ADAPTIVE SYNCHRONIZATION USING VSC FOR DELAYED COMPLEX DYNAMICAL NETWORKS

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Abstract: The synchronization problem for a class of delayed complex dynamical networks via employing variable structure control has been explored and a solution proposed. The synchronization controller guarantees the state of the dynamical network is globally asymptotically synchronized to arbitrary state. The switching surface has been designed via the left eigenvector function of the system, and assures the synchronization sliding mode possesses stability. The hitting condition and the adaptive law for estimating the unknown network parameters have been used for designing the controller hence the network state hits the switching manifold in finite time. Two illustrative examples along with the respective simulation results are given, which employ the designed variable structure controllers. *Copyright © 2007 IFAC*

Keywords: Complex dynamical networks; Lyapunov stability; synchronization; time delays; variable structure control.

1. INTRODUCTION [1]

Network structures have been subject of research for considerable time in mathematical science. Furthermore, it has been observed for some time that complex dynamic networks exist in all fields of science and humanities as well as in nowadays networked individuals, societies and technical and non-technical systems. Thus the latter have been studied extensively over the past decades. As it is well-known, traditional networks are mathematically represented by a graph, e.g. a pair $G = \{P, E\}$ in which P represents a set of N nodes (or vertices) $P_1, P_2, \cdots, P_N$ and E is a set of links (or arcs or edges) $L_1, L_2, \cdots, L_N$ each of which connects two elements of P. The well known chains, grids, lattices and fully connected graphs have been formulated to represent the so-called completely regular networks.

In the course of development of the mathematical theory of dynamical netwroks, the theory of random graphs (Figure 1-a) was first introduced by Paul Erdos and Alfred Rényi [1], who discovered the probabilistic methods were often useful to tackle problems in graph theory. In recognition to their work, now these are known as ER random graph models. The ER random graph models have served as idealized coupling architectures for gene networks and the spread of infectious diseases for a long time,

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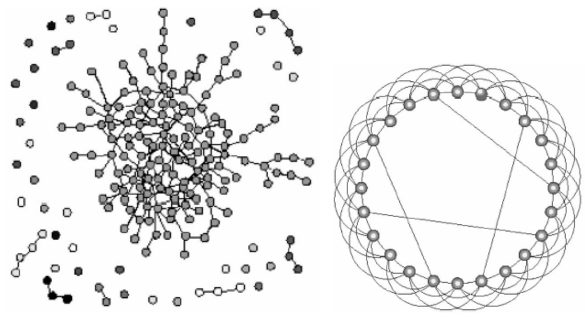
and recently for studying the spread of computer viruses too.



(a) (b)

Fig. 1 A random graph network (a), and a small-world network (b) [7]

In recent studies, Watts and Strogatz [2] introduced the so-called small-world networks (Figure 1-b), or so-called WS networks, in order to describe the transition from a regular network to a random network [3], [7]. Subsequently, in their studies [4-6], Barabasi and Albert have argued that the scale-free nature (Figure 2) of real-world networks is rooted in two general mechanisms: growth and preferential attachment respectively. It thus gives rise to dynamical nodes in networks and not solely static ones.

In the real world at large, many real systems such as biological, technological and social systems can be described by various models of complex networks [8], [9]. One of the interesting and rather significant phenomena in complex dynamical networks [10] is the synchronization of all dynamical nodes as well as the appearance of chaotic modes. Hence dynamical networks may be rather complex and the respective phenomena taking place within them are rather rich and often appear and disappear unexpectedly. The present study is devoted to such networks.

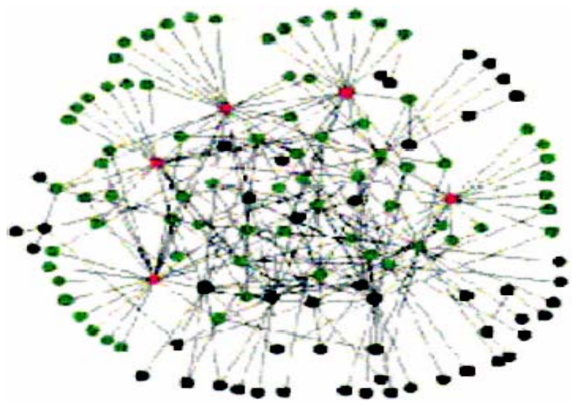


Fig. 2 A Scale-free network [7]

In this paper, a class of general complex dynamical network models with coupling delays is explored with regard to controlled synchronization via variable structure control. Here, we investigate the synchronization properties of these models with matched conditions uncertainty and unmatched conditions uncertainty by using of variable structure control. Via Lyapunov functional method and formulation of an adequate proper adaptive law, we derive synchronization conditions for both cases.

Section II presents a selected survey, a continuous-time dynamical network model with coupling time-delays and some preliminaries. In Section III, the switching surface is constructed by using the left eigenvector function method. The stabilities of the network synchronized states in both cases with known bound and with unknown boundary on nonlinear terms are investigated in Section IV and in Section V, respectively. Section VI presents the results and simulation experiments for two of benchmark examples. Concluding section and references follow thereafter.

2. DYNAMICAL NETWORK MODEL FORMULATION AND APPLICATION OF VSC

The synchronization in networks of coupled chaotic systems has received a great deal of attention during the last decade or so, e.g. see [10]-[18] for instance. In their work [10], Wang and Chen have established a uniform dynamical network model for such studies; also they explored its synchronization and control. Although, the model of Wang and Chen reflects the complexity from the network structure, still it is a fairly simple uniform dynamical network. A new model and chaos synchronization of general complex dynamical networks was also explored by Hu and Chen in [11], and by Lu and co-authors in [12]. Further, Wang and Chen [13] explored the synchronization problem in small-world dynamical networks, and similarly Barhona and Pecore studied the synchronization in heir small world systems in [14]. In [15], Wang and Chen investigated the synchronization in scale-free networks with regard to robustness and fragility. Subsequently, X. Li and Chen discussed synchronization and de-synchronization of complex dynamical networks from an engineering point of view in [16].

More recently, in works [17]-[21], the complex dynamical networks with time-delays have received particular attention more attentions because its presence is frequently a source of instability. For, time-delays commonly, or even unavoidably, exist in various network-like systems due to some inherent mechanism and/or the finite propagation speed of information carrying signals. Z. Li and Chen proposed in [17] a linear state feedback controller design to realize the synchronization for the networks with coupling delays. Similarly, C. Li and Chen proposed a solution in the case with coupling delays in work [18]. Further, P. Li and co-authors explored in [19] one way of global synchronization in delayed networks, and Z. Li and co-authors in [20] solved the same with regard to desired orbit. It should be noted controlled synchronization in complex dynamical networks with either nonlinear delays or with coupling delays in [18]-[21] was studied via the methodology of Lyapunov stability analysis. In parallel, also the design of robust decentralized control for large-scale systems with time-varying or uncertain delays has been revisited via several approaches and feasible designs derived in [22]-[25]. These studies too have been carried out via Lyapunov stability analysis and synthesis. The approach via variable structure control (VSC) is to be noted for their efficiency in dealing with all sorts of time-delay and uncertainty phenomena in dynamical systems.

We consider a complex dynamical network consisting of N identical nodes (n dimensional dynamical systems) with time varying delay coupling

$$\dot{x}_i = Ax_i + f(x_i,t) + \sum_{j=1}^{N} A_{ij}x_j(t-\tau_{ij}(t)) + B_i u_i, \quad i = 1,2,\cdots,N \tag{1}$$

Where: $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in R^n$, represents the state vector of the i-th node; $f(x_i,t): R^n \times R \to R^n$ are smooth nonlinear vector function; $\tau_{ij}(t)$ is bounded time varying delay and differentiable too satisfying $0 \le \tau_{ij}(t) \le \tau_{ij} < \infty$, where τ_{ij} is positive scalar; and $\sum_{j=1}^{N} A_{ij}x_j(t-\tau_{ij}(t))$ represent the uncertain interconnections with time delay. Furthermore, $A \in R^{n\times n}$, $B_i \in R^{n\times m}$ are constant system matrices of appropriate dimensions, and $u_i \in R^m$ represents the control input.

When the network achieves synchronization, namely, the state $x_1 = x_2 = \cdots = x_N$, as $t \to \infty$, the coupling control terms should vanish: $\sum_{j=1}^{N} A_{ij}x_j(t-\tau_{ij}(t)) + B_i u_i = 0$. This ensures that any solution $x_i(t)$ of a single isolate node is also a solution of the synchronized coupled network.

Let $s(t)$ be a solution of the isolate node of the network, which is assumed to exist and is unique, satisfying:

$$\dot{s} = As + f(s, \quad t). \tag{2}$$

In here $s(t)$ can be an equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit. The objective of control here is to find some smooth controllers $u_i \in R^m$ such that the solution of systems (1) asymptotically synchronize with the solution of (2), in the sense that

$$\lim_{x\to\infty}\|x_i(t) - s(t)\| = 0, \; i = 1,2,\cdots,N \tag{3}$$

Let it be defined $e_i = x_i(t) - s$. Then subtracting (2) from (1) yields the error dynamical system

$$\dot{e}_i = Ae_i + \widetilde{f}(x_i,s) + \sum_{j=1}^{N} A_{ij}x_j(t-\tau_{ij}(t)) + B_i u_i \tag{4}$$

where

$$\widetilde{f}(x_i,s) = f(x_i,t) - f(s,t).$$

For deriving the proofs given in sequel, certain convenient assumptions are given next.

Assumption 1: The matrix pair $(A, \quad B_i)$ is controllable.

Assumption 2: Each input matrix B_i is of full rank.

Assumption 3: The nonlinear function f satisfying

$$\|f(x_i,t) - f(x_j,t)\| \le \mu_i \|x_i(t) - x_j(t)\| \tag{5}$$

where $\mu_i > 0$ are constants, $i, j = 1,2,\cdots,N$.

Assumption 4: Suppose the interconnection matrix satisfy matching condition as follow:

$$A_{ij} = B_i H_{ij} \tag{6}$$

Assumption 5: The time delay terms in system (4) satisfy

$$\|x_j(t-\tau_{ij}(t))\| \le x_{j\max}(t) \tag{7}$$

where

$$x_{j\max}(t) = \max\|x_j(t)\|.$$

Therefore equation (4) can be rewritten as follows:

$$\dot{e}_i = Ae_i + \widetilde{f}(x_i,s) + \sum_{j=1}^{N} B_i H_{ij}x_j(t-\tau_{ij}(t)) + B_i u_i \tag{8}$$

3. APPLICATION OF VSC AND CONSTRUCTING THE SWITCHING SURFACE

The composite sliding surface of system (8) is defined by letting the composite sliding vector $\sigma(e)$ in the state space be zero. This is to say that

$$\sigma(e) = \left[\sigma_1^T(e_1), \quad \sigma_2^T(e_2), \quad \cdots \quad \sigma_N^T(e_N)\right] \tag{9}$$

where

$$\sigma_i(e_i) = C_i e_i = 0, i = 1,\cdots,N \tag{10}$$

are called the local sliding surface and $e = \left[e_1^T, \quad \cdots, \quad e_N^T\right] \in R^N$, while C_i are $m \times n$ constant matrices to be determined in due course.

In order to construct the controller sought, the following two relevant lemmas are needed as well.

Lemma 1 [24]. Suppose β and $b_1, b_2, \cdots, b_N$ be arbitrary vectors, then

$$2a^T(\sum_{i=1}^{N} b) \le \beta a^T a + \frac{1}{\beta}\sum_{i=1}^{N} b_i^T b_i \tag{11}$$

where $\beta > 0$ is a positive constant.

Lemma 2[24]. Suppose matrix $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ is inverse, and $\|D_{22}\| \neq 0$, then

$$D^{-1} = \begin{pmatrix} D_{11,2}^{-1} & -D_{11,2}A_{12}A_{22}^{-1} \\ -D_{22}^{-1}D_{21}D_{11,2}^{-1} & \Psi \end{pmatrix} \quad (12)$$

where $D_{22}^{-1} + D_{22}^{-1}D_{21}D_{11,2}^{-1}D_{12}D_{22}^{-1} = \Psi$ and $D_{11,2}^{-1} = D_{11} - D_{12}D_{22}^{-1}D_{21}$ is the inverse.

Further, let the isolate subsystem as follows

$$\dot{e}_i = Ae_i + B_i u_i \quad (13)$$

Be selected. Because $(A, \ B_i)$ is controllable, there exists matrix $K_i \in R^{m\times n}$ that can make the matrix $\widetilde{A}_i = A + B_i K_i$ be stable. And B_i is full rank matrix, we can assume $B_i = \begin{pmatrix} 0 \\ \widetilde{B}_i \end{pmatrix}$, $\widetilde{B}_i \in R^{m\times m}$. When the controller $u_i = K_i e_i + v_i$ is substituted in to (13), the equation is transformed to

$$\dot{e}_{i1} = \widetilde{A}_{i11}e_{i1} + \widetilde{A}_{i12}e_{i2} \quad (14)$$

$$\dot{e}_{i2} = \widetilde{A}_{i21}e_{i1} + \widetilde{A}_{i22}e_{i2} + \widetilde{B}_i v_i \quad (15)$$

Assume the stability eigenvalues of $\widetilde{A}_i$ are $\lambda_{i1}, \cdots, \lambda_{im}, \mu_{i1}, \cdots, \mu_{in-m}$, and then define:

$$\Lambda_{i1} = \begin{pmatrix} \mu_{i1} & & \\ & \ddots & \\ & & \mu_{in-m} \end{pmatrix}, \Lambda_{i2} = \begin{pmatrix} \lambda_{i1} & & \\ & \ddots & \\ & & \lambda_{im} \end{pmatrix} \quad (16)$$

The corresponding eigenvectors constitute the eigenvector matrix as $\begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix}$. The eigenvector matrix is the inverse through the pole placement, so that the following equation holds

$$\begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix}\begin{pmatrix} \widetilde{A}_{i11} & \widetilde{A}_{i12} \\ \widetilde{A}_{i21} & \widetilde{A}_{i22} \end{pmatrix} = \begin{pmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{pmatrix}\begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix} \quad (17)$$

$$\begin{pmatrix} \widetilde{A}_{i11} & \widetilde{A}_{i12} \\ \widetilde{A}_{i21} & \widetilde{A}_{i22} \end{pmatrix}\begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix}^{-1} = \begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix}^{-1}\begin{pmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{pmatrix} \quad (18)$$

$$\begin{pmatrix} G_{i1} & G_{i2} \\ V_{i1} & V_{i2} \end{pmatrix}^{-1} = \begin{pmatrix} \xi_{i1} & \xi_{i2} \\ \eta_{i1} & \eta_{i2} \end{pmatrix} \quad (19)$$

$$\begin{pmatrix} \widetilde{A}_{i11} & \widetilde{A}_{i12} \\ \widetilde{A}_{i21} & \widetilde{A}_{i22} \end{pmatrix}\begin{pmatrix} \xi_{i1} & \xi_{i2} \\ \eta_{i1} & \eta_{i2} \end{pmatrix} = \begin{pmatrix} \xi_{i1} & \xi_{i2} \\ \eta_{i1} & \eta_{i2} \end{pmatrix}\begin{pmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{pmatrix} \quad (20)$$

Therefore

$$\widetilde{A}_{i11}\xi_{i1} + \widetilde{A}_{i12}\eta_{i1} = \xi_{i1}\Lambda_{i1} \quad (21)$$

From the above, we know that $\begin{pmatrix} \xi_{i1} & \xi_{i2} \\ \eta_{i1} & \eta_{i2} \end{pmatrix}$ is the right eigenvector matrix of $\begin{pmatrix} \widetilde{A}_{i11} & \widetilde{A}_{i12} \\ \widetilde{A}_{i21} & \widetilde{A}_{i22} \end{pmatrix}$. If we select

$$C_i = \begin{bmatrix} V_{i1} & V_{i2} \end{bmatrix} \quad (22)$$

when the system trajectory hit the sliding mode, i.e. $\sigma_i = C_i e_i = V_{i1}e_{i1} + V_{i2}e_{i2}$, then

$$e_{i2} = -V_{i2}^{-1}V_{i1}e_{i1} \quad (23)$$

Substitution of (23) into (14) yields the sliding mode equation

$$\dot{e}_{i1} = (\widetilde{A}_{i11} - \widetilde{A}_{i12}V_{i2}^{-1}V_{i1})e_{i1} \quad (24)$$

Because $V_{i2} \in R^{m\times m}$ and $(G_{i1} - G_{i2}V_{i2}^{-1}V_{i1})$ are inverse, and due to Lemma 2, it follows

$$\eta_{i1} = -V_{i2}^{-1}V_{i1}\xi_{i1} \quad (25)$$

Also upon substitution of (25) into (21) yields

$$(\widetilde{A}_{i11} - \widetilde{A}_{i12}V_{i2}^{-1}V_{i1})\xi_{i1} = \xi_{i1}\Lambda_{i1} \quad (26)$$

From the above we can know the eigenvalues of the sliding mode equation of system (13) represent the desired $n-m$ stable eigenvalues. It is obvious that the sliding mode equation (14) is stable. From the above analysis, C_i is the left eigenvector of $\widetilde{A}_i$ with desired m stable eigenvalues, then

$$C_i\widetilde{A}_i = \Lambda_{i2}C_i \quad (27)$$

For the error complex system (4), because the coupling term is satisfying the matching condition, the sliding mode equation of system (4) is still satisfying equation (24), which has the desired eigenvalues. If the nonlinear and the coupling terms do not satisfy the matching condition, the system (4) can be written as follows

$$\dot{e}_{i1} = \widetilde{A}_{i11}e_{i1} + \widetilde{A}_{i12}e_{i2} + \widetilde{f}_{i1}(x_i, s) \quad (28)$$

$$\dot{e}_{i2} = \widetilde{A}_{i21}e_{i1} + \widetilde{A}_{i22}e_{i2} + \widetilde{f}_{i2}(x_i, s) + \sum_{j=1}^{N}\widetilde{B}_i\widetilde{H}_{ij}x_j(t - \tau_{ij}(t)) + \widetilde{B}_i v_i \quad (29)$$

When $\sigma_i = 0$, then $e_{i2} = -V_{i2}^{-1}V_{i1}e_{i1}$, so the above sliding mode equation of system (28) is

$$\dot{e}_{i1} = \overline{A}_i e_{i1} + \widetilde{f}_{i1}(x_i, s) \quad (30)$$

where $\overline{A}_i = \widetilde{A}_{i11} - \widetilde{A}_{i12}V_{i2}^{-1}V_{i1}$. It is obvious, matrix $\overline{A}_i$ is stable.

The synchronization condition for the complex network with unmatched uncertainty is given according the next theorem, the first novel result.

Theorem 1. If the nonlinear term $\widetilde{f}(x_i, s)$ is unmatched, then the decentralized sliding mode of the interconnected system (30) is asymptotically stable, if and only if the inequality

$$k_1\mu_i < -\beta_i(\lambda_i + \beta_i) \tag{31}$$

holds true, where $k_1 > 0$ is constant, and $\lambda_i = \max\{\lambda_{i1}, \ \cdots \ \lambda_{im}\} < 0$.

Proof: Let V be a candidate Lyapunov function for the dynamic system (30),

$$\dot{V} = \sum_{i=1}^{N} e_{i1}^T e_{i1} \tag{32}$$

Taking the derivative of V along the trajectory of system (30) yields

$$\dot{V} = \sum_{i=1}^{N} 2e_{i1}^T\left[\overline{A}e_{i1} + \widetilde{f}_{i1}(x_i, s)\right] \tag{33}$$

Due to Assumption 3, there exist positive constant k_1 that makes $\left\|\widetilde{f}_{i1}(x_i, s)\right\| \le k_1\mu_i\|e_{i1}\|$. Thus

$$\begin{aligned}\dot{V} &\le \lambda_i\|e_{i1}\|^2 + \beta_i\|e_{i1}\|^2 + \frac{1}{\beta_i}k_1\mu_i\|e_{i1}\|^2 \\ &= \left(\lambda_i + \beta_i + \frac{1}{\beta_i}k_1\mu_i\right)\|e_{i1}\|^2\end{aligned} \tag{34}$$

due to Lemma 1. And then because $\lambda_i < 0$, if $\beta_i + \frac{1}{\beta_i}k_1\mu_i < -\lambda_i$ it follows $\dot{V} < 0$ at once.

4. DESIGNING SYNCHRONIZATION CONDITION: CASE WITH KNOWN BOUNDS ON NONLINEAR TERMS

Although we have a set of stable sliding surfaces, unless the initial states and all system dynamics are always ensured to stay on the surface for all time, a set of decentralized sliding controllers is required, such that the global robust stability of the surface is assured. Traditionally, the hitting condition for small-scale system is

$$\sigma(t)^T\dot{\sigma}(t) < 0 \tag{35}$$

where $\sigma(t) = 0$ is the sliding surface of some small-scale systems. Since the existence of interconnections and the lack of global information, equation (35) is not easily satisfied for the interconnected system. Hence, we require a global hitting condition of the sliding surface

$$\sum_{i=1}^{N}\frac{\sigma_i^T(e_i)\dot{\sigma}_i(e_i)}{\|\sigma_i(e_i)\|} < 0 \tag{36}$$

If $V = \sum_{i=1}^{N}\|\sigma_i\|$, the condition is readily derived from the stability theory of Lyapunov .

Theorem 2: The motion of the system (4) asymptotically converges to the composite sliding surface $\sigma(e) = 0$, if and only if the following condition

$$u_i = K_ie_i - (C_iB_i)^{-1}R_i\frac{\sigma_i}{\|\sigma_i\|}\|e_i\|, \tag{37}$$

where

$$R_i = \mu_i\|C_i\| + \sum_{j=1}^{N}\|C_iB_i\|\|H_{ij}\|\|x_{j\max}(t)\| + \varepsilon$$

and $\varepsilon > 0$ is constant, is satisfied.

Proof: From (4) and (10), the sliding dynamics can be written as

$$\begin{aligned}\dot{\sigma}_i &= C_i\dot{e}_i \\ &= C_iAe_i + C_i\widetilde{f}(x_i, s) + C_i\sum_{j=1}^{N}B_iH_{ij}x_j(t-\tau) + C_iB_iu_i \\ &= \Lambda_{i2}\sigma_i - C_iB_iK_ie_i + C_iB_iu_i \\ &\quad + C_i\widetilde{f}(x_i, s) + C_i\sum_{j=1}^{N}B_iH_{ij}x_j(t-\tau)\end{aligned} \tag{38}$$

Upon substitution of (37) into (38), the (38) can be written down as

$$\begin{aligned}\dot{\sigma} &= \Lambda_{i2}\sigma_i - R_i\frac{\sigma_i}{\|\sigma_i\|}\|e_i\| + C_i\widetilde{f}(x_i, s) \\ &\quad + C_i\sum_{j=1}^{N}B_iH_{ij}x_j(t-\tau).\end{aligned} \tag{39}$$

Then construct Lyapunov function as

$$V = \sum_{i=1}^{N}d_i\|\sigma_i\| \tag{40}$$

and obtain the time derivative of (40) as

$$\dot{V} = \sum_{i=1}^{N}d_i\frac{\sigma_i^T\dot{\sigma}_i}{\|\sigma_i\|} \tag{41}$$

Substitution of (37) and (39) into (41) yields

$$\dot{V} = \sum_{i=1}^{N}d_i\frac{\sigma_i^T}{\|\sigma_i\|}[V_\Sigma] \tag{42-a}$$

$$[V_\Sigma] = \left[\Lambda_{i2}\sigma_i + C_i\widetilde{f}(x_i, s) + \sum_{j=1}^{N}C_iB_iH_{ij}x_j(t-\tau_{ij}) - R_i\frac{\sigma_i}{\|\sigma_i\|}\|e_i\|\right] \tag{42-b}$$

Because of $\lambda_i = \max\{\lambda_{i1}, \cdots, \lambda_{im}\} < 0$ and Assumption 2, it follows that

$$\dot{V} \le \sum_{i=1}^{N} \begin{bmatrix} d_i \lambda_i \|\sigma_i\| + d_i \|C_i\| \mu_i \|e_i\| \\ + d_i \sum_{j=1}^{N} \|C_i B_i\| \|H_{ij}\| \|x_j(t-\tau_{ij})\| - d_i R_i \|e_i\| \end{bmatrix}$$

$$= \sum_{i=1}^{N} [d_i \lambda_i \|\sigma_i\|]$$
$$- \sum_{i=1}^{N} d_i \left[-\|C_i\| \mu_i \|e_i\| - \sum_{j=1}^{N} \|C_i B_i\| \|H_{ij}\| \|x_j(t-\tau_{ij})\| + R_i \|e\| \right]$$

$$= \sum_{i=1}^{N} d_i \lambda_i \|\sigma_i\| + \sum_{i=1}^{N} \sum_{j=1}^{N} d_i \|C_i B_i\| \|H_{ij}\| \|x_j(t-\tau_{ij})\|$$
$$- \sum_{i=1}^{N} \sum_{j=1}^{N} d_i \|C_i B_i\| \|H_{ij}\| \|x_{j\max}(t)\| - \varepsilon < 0. \tag{43}$$

Thus, on the grounds of the designed controller according to Theorem 2, the motion of the system (4) asymptotically converges to the composite sliding surface.

5. Designing Synchronization Condition: Case with Unknown Bounds on Nonlinear Terms

In practical terms, there exist $\|\tilde{f}(x_i, s)\| \le \mu_i \|e_i\|$, where μ_i represents unknown parameters. In this section, we will design robust adaptive controller with unknown parameters. In order to derive the proof, conveniently, first another two assumptions are presented.

Assumption 6: Let $rank(\tilde{f}, \quad B_i) = rank(B_i)$.

Assumption 7: Let $B_i = \begin{pmatrix} 0 \\ B_{2i} \end{pmatrix}$, where $B_{2i} \in R^{m \times m}$ is an nonsingular matrix.

When the transformation $u_i = K_i e_i + v_i$ is selected, then the sliding mode equation becomes

$$\dot{e}_{i1} = (\tilde{A}_{i11} - \tilde{A}_{i12} V_{i2}^{-1} V_{i1}) e_{i1}. \tag{44}$$

It is easy to prove the asymptotic stability of the sliding mode trajectory by the constructing switching function. Therefore the main task here is to design a robust controller that guarantees the system trajectory shall reach the sliding surface from arbitrary initial state.

Theorem 3 Let Assumption 4 and Assumption 5 hold true. Then with

$$\dot{\hat{\mu}}_i = \|C_i\| \|e_i\|, \; \tilde{\mu}_i = \hat{\mu}_i - \mu_i, \tag{45-a}$$

the following robust adaptive controller

$$u_i = K_i e_i - (C_i B_i)^{-1} \begin{bmatrix} C_i \hat{\mu}_i \|e_i\| \\ + \sum_{j=1}^{N} \|C_i B_i\| \|H_{ij}\| \|x_{j\max}\| + \varepsilon_i \operatorname{sgn} \sigma_i \end{bmatrix}, \tag{45-a}$$

where $\hat{\mu}_i$ is the estimate of the unknown parameter, and μ_i, $\varepsilon_i > 0$ are constants, uniformly, asymptotically stabilize the system (4) in the large.

<u>Proof</u>: Consider the Lyapunov function as follows

$$V = \sum_{i=1}^{N} \|\sigma_i\| + \frac{1}{2} \sum_{i=1}^{N} \tilde{\mu}_i^2 \tag{46}$$

The time derivative of (46) is as follows:

$$\dot{V} = \sum_{i=1}^{N} \frac{\sigma_i^T(e_i) \dot{\sigma}_i(e_i)}{\|\sigma_i\|} + \sum_{i=1}^{N} \tilde{\mu}_i \dot{\hat{\mu}}_i$$

$$= \sum_{i=1}^{N} \frac{\sigma_i^T}{\|\sigma_i\|} \begin{bmatrix} \Lambda_{i2} \sigma_i - C_i B_i K_i e_i \\ + C_i B_i u_i + C_i \tilde{f}(x_i, s) \\ + C_i \sum_{j=1}^{N} B_i H_{ij} x_j(t-\tau_{ij}) \end{bmatrix} + \sum_{i=1}^{N} \tilde{\mu}_i \dot{\hat{\mu}}_i$$

$$\le \sum_{i=1}^{N} \begin{bmatrix} \lambda_i \|\sigma_i\| - C_i B_i K_i e_i + C_i \mu_i \|e_i\| \\ + \sum_{j=1}^{N} \|C_i B_i\| \|H_{ij}\| \|x_j(t-\tau_{ij})\| + C_i B_i u_i \end{bmatrix}$$
$$+ \sum_{i=1}^{N} \tilde{\mu}_i \dot{\hat{\mu}}_i \le \sum_{i=1}^{N} \lambda_i \|\sigma_i\| - \varepsilon_i. \tag{47}$$

Because of $\lambda_i < 0$, $\varepsilon_i > 0$, apparently (47) is negative. Thus, the system (4) can be stabilized by means of the controller (45-a, b) , which is designed according to Theorem 3.

6. Illustrative Examples and Simulation Results

6.1. The complex network system with known bound on nonlinear term

The chaotic Chua circuit is assumed in the nodes of the complex dynamical network. A singular Chua circuit is described by the piecewise-linear system

$$\dot{x} = p(-x + y - f(x)), \; \dot{y} = x - y + z, \; \dot{z} = -qy$$

where

$$f(x) = m_0 x + \frac{1}{2}(m_1 - m_0)(|x+1| - |x-1|)$$

with constants $m_0 < 0$ and $m_1 < 0$, $p = 10$, $q = 14.87$, $m_0 = -1.27, m_1 = -0.68$. Let it be $x_1 = x, x_2 = y, x_3 = z$. Then this Chua circuit can also be represented as follows:

$$\dot{x}_1 = p(-x_1 + x_2 - f(x_1)), \dot{x}_2 = x_1 - x_2 + x_3,$$
$$\dot{x}_3 = qx_2.$$

The corresponding complex network with coupling time-delay is represented by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} -p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} -pf(x_{i1}) \\ 0 \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} \sum_{k=i}^{i+2} x_{k,1}(t - \tau_{ij}) \\ 0 \\ \sum_{k=i}^{i+2} x_{k,3}(t - \tau_{ij}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_i$$

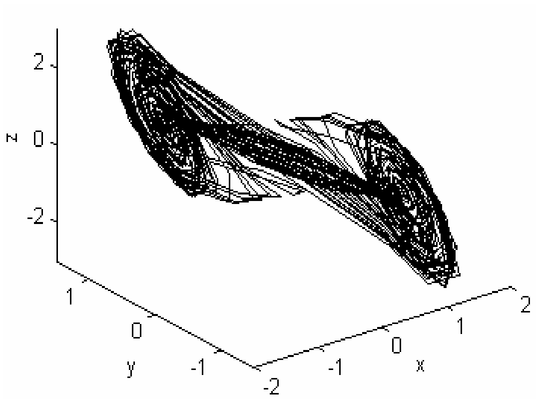


Fig. 3 Trajectories of chaotic Chua circuit in its 3D phase space.

In order to simulate it conveniently, it has been assumed $\tau_{ij} < 0.02$. On the grounds of Theorem 1, the synchronization error trajectory of chaotic Chua circuit can be computer simulated to give the results on synchronization errors as depicted in 3 different figures; only Figure 4 is given due to paper size limits. These results show the synchronization has been enforced rather efficiently by synchronisation employing the proposed variable structure control design.

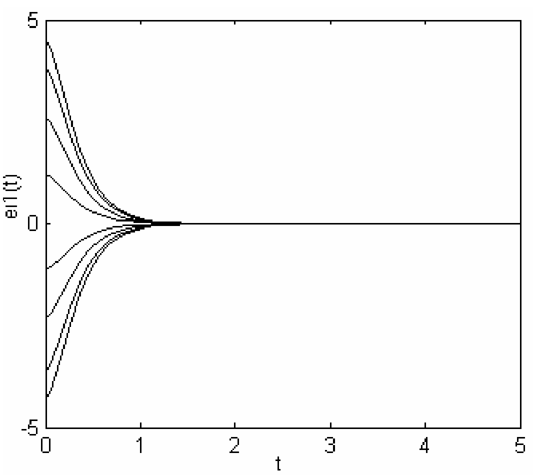


Fig. 4 Synchronization errors e_{i1} of chaotic Chua circuit.

6.2. The complex network system with unknown bound nonlinear term

The Duffing forced-oscillation system is used as nodes in a dynamical network. A singular Duffing forced-oscillation system is described as

$$\dot{x} = y, \ \dot{y} = -0.1y - x^3 + 12\cos t$$

Let it be defined $x_1 = x$, $x_2 = y$. Then the Duffing forced-oscillation system can also be expressed as follows:

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -0.1x_2 - x_1^3 + 12\cos t$$

Its phase portrait is depicted in the next figure.

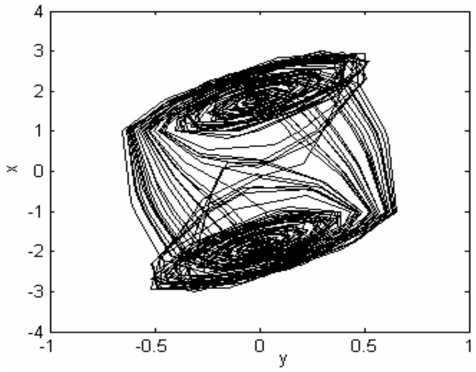


Fig. 5 Trajectory of Duffing forced-oscillation system in its 2D phase space.

The corresponding network is described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}^3 + 12\cos t \end{pmatrix}$$
$$+ \begin{pmatrix} \sum_{k=i}^{i+2} x_{k,1}(t - \tau_{ij}) \\ \sum_{k=i}^{i+2} x_{k,1}(t - \tau_{ij}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i$$

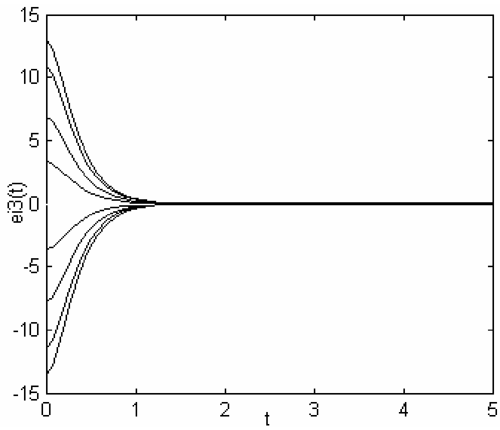


Fig. 6 The synchronization errors of the Duffing forced-oscillation system.

By using making use of the controller in Theorem 3, the synchronization error trajectories of Duffing forced-oscillation system has been simulated and the error signals are depicted in Figure 8. These simulation results show the network synchronization by the designed variable structure controller is enforced rather efficiently.

7. CONCLUSION

The synchronization problem in coupled complex dynamic delay network has been explored. Both systems with known bound and with unknown bound on nonlinear terms have been successfully taken into consideration. Stability solutions to synchronization are proposed that employ variable structure control theory along with constructive design of the sliding surfaces and the switching. The switching surface has been designed via the left eigenvector function of the system, which can assure the synchronization sliding mode possesses stability. The hitting condition and the adaptive law for estimating the unknown network parameter have been used for designing the controller, which can assure the network state hitting the switching manifold in finite time.

The two benchmark examples, based on employing chaotic Chua circuit and on Duffin's oscillator at the nodes, have been used for illustrating the achieved performance via synchronization error trajectories. The respective computer simulations demonstrated both efficient network synchronization as well as quality transient performance.

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