International Physics and Control Conference, Facolta di Ingegneria Universita degli Studi di Catania, September 1-4, 2009

Asymptotical Symmetrization of Hamilton Systems

A-G. Petrov*
IPMecRAS
Moscow, Russia

Abstract— *We discuss a new algorithms of calculations of Hamiltonian normal form. A normal form of a Hamilton system has two main properties: a) Tailor expansion of the normal form has the simplest form; b) its linear part commutates with a nonlinear one. Property a is used for the normalization procedure. Property b) is used to build asymptotic solutions. For this purpose, instead of the normal form we define symmetrical form: a form satisfying property b). Symmetrization algorithm is reduced to sequential calculations of the quadrature in the approximation of each order and is essentially simpler than all the classical normalization procedures.*

Keywords: Hamilton system, normal form, nonlinear oscillations

I. A normal form of a Hamilton system [2].

To simplify our reasoning, we shall limit ourselves by two degrees of freedom, although all the conclusions are extended to the case of finite degrees of freedom. Let $(\mathbf{q},\mathbf{p}) \stackrel{\text{def}}{=} (q_1,q_2,p_1,p_2)$: be dependent variables, $H = H(\mathbf{q},\mathbf{p})$: be Hamilton function of Hamilton system

$$\dot{q}_i = \partial H/\partial p_i, \;\; \dot{p}_i = -\partial H/\partial q_i, \;\; i = 1,2, \qquad (1)$$

where the dot means d/dt. Let $\mathbf{q} = \mathbf{p} = 0$: be a fixed point of system (1) and function $H = H(\mathbf{q},\mathbf{p})$: in it be analytical. Then function H can be represented as an expansion in powers of q,p which starts with quadratic terms while power expansions of the right parts of system (1) start with linear members. Let R be a matrix of a linear part of system (1). Eigenvalues $\lambda_1,\ldots,\lambda_4$: of matrix R are split into pairs $\lambda_{j+2} = -\lambda_j$, $j = 1,2$:. By means of a canonic linear variable change: $(\mathbf{q},\mathbf{p})^* = B(\mathbf{x},\mathbf{y})^*$, where * means transposition, Matrix R can be reduced to Jordan complex-valued normal form, in which eigenvalues $\lambda_1,\ldots,\lambda_4$: are located diagonally. Then: $H(\mathbf{q},\mathbf{p}) = \tilde{H}(\mathbf{x},\mathbf{y})$. Let the canonic linear complex variable change:

$$(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v}) + \mathbf{N}(\mathbf{u},\mathbf{v}), \qquad (2)$$

where $\mathbf{N} \stackrel{\text{def}}{=} (N_1,\ldots,N_4)$, $N_j(\mathbf{u},\mathbf{v})$ are power series without constant and linear terms reduce Hamilton function: $\tilde{H}(\mathbf{x},\mathbf{y})$ to

$$h(\mathbf{u},\mathbf{v}) \stackrel{\text{def}}{=} \sum h_{\mathbf{s}}\, \mathbf{w}^{\mathbf{s}}, \qquad (3)$$

where $\mathbf{w} \stackrel{\text{def}}{=} (w_1,\ldots,w_4) \stackrel{\text{def}}{=} (\mathbf{u},\mathbf{v})$, $\mathbf{s} = (s_1,\ldots,s_4)$, $\mathbf{w}^{\mathbf{s}} \stackrel{\text{def}}{=} w_1^{s_1}\ldots w_4^{s_4} \stackrel{\text{def}}{=} u_1^{s_1}u_2^{s_2}v_1^{s_3}v_2^{s_4}$. Hamilton formal function is called *a complex normal form* provided that:

1) the linear part matrix of the relevant Hamilton system is of normal form with eigenvalues: $\lambda_1,\lambda_2,-\lambda_1,-\lambda_2$, located diagonally

2) in expansion (3) there are only resonance terms for which

$$(s_1 - s_3)\lambda_1 + (s_2 - s_4)\lambda_2 = 0. \qquad (4)$$

[1] proves that for every system (1) there is a formal change (2) which leads Hamilton function: $H(\mathbf{q},\mathbf{p})$ to a normal form (3), (4). In accordance with [4], if the initial system (1) is real, there is a real normal form which can be reduced to a complex normal form (3), (4) by a standard linear change of coordinates.

Special cases of such a normal form are those of Birkhoff [3] and of Cherry and Gustavson. Birkhoff [3] analyzed a case with all eigenvalues incommensurable, so equation (4) in which $s_1,\ldots,s_4$ are integers, has a trivial solution $s_1 - s_3 = s_2 - s_4 = 0$. In this case expansion (4) is a series of products: $(u_1v_1)^{s_1}(u_2v_2)^{s_2}$ and every such a product is a formal integral of the relevant Hamilton system. Cherry considered a case where eigenvalues λ_1 and λ_2 are different. Gustavson came to the same result. Belitskii suggested an expanded normal form in which Jordan boxes of matrix linear part are used to further reduce the number of nonlinear members. A more detailed review of other expanded normal forms is given in [4],[2].

II. Methods of normal form computation

Computation algorithms of canonical normalizing transformations (2) and normal forms (3), (4) are classified into three groups according to the form of canonical transformation. There are now three forms of canonical transformations: A. by means of a generating function; B. by means of Lie series; C. parametric. Thus, we will refer algorithms to one of the three groups above depending on the canonical transformation used.

*E-mail: petrov@ipmnet.ru

Description of algorithms.

A. The generating function to compute a normal form was first introduced by Jacobi [3]-[8]. According to this method, vector series: $\mathbf{N}(\mathbf{u},\mathbf{v})$ in nonlinear formal transformation is computed using generating function $g(\mathbf{x},\mathbf{v}) = x_1v_1 + \ldots + x_2v_2 + \ldots$ of mixed variables $\mathbf{x} = (x_1, x_2) \quad \mathbf{v} = (v_1, v_2)$, while

$$\begin{aligned} u_j = \partial g/\partial v_j = x_j + \ldots, \\ y_j = \partial g/\partial x_j = v_j + \ldots, j = 1, 2. \end{aligned} \tag{5}$$

If the generating series $g(\mathbf{x},\mathbf{v})$ is computed, it is necessary to express x_j with the help of $\mathbf{u},\mathbf{v}$ to obtain transformation (2), thus to invert power series for u_j. This results in a highly complicated computation, however always applicable (with no matrix R limitations).

B. For normalizing with the help of Lie series, scaling $\mathbf{q} = \varepsilon\mathbf{q}'$, $\mathbf{p} = \varepsilon\mathbf{p}'$ and $\mathbf{x} = \varepsilon\mathbf{x}'$, $\mathbf{y} = \varepsilon\mathbf{y}'$, $t' = \varepsilon^2 t$, $\mathbf{w} = \varepsilon\mathbf{w}'$ is usually applied in which case $\tilde{H}(\mathbf{x},\mathbf{y})$: Hamiltonian, $h(\mathbf{w})$: normal form, and Lie $G(\mathbf{w})$ generator can be considered a series on a small parameter ε

$\tilde{H}(\mathbf{x}',\mathbf{y}') = \sum\limits_{k=0}^{\infty} \varepsilon^k \tilde{H}_k(\mathbf{x}',\mathbf{y}')$, $h(\mathbf{w}') = \sum\limits_{k=0}^{\infty} \varepsilon^k G_k(\mathbf{w}')$

The normalizing coordinate transformation and the normal form $h(\mathbf{w}')$: can be found in the form of Lie series

$\mathbf{z}' = \mathbf{w}' + \varepsilon\{\mathbf{w}', G\} + \frac{\varepsilon^2}{2!}\{\{\mathbf{w}', G\}, G\} + \ldots,$

$h(\mathbf{w}') = \tilde{H}(\mathbf{w}') + \varepsilon\{\tilde{H}, G\} + \frac{\varepsilon^2}{2!}\{\{\tilde{H}, G\}, G\} + \ldots$, where curly brackets mean Puasson brackets.

Functions h_k and G_{k-1} in their turn are computed successively following k growth with the help of a homologous equations

$$h_k(\mathbf{w}) = \{\tilde{H}_0(\mathbf{w}), G_{k-1}(\mathbf{w})\} + M_k(\mathbf{w}), \tag{6}$$

There are two algorithms to solve homologous equations (6), and consequently two normalization algorithms.

B.1. Equation (6) is computed as a system of linear equations for form $h_k \quad G_{k-1}$ coefficients. This method was developed by Hori and Deprit. Similar to the previous method, there are no R matrix limitations in this method either.

B.2. Zhuravlev [8], [6] proposed to solve the homologous equation by means of integration. If matrix R is diagonalizable $\{H_0, G_{k-1}\} = dG_{k-1}/dt$, Puasson bracket equals to the derivative of G with respect to t along the solution of system $\dot{\mathbf{q}} = \partial H_0/\partial\mathbf{p}$, $\dot{\mathbf{p}} = -\partial H_0/\partial\mathbf{q}$. Therefore, h_k is an average of M_k function along the solutions of the system, and function minus G_{k-1} is a constant in $\int\limits_0^t M_k\,dt$ integral. Thus, for the first two approximations, functions $M_k(\mathbf{w})$ are as follows

$M_1 = H_1$,

$M_2 = H_2 + \{H_1, G_1\} + \frac{1}{2}\{\{H_0, G_1\}, G_1$.

C. Petrov [9], [10] proposed a parametrical form of canonical transformation $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$. The general result concerning the parametrization of canonical change of variables can be stated as follows [10]

Theorem. *Suppose that transformation $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ of variables is represented in the parametric form*

$$\begin{aligned} \mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \quad \mathbf{Q} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \\ \mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, \quad \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \end{aligned} \tag{7}$$

where $\Psi(t,\mathbf{x},\mathbf{y})$ is a twice continuously differentiable function in a neighborhood of the point $(t_0,\mathbf{x}_0,\mathbf{y}_0)$. Then the following assertions are valid.

1) The Jacobians of two transformations $(\mathbf{x},\mathbf{y}) \to (\mathbf{q},\mathbf{p})$ and $(\mathbf{x},\mathbf{y}) \to (\mathbf{Q},\mathbf{P})$ identically coincide:

$$\frac{\partial(\mathbf{q},\mathbf{p})}{\partial(\mathbf{x},\mathbf{y})} = \frac{\partial(\mathbf{Q},\mathbf{P})}{\partial(\mathbf{x},\mathbf{y})} = J(t,\mathbf{x},\mathbf{y}). \tag{8}$$

2) For $J(t,\mathbf{x},\mathbf{y}) \neq 0$, there exists a neighborhood of the point $(t_0,\mathbf{x}_0,\mathbf{y}_0)$ in which the transformation (2) $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ brings a Hamiltonian $H(t,\mathbf{q},\mathbf{p})$ to Hamiltonian $\bar{H}(t,\mathbf{Q},\mathbf{P})$ such that

$$\Psi_t(t,\mathbf{x},\mathbf{y}) + H(t,\mathbf{q},\mathbf{p}) = \bar{H}(t,\mathbf{Q},\mathbf{P}),$$

where the arguments $\mathbf{q},\mathbf{p}$ and $\mathbf{Q},\mathbf{P}$ of the Hamiltonians H and $\bar{H}$ can be expressed via parameters $\mathbf{x}$ and $\mathbf{y}$ by formulas (7).

If a Hamiltonian system is autonomous, then the function $\Phi(\mathbf{q},\mathbf{p}) = \Psi\left(\frac{1}{2}(\mathbf{q} + \mathbf{Q}(\mathbf{q},\mathbf{p})), \frac{1}{2}(\mathbf{p} + \mathbf{P}(\mathbf{q},\mathbf{p}))\right)$ is a generating function, which Poincare introduced [13]. Thus, function $\Psi(t,\mathbf{x},\mathbf{y})$ can be called parametrical Poincare function.

Function $\Psi(\mathbf{x},\mathbf{y})$ and parametric canonical normalizing transformation of variables in the form of (7) is used instead of G generator in the algorithm of constructing a normal form [11], [12]. The first two approximations for G and Ψ are the same, while the ones that follow are different. To simplify the computation, it is possible to use integration similar to the method described in B.2.

Methods B.2 and C simplify the normal form computation significantly. In addition to this, there is a notion of Hamiltonian symmetrization introduced which expands the notion of the normal form. This is done using property of commutation of perturbed and non-perturbed parts only.

III. Hamilton symmetric form [6]

Definition. *Perturbed Hamiltonian* $H_0 + F$*: is a symmetric form if perturbation* $F(t, \mathbf{q}, \mathbf{p}, \varepsilon)$ *is the first integral of non-perturbed part* $\frac{\partial F}{\partial t} + \{H_0, F\} = 0$.

There are three advantages of this definition over the previous ones [3], [7]. They are as follows:

1. To solve the whole system of Hamilton equations in its symmetrical form, a superposition of solutions of a non-perturbed system and a solution of an autonomous Hamiltonian, which equals $F(0, \mathbf{q}, \mathbf{p}, \varepsilon)$, is used.

2. The invariant character of the definition allows symmetrization both without a preliminary simplification of a non-perturbed part and specification of autonomous/non-autonomous, resonance/non-resonance cases.

3. Asymptotic of a normal form and transformation of variables which lead Hamiltonian to its normal form can be found by consequent quadratures of the functions known at every step (algorithms II.2 and III).

IV. Algorithm of symmetrization with the help of generating Hamiltonian [6]

Let the Hamiltonian under consideration has the following form

$$H(\mathbf{q}, \mathbf{p}, \varepsilon) = H_0(\mathbf{q}, \mathbf{p}) + F(\mathbf{q}, \mathbf{p}, \varepsilon),$$

$$F = \varepsilon F_1 + \varepsilon^2 F_2 + \ldots,$$

where H_0 and F are non-perturbed and perturbed parts of the Hamiltonian, ε is small parameter.

To construct generating Hamiltonian $G = \varepsilon G_1 + \varepsilon^2 G_2 +, \ldots$ (Lee generator) and symmetrical part $\bar{F} = \varepsilon \bar{F}_1 + \varepsilon^2 \bar{F}_2 +, \ldots$ we should
1. find solution $\mathbf{q}(t, \mathbf{Q}, \mathbf{P})$, $\mathbf{p}(t, \mathbf{Q}, \mathbf{P})$ of a non-perturbed system;

$$\dot{\mathbf{q}} = \frac{\partial H_0}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H_0}{\partial \mathbf{q}}, \quad \mathbf{q}(0) = \mathbf{Q}, \quad \mathbf{p}(0) = \mathbf{P},$$

2. find functions: $m_k(t, \mathbf{Q}, \mathbf{P}) =$
$M_k(\mathbf{q}(t, \mathbf{Q}, \mathbf{P}), \mathbf{p}(t, \mathbf{Q}, \mathbf{P}))$, $k = 1, 2, \ldots$, where
$M_1 = F_1$, $M_2 = F_2 + \{F_1, G_1\} + \frac{1}{2}\{\{H_0, G_1\}, G_1\}$, ($\{f, g\}$ are Poisson brackets).

3. Using identity of $\int_0^t m_k(t, \mathbf{Q}, \mathbf{P}) dt = t\bar{F}_k(\mathbf{Q}, \mathbf{P}) + G_k(\mathbf{Q}, \mathbf{P}) + f(t)$ we find asymptotics coefficients of the symmetrized part $\bar{F}_k(\mathbf{Q}, \mathbf{P})$ and generator $G_k(\mathbf{Q}, \mathbf{P})$ from the quadrature $\int_0^t m_k(t, \mathbf{Q}, \mathbf{P}) dt$. In particular, H_0 is a quadratic normal form, the symmetrization algorithm equals II.3 normal form algorithm. However, even in this classical case, the algorithm is significantly less complicated than classical algorithms I. and II.1. This can be demonstrated by the following examples.

Example 1. Zhuravlev V.F. [6]. Let Hamiltonian be a rational function [6] $H = \dfrac{1}{2}(p^2 + q^2) + \dfrac{\varepsilon}{1 + q^2}$. According to the algorithm, we find
1. solution of the non-perturbed systems ($\varepsilon = 0$): $q = Q\cos t + P\sin t$, $p = -Q\sin t + P\cos t$
2. function $m_1(t, \mathbf{Q}, \mathbf{P}) = 1/(1 + Q^2\cos^2 t + 2QP\cos t\sin t + P^2\sin^2 t)$
3. From quadrature $\int_0^t m_1(t, \mathbf{Q}, \mathbf{P}) dt$ we find the first coefficients of expansions of symmetric form $\bar{F}_1 = 1/\sqrt{1 + Q^2 + P^2}$ and Lee generator $G_1 = -\bar{F}_1[\arctan(\bar{F}_1 P/Q) - \arctan(P/Q)]$.

If we have used classical algorithms, we could have obtained only a few Taylor expansion terms of symmetric form $\frac{1}{2}(Q^2 + P^2) + \varepsilon/\sqrt{1 + Q^2 + P^2} = \frac{1}{2}(1 - \varepsilon)(Q^2 + P^2) + \frac{3}{8}\varepsilon(Q^2 + P^2)^2 - \frac{5}{16}\varepsilon(Q^2 + P^2)^3 + \ldots]$ and it require many more calculations.

Example 2 [2]. Let two bodies of masses m_1 and m_2 ($m_1 \geq m_2$) travel in the same plane along circular orbits around their center of mass. The motion of the third body, whose mass is negligible and which is attracted by the first two bodies, is described by the plane circular restricted three-body problem. In the rotating system of coordinates, this problem is an autonomous Hamiltonian system and possesses a triangular fixed point. In order to analyze the stability of this point, the normal form of the Hamiltonian function is calculated. The Hamiltonian function has the Taylor expansion $H = H_0 + F_1 + F_2 + \ldots$, In according symmetrization algorithm we obtained normal forms [2] for the two resonance cases $\omega_1 = 2\omega_2 = 2/\sqrt{5}$, $(\mu = (45 - \sqrt{1833})/90)$ $\omega_1 = 3\omega_2 = 3/\sqrt{10}$, $(\mu = (15 - \sqrt{213})/30)$:
$\bar{H} = (2r_1 - r_2)/\sqrt{5} + r_2\sqrt{r_1}[\alpha_1 \sin(\varphi_1 + 2\varphi_2) + \beta_1 \cos(\varphi_1 + 2\varphi_2)]$,
$\bar{H} = (r_1 - 3r_2)/\sqrt{10} + r_2\sqrt{r_1 r_2}[\alpha_2 \sin(\varphi_1 + 3\varphi_2) + \beta_2 \cos(\varphi_1 + 3\varphi_2)] + c_{20}r_1^2 + c_{11}r_1 r_2 + c_{02}r_2^2$,
$Q_j = \sqrt{2r_j/\omega_j}\sin\varphi_j$, $P_j = \sqrt{2r_j\omega_j}\cos\varphi_j$, $j = 1, 2$.

All coefficients were calculated exactly [2].

In the case of equal frequencies $\omega_1 = \omega_2 = 1/\sqrt{2}$, $\mu = (9 - \sqrt{69})/18$ normal form of third order was calculated exactly. Coefficient $59/864 \approx 0.068287$ differs from the earlier calculated value 0.603 [14] by an order of magnitude. In this problem, symmetric forms are identical to classical normal forms.

V. Algorithm of symmetrization with the help of parametric Poincare function [11,12]

1. Find solution of non-perturbed system
$\mathbf{q}(t, \mathbf{Q}, \mathbf{P})$, $\mathbf{p}(t, \mathbf{Q}, \mathbf{P})$.
2. Find functions: $M_1 = F_1$,
$M_2 = F_2 + \{F_1, \Psi_1\} + \frac{1}{2}\{\{H_0, \Psi_1\}, \Psi_1\}$,

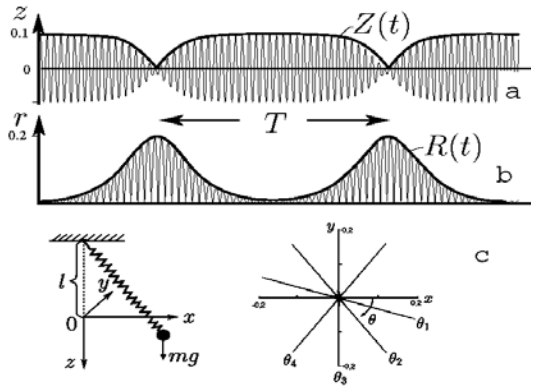

Fig. 1. Heavy point on the spring for resonance

$m_k(t,\mathbf{Q},\mathbf{P}) = M_k(t,\mathbf{Q}(t,\mathbf{Q},\mathbf{P}),\mathbf{P}(t,\mathbf{Q},\mathbf{P}))$

3. Find asymptotic coefficients of the symmetric form $\bar{F}_k(t,\mathbf{Q},\mathbf{P})$ and Poincare function $\Psi_k(t,\mathbf{Q},\mathbf{P})$ from the quadrature

$\int_{t_0}^{t} m_k(t,\mathbf{Q},\mathbf{P})dt = (t-t_0)\bar{F}_k(t_0,\mathbf{Q},\mathbf{P})+$
$\Psi_k(t_0,\mathbf{Q},\mathbf{P}) - \Psi_k(t,\mathbf{q}(\mathbf{t}),\mathbf{p}(\mathbf{t})), \quad k=1,2,\dots$

This algorithm also works in the non-autonomous case.

Example 3. We consider three dimensional oscillations of a heavy point on the spring for resonance 1:1:2 (fig 1.) [15]. The Hamiltonian function H has the following Taylor expansion

$$H = H_0 + F_1,$$
$$H_0 = \tfrac{1}{2}(u^2+v^2+w^2+x^2+y^2+4z^2),$$
$$F_1 = \tfrac{3}{2}z(x^2+y^2),$$

where x, y, z are coordinates of the heavy point.

According second algorithm we find

1. solution of the non-perturbed systems $x = X\cos t + U\sin t, \quad y = Y\cos t + V sint, \quad z = Z\cos 2t + (W/2)\sin 2t$
2. function $m_1(t) = F_1(x(t), y(t), z(t)$ and
3. from quadrature $\int_0^t m_1(t)dt$ we find first coefficients of expansions of symmetric form (identical normal form) $\bar{F}_1 = \frac{3}{8}Z(X^2+Y^2) - \frac{3}{8}Z(U^2+V^2) + \frac{3}{8}W(XU+YV)$, and Poincare function $\Psi_1 = \frac{3}{16}Z(XU+YV) + \frac{9}{64}W(U^2+V^2) + \frac{15}{64}W(X^2+Y^2)$.

The solution of the normal form equations in Birkhoff variables $z_1 = u+ix, \quad z_2 = v+iy, \quad z_3 = w/\sqrt{2}+iz\sqrt{2}$ is obtained, if we substitute in $z_1 = Z_1e^{it}, \quad z_2 = Z_2e^{it}, \quad z_3 = Z_3e^{2it}$ solutions $Z_1, Z_2 \quad Z_3$ of the following system of differential equations

$$\dot{Z}_1 = -\tfrac{3\sqrt{2}}{8}\bar{Z}_1Z_3, \quad \dot{Z}_2 = -\tfrac{3\sqrt{2}}{8}\bar{Z}_2Z_3,$$
$$\dot{Z}_3 = \tfrac{3\sqrt{2}}{16}(Z_1^2+Z_2^2).$$

For the following initial data $x(0) = \varepsilon_1\delta, \ \dot{x}(0) = 0, \ y(0) = 0, \ \dot{y}(0) = \varepsilon_2\delta, \ z(0) = \delta, \ \dot{z}(0) = 0$ $\delta \ll 1, \ \varepsilon_1 \ll 1, \ \varepsilon_2 \ll 1$ there is a simple asymptotic solution which describes a periodic process of alternate transformation of vertical oscillations energy into horizontal oscillations energy. The frequency of oscillations along the Z axis (fig. 1a) equals approximately 2, while these oscillations amplitude: equals the sum of soliton type functions, which are distant from each other by T-period [16], [17] (fig. 1a)

$$\frac{Z(t)}{\delta} = |\mathrm{th}\frac{3\delta}{4}(t-T/2)| +$$
$$|\mathrm{th}\frac{3\delta}{4}(t-3T/2)| - 1, \quad T = \frac{4}{3\delta}\ln\frac{32}{\varepsilon_1^2+\varepsilon_2^2}.$$

The frequency of horizontal oscillations is close to 1, while its amplitude equals $R(t)$ (fig. 1b)

$$\frac{R(t)}{2\delta} = \mathrm{sech}\frac{3\delta}{4}(t-T/2) + \mathrm{sech}\frac{3\delta}{4}(t-3T/2).$$

Following the law of energy conservation $R^2(t) + 4Z^2(t) = 4\delta^2$. The trajectory of the point movement in the configuration space, when $\delta = 0.1, \ \varepsilon_1 = 0.1, \ \varepsilon_2 = 0.03$ is presented in fig.1. Fig. 1a and 1b present numerically found dependencies $z(t)$ and $r(t) = \sqrt{x^2+y^2}$. As is clearly seen from the figures, they are modulated with a high degree of accuracy by the analytically found solitons with T-period.

As is seen from fig. 1c, the rotation angle of oscillation plane θ remains practically unchanged when the vertical oscillation amplitude is low. However, it changes drastically when the horizontal oscillations amplitude is low. Therefore, the oscillations projected on the horizontal plane are presented as intercepts located at equal angles from each other. The first oscillation plane is located at the angle of $\theta_1 = 24^\circ$ to axis x, while the consequent angles are terms of the arithmetic progression $\theta_2 = \theta_1 + 33^\circ = 57^\circ, \ \theta_3 = \theta_2 + 33^\circ = 90^\circ, \dots$.

Example 4. We consider nonautonomous system: Lagrange top on vibration base [6]. Hamiltonian has the form

$$H = H_0 + F(t,x,y,u,v),$$
$$H_0 = \tfrac{1}{2}(x^2+y^2+u^2+v^2) - xv + yu,$$
$$F = -\delta\,(xv-yu) +$$
$$\left(x^2+y^2\right)\left(2\delta - \varepsilon\left(1+2k\cos 2t\right)\right),$$

where x, y are coordinates, u, v are impulses, H_0 and F are non-perturbed and perturbed parts correspondently.

According second algorithm

1. we solve linear non-perturbed system

$\dot{x} = y+u, \quad \dot{y} = -x+v,$
$\dot{u} = -x+v, \quad \dot{v} = -y-u,$
$x(t_0) = X, \ y(t_0) = Y, \ u(t_0) = U, \ v(t_0) = V.$ The solution has the form $x = \frac{1}{2}(X+V)+$
$\frac{1}{2}(X-V)\cos 2\tau + \frac{1}{2}(Y+U)\sin 2\tau,$

$y=\frac{1}{2}(Y-U)+\frac{1}{2}(Y+U)\cos 2\tau+$
$\frac{1}{2}(-X+V)\sin 2\tau,$
$u=y-Y+U,\quad v=-x+X+V,\quad t=\tau+t_0.$
2. Using substitution $t,x,y,u,v\to\tau,X,Y,U,V$ we find function $m\,(t_0,\tau,X,Y,U,V)=$
$F(t,x,y,u,v).$
3. From quadrature $\int_{t_0}^{t} m(t_0,t'-t_0,X,Y,U,V)dt'$ we obtain normal form

$$\bar F(t,X,Y,U,V)=\delta\,(YU-XV)+\left(\delta-\tfrac{\varepsilon}{2}\right)\left(X^2+Y^2+U^2+V^2\right)+ +\tfrac{1}{2}\varepsilon k(-X^2-Y^2+V^2+U^2)\cos 2t+ \varepsilon k\,(YV+UX)\sin 2t$$

and Poincare function

$$\Psi\,(t,X,Y,U,V)=(\delta-\varepsilon/2)(UX+YV)- (1/4)\varepsilon k\,(YV+UX)\cos 2t+ +(1/8)\varepsilon k\left(3\left(U^2+V^2\right)+5\left(X^2+Y^2\right)\right)\sin 2t.$$

We find the stability conditions of periodic solution. We have proved that it is minimum point of the function $\bar F(0,X,Y,U,V)$. Matrix of the square form $\bar F(0,X,Y,U,V)$ has following eigenvalues:

$$\lambda_1=\lambda_2=\delta-\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\sqrt{(\delta^2+\varepsilon^2k^2)},$$
$$\lambda_3=\lambda_4=\delta-\tfrac{1}{2}\varepsilon-\tfrac{1}{2}\sqrt{(\delta^2+\varepsilon^2k^2)}.$$

The quadratic form $\bar F$ has minimum and vertical equilibrium position is stable if all eigenvalues are positive. It follows that stability condition has the form $(2\delta-\varepsilon)^2>\delta^2+\varepsilon^2k^2$.

Example 5. We apply this method to the investigation of the spherical pendulum with 3-dimensional vibration of the suspension point [18]. Hamiltonian in dimensionless variables has the form $H(\theta,\varphi,u,v)=H_0+\Phi$,

$$H_0=W_{t't'},\ \Phi=\varepsilon\left(\frac{1}{2}u^2+\frac{v^2}{2\sin^2\theta}-\cos\theta\right),$$
$$W=a_3\cos\theta+\sin\theta(a_1\cos\varphi+a_2\sin\varphi),$$
$$a_i(t')=\frac{\omega x_i}{\sqrt{gl}},\quad t'=\omega t,\quad \varepsilon=\sqrt{\frac{g}{l\omega^2}}.$$

where θ and φ are spherical coordinates of the heavy point, u and v are impulses, $x_i,\ i=1,2,3$ are Cartesian coordinates of the suspension point, l is length of pendulum, ω is oscillation frequency of the suspension point, $W_{t't'}$ is the second derivation of W with respect to dimensionless time t'.

The potential $H_0=W_{t't'}$ of inertial forces is a non-perturbed Hamiltonian and sum of kinetic energy and potential energy of gravity force Φ is perturbation. It is unusual way in normal form theory.

According second algorithm we find
1. solution of non-perturbed system

$$\theta=\theta_0,\quad \varphi=\varphi_0,\quad u=u_0-W_{t'\theta},\quad v=v_0-W_{t'\varphi};$$

2. function $m_1=\dfrac{1}{2}(u-W_{t'\theta})^2+\dfrac{(v-W_{t'\varphi})^2}{2\sin^2\theta_0}-\cos\theta_0$
and
3. from quadrature we obtain normal form

$$\bar\Phi(t_0,\theta_0,\varphi_0,u_0,v_0)=\varepsilon\left(\frac{1}{2}u_0^2+\frac{v_0^2}{2\sin^2\theta_0}+U(\theta_0,\varphi_0)\right),$$

$$U(\theta_0,\varphi_0)=\frac{1}{2}<W_{t'\theta}^2>+\frac{<W_{t'\varphi}^2>}{2\sin^2\theta_0}-\cos\theta_0,$$
$$<f>=\tfrac{1}{T}\int\limits_0^T f(t)dt.$$

We find the stability conditions of periodic solution by Lagrange theorem. It is minimum point of the function U.

VII. Conclusions

The examples analyzed clearly demonstrate significant advantages of symmetrization method over the other existing methods used to find asymptotic solutions.

Example 1 demonstrates that applying the new method the whole Taylor series of the first asymptotic approximation can be found by unique integration only, whereas standard methods allow finding only its first terms.

Example 2 shows the efficiency of the normal form computation in resonance cases. The exact normal form coefficients are found, and this disproved one of the previously obtained results.

Example 3 is a new analytical solution of the classical task of non-linear oscillations of the swinging spring in frequency resonance. The solution proposed does not only give a qualitative explanation of this complicated phenomenon of energy transformation from one degree of liberty to another, but also provides an exact quantitative description of these processes.

Example 4 of Lagrange top demonstrates the way to get the Hamiltonian symmetrical form by a unique integration without transformation of the lineal part of the system to its diagonal kind. With the help of thus obtained symmetric form, it is easy to get a periodic solution and its stability condition.

Example 5 gives an asymptotic solution to a problem of the spherical pendulum oscillations with periodic 3-d vibrations of the point of suspension, which solution is also achieved by a unique integration. In this example, the potential energy of inertial forces is taken as a non-perturbed part, whereas in the classical method, this energy is considered a perturbation. To sum it all up, it is worth mentioning that the symmetric form (as well as the normal form) is a highly convenient tool which does not only help to find additional system integrals and periodic solutions but also to analyze their stability.

This work was supported by the Russian Foundation for basic Research, project No. 07-01-00129.

References

[1] A.D. Bruno, *Analytic form of differential equations.* Tr. Mosk. Mat. Obshc. **26** (1972), 199-239.

[2] A.D. Bruno and A. G. Petrov, *Calculation of the Hamiltonian Normal Form.* Dokl. Akad. Nauk **410**(4), (2006), 474-478 [Dokl. Phys. **51**(10) (2006), 555-559].

[3] G.D. Birkhoff, Dynamical Systems (American Mathematical Society, New York, 1927).

[4] A.D. Bruno, *The Restricted Three Body Three Problem: Periodic Orbits* (Nauka, Moscow, 1990), [in Russian].

[5] V.M. Zhuravlev, D.M. Klimov, Applied Methods of Vibration Theory (Nauka, Moscow, 1988) [in Russian].

[6] V.F. Zhuravlev, *Invariant normalization of non-autonomus Hamiltonian systems.* Prikl. Mat. Mekh. **66**(3) (2002), 356-365.

[7] V.I. Arnold, Mathematical Methods of Classical Mechanics. (Nauka, Moscow, 1978), [in Russian].

[8] V.F. Zhuravlev, Introduction to Theoretical Mechanics. (Nauka Fizmatlit, Moscow, 1997) [in Russian].

[9] A.G. Petrov, *On average of Hamiltonian systems with periodic Hamiltonian.* Dokl. Akad. Nauk **368**(4), (1999), 483-488 [Dokl. Phys. **44**(10) (1999)].

[10] A.G. Petrov, *Asymptotic Methods for Solving the Hamilton Equations with the Use of a Parametrization of Canonical Transformations.* Differentsial'niya Uravnenia **40**(5), (2004), 626-638. [Differencial Equations **40**(5), (2004), 672-685.

[11] A.G. Petrov, *Modification of invariant normalization method of Hamiltonians with help parametrization of the canonical transformations.* Dokl. Akad. Nauk. **386**(4) (2002) 343-345 [Dokl. Phys. **47**(10) (2002), 742-744].

[12] A.G. Petrov, *Invariant normalization of non-autonomus Hamiltonian systems.* Prikl. Mat. Mekh. **68**(3) (2004), 402-413. [J. Appl. Math. Mech. **68** (3) (2004), 357-367].

[13] H. Poincare, Les methods nouvelles de la mecanique celeste. Paris: Gauthier-Villars, 1892, 385 p.; 1893, 472 p.; 1899, 414 p.

[14] A.P. Markeev, Libration Points in Celestial Mechanics and Astrodynamics (Fizmatlit, Moscow, 1978) [in Russian].

[15] A.G. Petrov, A.V. Fomichev, *On Nonlinear threedimentional oscillations of a havy swinging Spring.* (unpublished)

[16] M.N. Zaripov, A.G. Petrov, *Nonlinear Oscillations of a Swinging Spring.* Dokl. Phys. **49**(11) (2004), 691-696 [Dokl. Akad. Nauk **399**(3) (2006), 347-352].

[17] A.G. Petrov, *Nonlinear oscillations of a swinging Spring for resonance.* Mechanics of Solids **40**(5) (2006) [Mekh. Tverd. Tela (5) (2006), 18-28]

[18] A.G. Petrov, *On the Equations of Motion of a Spherical Pendulum with a Fluctuating Support.* Dokl. Phys. **50**(11) (2005), 588-591 [Dokl. Akad. Nauk V. 405, No 1, 51 (2005)].