fluctuating around the dashed curve which is the same as the line ABC in Fig.1*a*. For Eq.(1), we use the value $\sqrt{u^2 + \dot{u}^2}$ as the amplitude of oscillations. It is close to $A(t)$ for rather small amplitudes according to the relation (3). One notes that the spreading of line 2 owing to high frequency fluctuations is caused by the difference between proposed circular orbits (3) and real ones, which is appreciable for high amplitudes. Such growth of the amplitude have been observed early [3] where it was shown that it is a result of the phase-locking ("autoresonance" effect). In contrast, when α exceeds some threshold value, the amplitude is saturated (lines 3,4) on a small level and high amplitudes associated with the range BC in Fig.1 became unattainable. The close behaviour of the system have been observed in the case when the initial time was $t_0 = 0$.

It is convenient to characterize the threshold phenomena by the dimensionless parameter

$$\varkappa = \frac{\alpha_{cr}}{m^{2/3} e^{4/3}}. \qquad (20)$$

Thus, the threshold value, found analytically in (19), was $\varkappa = 2.062$. The threshold value found numerically for the system (4), (5) at $t_0 = -400$ was

$$\varkappa \approx 3.262 \qquad (21)$$

and for the original equation (1) it was $\varkappa \approx 2.945$. In the second case, when $t_0 = 0$, the critical value was found to be

$$\varkappa \approx 1.941 \qquad (22)$$

for the system (4), (5) and $\varkappa \approx 1.826$ for the original equation (1).

One notes that the same threshold (21) have been found in Ref.[13]. It is greater then the value $\varkappa = 2.062$ found analytically. On the other hand, at $t_0 = 0$, the critical value (22) is rather close to the analytical value. To comprehend these results we should analyze dynamics of the phase $\Phi(t)$ in the system (4), (5). It is shown in Fig.3 where trajectories $(\Phi(t), A(t))$ of the solutions are presented on a background of the effective potential $U(A,\Phi)$ (14). The dashed lines in the figures are coordinates Φ where potential $U(A,\Phi)$ has minima at a given value of A. The dotted lines indicate maxima value of the potential. Because in Figs.3a,b of $\varkappa$ is greater than the value 2.062, there are gaps where the potential has no extreme points. On the contrary, in Fig.3c, parameter $\varkappa < 2.062$ and the potential has the extreme points in all range of A. Solid lines in Fig.3a and Fig.3b are dynamics of the solutions shown in another terms as lines 1 and 3 respectively in Fig.2. The main feature of the dynamics is that at the initial stage of the process before crossing of the resonance ($t < 0$), the system is phase-locking in the potential mimima near $\Phi \approx 0$. It gives the system a high probability to overcome the gaps. If $\varkappa < 3.262$ (see Fig.3a), the gap is not too long and the system can overcome sucessfully the gap and then do phase-lock in the minima of potential at high amplitudes. It was observed in Fig.2 as infinite growth of the amplitude (line 1). In opposite case, when $\varkappa > 3.262$ (see Fig.3b), the gap is so long that the system, even being phase-locked at the initial stage, cannot be captured in the potential minima at high amplitudes. Now phase-locking is destroyed and saturation of excitation is observed (line 3, Fig.2). It is obvious that namely phase-locking at initial stage causes appropriate preparation of the system to overcome the gap, which explain some increasing of the critical value above the limit (19).

In the second case, at $t_0 = 0$, the system starts just in the resonance (Fig.3). There is no initial phase-locking and the trajectory starts near unstable region of maximum of the potential (at $\Phi \approx -\pi/2$, see Fig.3c). In this case the dynamics becames very sensitive to structure of the effective potential. Now the potential should has global minima for any A in order to "quasiparticle" can be phase-locked. As a result we have the threshold (22) close to the analytical value $\varkappa = 2.062$.

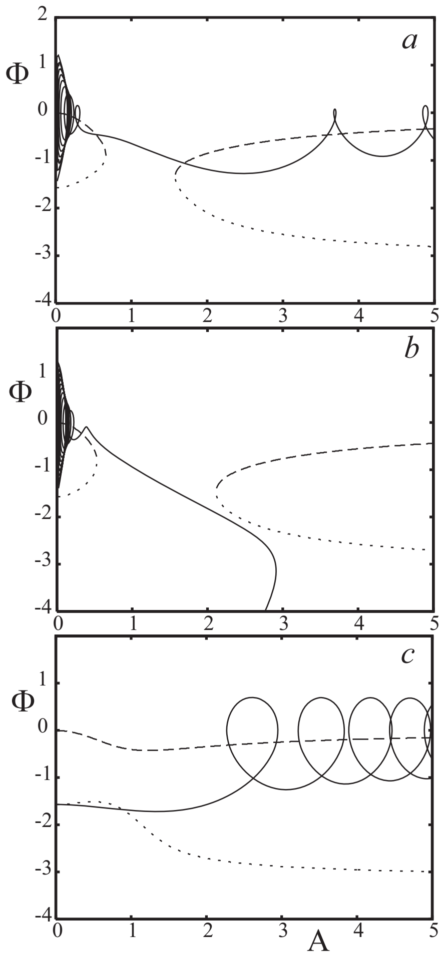

FIG. 3: The trajectories $(\Phi(t), A(t))$ of the system (4), (5) (solid lines) at $m = 0.01$, $g = 0$, $e = 0.02$. (a) $t_0 = -400$, $\varkappa = 2.580$ ($\alpha = 0.00065$); (b) $t_0 = -400$, $\varkappa = 3.373$ ($\alpha = 0.00085$); (c) $t_0 = 0$, $\varkappa = 1.191$ ($\alpha = 0.0003$). Dashed lines – minima of the effective potential $U(\Phi, A)$ at fixed A; Dotted lines – maxima of the effective potential.

IV. OSCILLATOR WITH LINEAR DISSIPATION

The minima of the effective potential for the oscillator with $g < 0$ and $\sigma = 0$ are defined by the equation

$$\frac{dU}{d\Phi} = \alpha + \left(2mA + \frac{e}{A^2}\cos\Phi\right)(e\sin\Phi - 4gA) = 0. \quad (23)$$

As it was made in the previous section we can introduce the auxiliary function $F(A) = \max_\Phi f(\Phi, A)$, where

$$f(\Phi, A) = -e\left(2mA + \frac{e}{A^2}\cos\Phi\right)\left(\sin\Phi - \frac{4gA}{e}\right). \quad (24)$$

The sufficient condition when potential has minima takes the form

$$\alpha < F(A). \quad (25)$$

The typical view of the function $F(A)$ is shown in Fig.4. It tends to infinity at $A \to 0$ and becames zero when the amplitude achieves maximum value at the point C: $A = A_C$ (see Fig.1b). In contrast with the dissipationless case, now the function $F(A)$ has no absolut minimum and the condition (25) will define only a range of amplitudes where the effective potential can have minima. It is seen from the Fig.4 that in the range $F(A) \gtrsim 0.0004$ the minima of the potential are possible only for small amplitudes. On the other hand for $F(A) \lesssim 0.0004$ the possible range of the minima rapidly enlarge up to the maximum value of amplitudes $A_C = 4.17$. Our goal of excitation is to attain amplitudes in the range of line BC (Fig.1b), that is in the range $A_B = 1.96 < A < A_C = 4.17$. One can propose that such excitation will be possible only when the range of minima of the potential covers high amplitudes making available the phase-locking, that is threshold condition should be

$$\alpha \lesssim 0.0004. \quad (26)$$

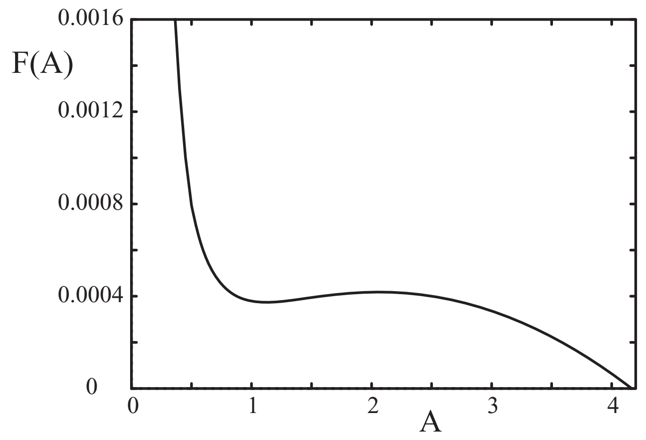

FIG. 4: The auxiliary function $F(A)$ for $m = 0.01$, $g = -0.0012$, $\sigma = 0$, $e = 0.02$.

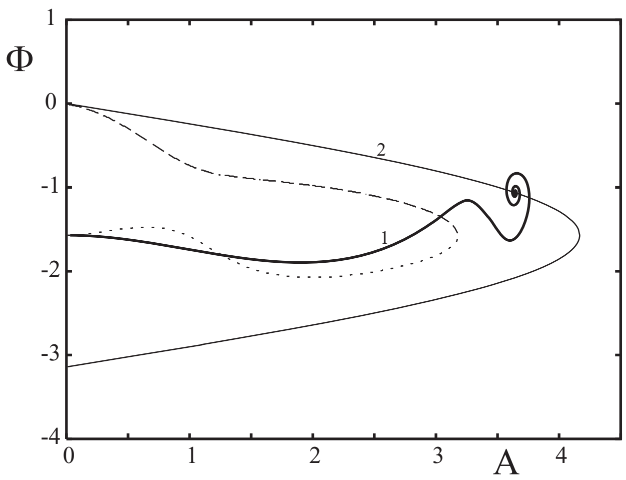

FIG. 5: Dynamics of the system (4),(5) in coordinates $(\Phi(t), A(t))$ (solid line 1) under the controlling path (27) for $t_0 = 0$, $\Lambda_0 = 0.13$ and $m = 0.01$, $g = -0.0012$, $\sigma = 0$, $e = 0.02$, $\alpha = 0.0003$. Solid line 2 – stationary solutions $(A_0(\Lambda), \Phi_0(\Lambda))$ of the system (8), (9). Dashed line – minima of the effective potential $U(\Phi, A)$ at fixed A. Dotted line – maxima of the effective potential.

To test this threshold numerically we will use the following controlling path:

$$\Lambda(t) = \begin{cases} \alpha t, & t_0 < t \leq \Lambda_0/\alpha \\ \Lambda_0, & t > \Lambda_0/\alpha \end{cases} \quad (27)$$

That is at initial times we use the linear chirping of the frequency and then, at $t > \Lambda_0/\alpha$, the chirping is switched off. Parameter Λ_0 is chosen so that the corresponding amplitude $A_0(\Lambda_0)$ lies in the curve BC (Fig.1*b*). Because BC is the range of attractive focuses of the system (4),(5), the path after switching off the chirping should excite a stable cycle which is a solution of Eqs.(8),(9) at $\Lambda = \Lambda_0$. Fig.5 illustrates dynamics of the system (4),(5) under the controlling path (27) when the threshold condition (26) is fulfilled.

Using the controlling path (27) and varying Λ_0 we have studied an important problem what maximum of the amplitude of oscillations can be reached for a given α. The results are collected in Fig.6 for $t_0 = -400$ (solid line) and $t_0 = 0$ (dashed line). The curves have confirmed that the proposed value (26) is the good threshold to reach high amplitude excitations. Especially it is well seen in the line at $t_0 = 0$ (dashed line) when the attainable amplitudes undergo the jump as α crosses the critical value about 0.0004. As it was discussed in previous section, for initial condition with $t_0 = 0$, the dynamics of the system is very sensitive to destribution of the effective potential, which became apparent in the jump of amplitudes at the threshold.

The main characteristic of the phase-locking phenomena is dependence of the critical value α_{cr} from the amplitude of forcing e. In the dissipationless case it was an exponential function $\alpha_{cr} \propto e^{4/3}$. To studying the dependence in the dissipation case we define the function

$$\alpha_0 = \min_{[0,A_0]} F(A), \quad (28)$$

where A_0 is the amplitude of the stationary solution corresponding to Λ_0 in the controlling path (27). Then we

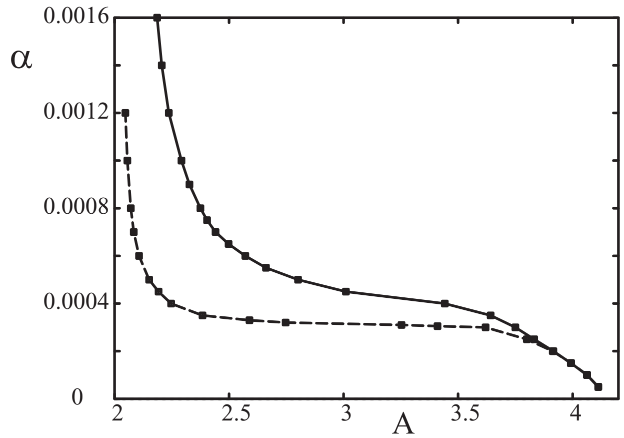

FIG. 6: The maxima of the amplitude A_0 attainable at a given α. Solid line: $t_0 = -400$, dashed line: $t_0 = 0$; $m = 0.01$, $g = -0.0012$, $\sigma = 0$, $e = 0.02$

may suppose that the correct threshold condition when the amplitude can attain the value A_0 reads

$$\alpha < \alpha_{cr} = \alpha_0. \quad (29)$$

The comparison of the theoretical value of the threshold (28), (29) with computations in the averaging system (4),(5) and the original equation (1) is given in Fig.7. It is well seen that the theoretical value tends to exponent, but with the index different from the dissipationless case: $\alpha_{cr} \propto e^{1.43}$. The systems (4),(5) and (1) exhibit close behavior except the edges of the range studied. One notes that the results above slightly depend on another paremeters g and m while the inequality (11) is fulfilled.

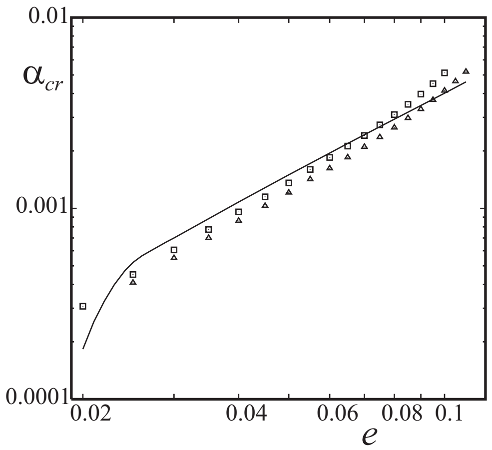

FIG. 7: The dependence of the critical value α_{cr} on the amplitude of forcing e for the path (30) with $\Lambda_0 = 0.13$ and $t_0 = 0$; $m = 0.01$, $g = -0.0012$, $\sigma = 0$; $\square$ – system (4),(5), $\triangle$ – equation (1). The solid line is $\alpha_{cr} = \alpha_0(e)$.

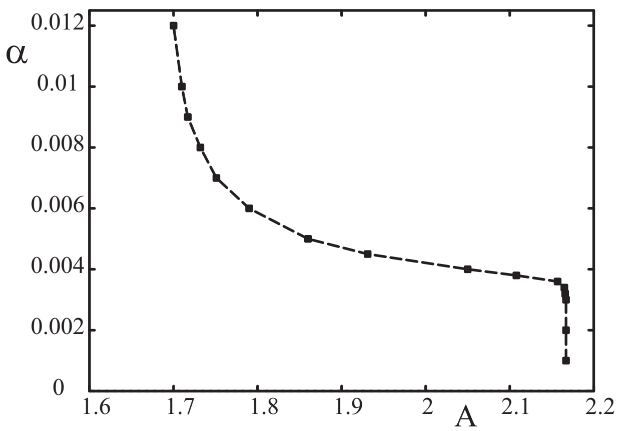

FIG. 8: The maxima of the amplitude A attainable at a given α for the path (30) with $t_0 = 0$; $m = 0.05$, $g = 0.0001$, $\sigma = 50$, $e = 0.05$.

V. VAN DER POL - DUFFING OSCILLATOR

The minima of the effective potential for the oscillator with $g > 0$ and $\sigma > 0$ are defined by the equation

$$\frac{dU}{d\Phi} = \alpha + \left(2mA + \frac{e}{A^2}\cos\Phi\right)\left(e\sin\Phi - gA(4-\sigma A^2)\right) = 0. \tag{30}$$

Treating as in the prevous sections, we can find the sufficient condition when the effective potential has minima

$$\alpha < F(A). \tag{31}$$

At the small g (12), the typical shape of the function $F(A)$ is close to Fig.4. Just the same as with linear dissipation, $F(A)$ becames zero when the amplitude achieves the maximum value at the point C (see Fig.1c). Our goal of excitation is to attain amplitudes in the range of the line BC (Fig.1c). The results collected in Fig.8. We use the controlling path (27) and varying Λ_0 to reach maximum of the amplitude of oscillations for a given α. We use initial conditions only with $t_0 = 0$ because the range of variation of Λ in this case is bounded around zero by the stability condition.

As in the previous section we introduce the correct threshold condition (28), (29). The comparison of this theoretical value with computations in the averaging system (4),(5) and the original equation (1) have shown the similar asymptotic behavior $\alpha_{cr} \propto e^{2.1}$ with index differed significantly from the dissipationless case.

In conclusion the authors thanks to L. Friedland for helpful discussions. The work was supported by INTAS (Grant No. 03-51-4286).

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