A consensus protocol for multi-agent systems with diverse communication delays

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***Abstract*— In this paper, a protocol is proposed to solve the consensus problem of multi-agent systems with diverse communication delays. Sufficient conditions for convergence to a consensus are obtained based on the frequency-domain analysis and matrix theory. The conditions depends on each agent's self-delay, the weights of the edges to each agent's neighbors, and the interconnection topology of the network. Under the proposed protocol the communication delays do not influence the convergence; but prolong the converging time. Simulation results illustrate the correctness of the results.**

I. Introduction

The consensus problem for multi-agent systems has attracted more and more attention. Vicsek *et al.* proposed a simple discrete-time model of multi-autonomous agents, and provided various simulations which demonstrated the phenomena: without any central coordination control, all the agents in the model move in the same direction when the density is large and the noise is small [9]. Jadbabaie *et al.* studied the linearized Vicsek's Model and proved that all the agents converge to a common steady state provided that the diagraph formed by the agents is jointly connected, i.e., the agents are all "linked together" via their neighbors with sufficient frequency as the system evolves [2].

With non-negligible communication delays, the consensus problem becomes much more difficult. It is a natural idea to introduce self-delays in the consensus protocol and the self-delays are usually chosen to be equal to the communication delays (see, e.g., [8]. But such a protocol cannot be robust because the measurement of communication delays always contain some uncertainty. Moreover, the analysis of the stability or the convergence of the protocol is very difficult. Some stability results were obtained only for the multi-agent system with an identical communication delay [8], [6]. Based on the contraction theory and wave variable method, Wang and Slotine studied the consensus problem for the system with multi-variable agents under diverse communication delays [10]. They proposed a simple consensus protocol with zero self-delay, which is robust to arbitrary communication delays. However, the topology graph in their analysis should connected and bidirectional or unidirectional formed in closed rings.

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In this paper, we propose a new consensus protocol to solve the consensus problem of the multi-agent system with diverse communication delays. In our protocol the self-delay is uniform but it can be non-zero. We show that introducing a non-zero self-delay can speed up the convergence rate for the system with non-zero communication delays. Moreover, we show the protocol can be applied to networks with directed topology graphs and nonsymmetric weights. With the help of the frequency-domain method developed in [3], we analyzed the effect of both the communication delays and the self-delay on the convergence of the multi-agent system. Sufficient conditions for the multi-agent system converging to a consensus are obtained. These conditions depends on each agent's self-delay, the weights of the edges to each agent's neighbors, and the interconnection topology of the network. Under the proposed protocol the communication time-delays do not influence the convergence; but they prolong the converging time.

II. Preliminary

A weighted directed graph (digraph) $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order n consists of a set of vertices $\mathcal{V} = \{v_1, ..., v_n\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ with nonnegative adjacency elements a_{ij}. The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, ..., n\}$. An edge of the weighted diagraph $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j) \in \mathcal{E}$, i.e., e_{ij} is a directed edge from v_i to v_j. We assume that the adjacency elements associated with the edges of the digraph are positive, i.e., $a_{ij} > 0 \Leftrightarrow e_{ij} \in \mathcal{E}$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$.

In the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the out-degree of node i is defined as follows:

$$\deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij}.$$

Let $\mathcal{D}$ be the diagonal matrix with the out-degree of each node along the diagonal and call it the degree matrix of $\mathcal{G}$. The Laplacian matrix of the weighted digraph is defined as $L = \mathcal{D} - \mathcal{A}$.

Following [4] we introduce some important notions for digraph. A *path* on a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of length N from v_{j0} to v_{jN} is an ordered set of distinct nodes $\{v_{j_0}, v_{j_1}, ..., v_{j_N}\}$

such that $(v_{j_{i-1}}, v_{j_i}) \in \mathcal{E}, \forall i = 1, 2, ...N$. If there is a path in $\mathcal{G}$ from one node v_i to another node v_j, then v_j is said to be *reachable* from v_i, written $v_i \rightarrow v_j$. If not, then v_j is said to be not reachable from v_i, written $v_i \nrightarrow v_j$. If a node is reachable from every other node in the digraph, then we say it *globally reachable*. A digraph is *strongly connected* if every two of its nodes, say v and u, are such that v is reachable from u and u is reachable from v. Thus, a globally reachable node is precisely the degree of connectedness required and is much weaker than strong connectedness of the digraph.

In this paper, we just consider static topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, i.e., the connection of the nodes in the diagraph $\mathcal{G}$ does not change with time.

III. Consensus Protocol

In a multi-agent system with n agents, each agent can be considered as a node in a digraph, and the information flow between two agents can be regarded as a directed path between the nodes in the digraph. Thus, the interconnection topology in a multi-agent system can be described as a diagraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$.

Consider a discrete-time model of integrator agents

$$x_i(k+1) = x_i(k) + u_i(k), i \in \mathcal{I}, \tag{1}$$

where $x_i(k) \in R$ and $u_i(k) \in R$ denote the state and the control input of agent i, respectively.

The following consensus protocol for the multi-agent system (1) has been extensively studied in the literature (see, e.g., [8])

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j - x_i),$$

where N_i denotes the neighbors of agent i, and $a_{ij} > 0$ is the adjacency element of $\mathcal{A}$ in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$.

For networks with non-negligible communication delays, the following time-delayed consensus protocol was adapted in [8]

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k - D_{ij}) - x_i(k - D_{ij})), \tag{2}$$

where communication delay $D_{ij} > 0$ corresponds to information flow from agent j to agent i, i.e., the edge $e_{ij} \in \mathcal{E}$ in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. However, the consensus condition was obtained only for the identical communication delays, i.e., $D_{ij} = d$. Reference [10] proposed another consensus protocol without self-delay

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k - D_{ij}) - x_i(k)). \tag{3}$$

But they only analyzed the connected and bidirectional topology graph with symmetric weights (i.e., $a_{ij} = a_{ji}$) and the unidirectional graph formed in closed rings with identical weights.

Differing from the protocols above, we propose a consensus protocol with a uniform self-delay to solve the consensus problem of the multi-agent system (1) with diverse communication delays, which is given by

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k - D_{ij}) - x_i(k - D)), \tag{4}$$

where $D \geq 0$ is the self-delay which is uniform for all the agents. Obviously, protocol (4) is a compromise between (2) and (3).

With the consensus protocol (4), the closed form of the multi-agent system (1) is

$$\begin{aligned} &x_i(k+1) = x_i(k) + \\ &\textstyle\sum_{v_j \in N_i} a_{ij}(x_j(k - D_{ij}) - x_i(k - D)), \quad i \in \mathcal{I}. \end{aligned} \tag{5}$$

Theorem 1. Consider a network of coupled n agents (5) with a static interconnection topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that has a globally reachable node. If

$$\sum_{v_j \in N_i} a_{ij} < \frac{1}{2D+1}, \forall\ i \in \mathcal{I}, \tag{6}$$

then system (5) has an asymptotic consensus, i.e.,

$$\lim_{k \to \infty} x_i(k) = c, \forall\ i \in \mathcal{I},$$

where c is a constant.

Applying Theorem 1 to typical discrete-time models of multi-agent systems, such as linearized Vicsek's model [2] and Moreau model [7], we can extend some existing results on the consensus problem to the case with communication delays.

With communication delays, the linearized Vicsek's Model proposed by [2] becomes

$$x_i(k+1) = \frac{1}{1+n_i}(\sum_{v_j \in N_i} x_j(k - D_{ij}) + x_i(k)), \quad i \in \mathcal{I}, \tag{7}$$

where n_i denotes the number of the neighbors of agent i.

Corollary 1. If the interconnection topology of (7) has a globally reachable node, then the system (7) has an asymptotic consensus.

Similarly, one can extend the Moreau's Model [7] to the case with communication delays

$$\begin{aligned} &x_i(k+1) = \\ &\tfrac{1}{1+\sum_{v_j \in N_i} w_{ij}}(\textstyle\sum_{v_j \in N_i} w_{ij} x_j(k - D_{ij}) + x_i(k)), \\ &i \in \mathcal{I}, \end{aligned} \tag{8}$$

where w_{ij} denotes the positive weight corresponding to the edge e_{ij} in the diagraph $\mathcal{G}$.

Corollary 2. If the interconnection topology of 8) has a globally reachable node, then the system (8) has an asymptotic consensus.

Example 1: Consider a network of six agents described by (5). The interconnection topology is sketched in Fig. 1.

Based on the definition, the globally reachable node set of the diagraph in Fig. 1 is $\{2, 3, 6\}$, but the diagraph is not strongly connected. The weights of the directed paths are: $a_{12} = 0.1$, $a_{16} = 0.05$, $a_{23} = 0.15$, $a_{36} = 0.1$, $a_{43} = 0.05$, $a_{45} = 0.1$, $a_{56} = 0.15$, $a_{62} = 0.15$, and the corresponding communication delays are: $D_{12} = 5$(step), $D_{16} = 3$(step),

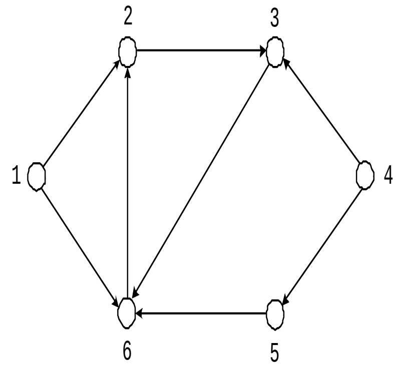

Fig. 1. The digraph of a group of 6 agents.

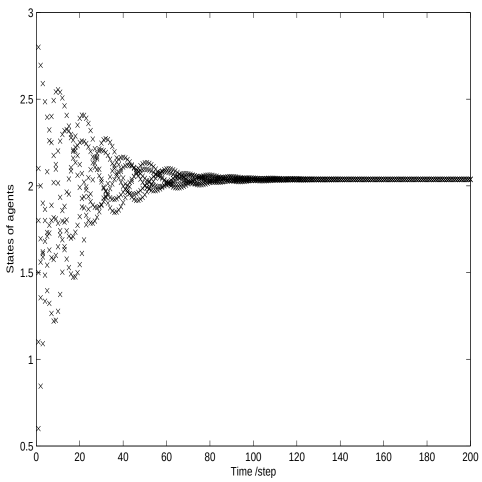

Fig. 2. Consensus with communication delays.

$D_{23} = 4$(step), $D_{36} = 4$(step), $D_{43} = 4$(step), $D_{45} = 6$(step), $D_{56} = 6$(step), $D_{62} = 5$(step). From Theorem 1, we get the delay $D \leq 2$(step) for every agent, and we choose $D = 2$(step) in this simulation. The initial states are generated randomly, and the multi-agent system converges to a consensus as in Fig. 2.

Theorem 1 implies that the multi-agent system (5) converges to a consensus without any relationship with the communication delays, but these delays have an impact on the dynamical performance for given initial states (e.g., the rate of the converging). For example, we change the delay D_{62} with the initial states as: $x(k) = [x_1(k), x_2(k), ..., x_6(k)]^T = [0.6, 2.8, 2.1, 1.5, 1.8, 1.1]^T$, $-\max\{D_{ij}, D\} \leq k \leq 0$, and the time of converging (i.e., the time when the algebraic sum of the absolute neighboring states' errors begins to be always less than 10^{-2}) is ranked in the following Table. 1.

TABLE I

INFLUENCE OF THE COMMUNICATION DELAYS

Delay D_{62}(step)	7	10	15	20	30	$\cdots$
Converging Time(step)	167	222	317	444	717	$\cdots$

As in the continuous-time model, we get the same conclusion from Table. 1 that the rate of the converging decreases as the communication delays increase.

IV. PROOF OF THEOREM 1

Before giving the proof of Theorem 1, we consider a discrete-time system with time delays

$$x(k+1) = x(k) + \sum_{i=1}^{n_d} A_i x(k - D_i), \tag{9}$$

where $x(k) \in R^n$, $A_i \in R^{n\times n}$ and $D_i \in R$. Taking the z-transformation, and we get the characteristic equation of system (9) as

$$\det((z-1)I - \sum_{i}^{n_d} A_i z^{-D_i}) = 0. \tag{10}$$

Lemma 1. If the roots of equation (10) have modulus less than unity except for a root at $z = 1$, then the equilibrium set $E = \{x \in R^n : (\sum_{i=1}^{n_d} A_i)x = 0\}$ of system (9) is asymptotically stable.

Proof. By the set stability theory, we need to show the Lyapunov stability and asymptotic attractivity of the set E. Since all the roots of the characteristic equation have nonnegative parts, the Lyapunov stability is obvious. Now we show its asymptotic attractivity.

Denote the roots of equation (10) by λ_i, $i = 0, 1, \cdots, m$, which satisfy $\lambda_0 = 1$ and $|\lambda_i| < 1, i = 1, \cdots, m$. Then, any solution of system (9) is given by

$$x(k) = c_0 P_0 \lambda_0^k + \sum_{i=1}^{m} c_i P_i \lambda_i^k, \tag{11}$$

where $P_i \in R^n$ is the eigenvector of λ_i, and $c_i \in R$ is a constant determined by initial conditions. Since $\lambda_0 = 1$ and $|\lambda_i| < 1, i = 1, \cdots, m$, the solution $x(k)$ given by (11) converges $c_0 P_0$ asymptotically as $k \to \infty$. Because $c_0 P_0 \lambda_0^k$ is also a solution of system (9), we obtain

$$(\sum_{i=1}^{n_d} A_i) c_0 P_0 = 0. \tag{12}$$

This implies that $c_0 P_0 \in E$. The asymptotic asttactivity of E is thus proved. Therefore, the set E is asymptotically stable. $\square$

To complete the proof of Theorem 1, we need the following two lemmas.

Lemma 2. $\sin(\frac{\pi}{2(2D+1)}) \geq \frac{1}{2D+1}$ holds for any nonnegative integer D.

Proof. Denote $x = \frac{1}{2D+1}$. Then we have $x \in (0, 1]$ for any nonnegative integer D. Thus, Lemma 2 is equivalent to the well-known inequlity $\sin(\frac{\pi}{2}x) \geq x$, $x \in (0, 1]$. $\square$

Lemma 3. The following inequality

$$\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \leq 2D + 1$$

holds for all nonnegative integers D and all $\omega \in [-\pi, \pi]$.

Proof. First of all, we note that

$$\lim_{\omega\to 0}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})}=2D+1.$$

In the following we prove

$$\frac{\sin(\frac{2D+1}{2}\omega)}{\sin\frac{\omega}{2}}\leq 2D+1 \tag{13}$$

holds for all nonnegative integers D and $\omega\in[-\pi,0)\cup(0,\pi]$. Since

$$\frac{\sin(-\frac{2D+1}{2}\omega)}{\sin(-\frac{\omega}{2})}=\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})},$$

we just need to prove (13) for all $\omega\in(0,\pi]$.

When $\omega\in(0,\frac{\pi}{2D+1}]$, let $h(\omega)=\sin(\frac{2D+1}{2}\omega)-(2D+1)\sin(\frac{\omega}{2})$. Calculating the derivative of $h(\omega)$ on ω yields

$$\dot{h}(\omega)=\frac{2D+1}{2}(\cos(\frac{2D+1}{2}\omega)-\cos(\frac{\omega}{2})).$$

Since $\omega\in(0,\frac{\pi}{2D+1}]$, we have $0<\frac{\omega}{2}\leq\frac{2D+1}{2}\omega\leq\frac{\pi}{2}$ for all nonnegative integers D. Thus, $\cos(\frac{2D+1}{2}\omega)\leq\cos(\frac{\omega}{2})$ holds for $\omega\in(0,\frac{\pi}{2D+1}]$. Therefore, $\dot{h}(\omega)\leq 0$, i.e., $h(\omega)$ is not increasing for all $\omega\in(0,\frac{\pi}{2D+1}]$. Since $h(0)=0$, we have $h(\omega)\leq 0$, i.e., $\sin(\frac{2D+1}{2}\omega)\leq(2D+1)\sin(\frac{\omega}{2})$ for all $\omega\in(0,\frac{\pi}{2D+1}]$. Because $\sin(\frac{\omega}{2})>0$ for $\omega\in(0,\frac{\pi}{2D+1}]$, we get $\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})}\leq 2D+1$ for all $\omega\in(0,\frac{\pi}{2D+1}]$ and all nonnegative integers D.

When $\omega\in(\frac{\pi}{2D+1},\pi]$, we have $\sin(\frac{\omega}{2})>\sin(\frac{\pi}{2(2D+1)})>0$ for all nonnegative integers D. So, from Lemma 2, we get

$$\begin{aligned}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} &\leq \frac{1}{\sin(\frac{\omega}{2})}\\ &< \frac{1}{\sin(\frac{\pi}{2(2D+1)})}\\ &\leq 2D+1\end{aligned}$$

for all $\omega\in(\frac{\pi}{2D+1},\pi]$ and all nonnegative integers D.

Lemma 3 is proved. □

Now, we give the proof of Theorem 1 as follows.

Taking the z-transformation of the system (5), we get

$$\begin{aligned}&zX_i(z)=X_i(z)+\\ &\textstyle\sum_{v_j\in N_i}a_{ij}(X_j(z)z^{-D_{ij}}-X_i(z)z^{-D}),\\ &i\in\mathcal{I},\end{aligned} \tag{14}$$

where $X_i(z)$ is the z-transformation of $x_i(k)$. Define a $n\times n$ matrix $\tilde{L}(z)=\{\tilde{l}_{ij}(z)\}$ as follows:

$$\tilde{l}_{ij}(z)=\begin{cases}-a_{ij}z^{D-D_{ij}}, & v_j\in N_i;\\ \sum_{v_j\in N_i}a_{ij}, & j=i;\\ 0, & \text{otherwise.}\end{cases}$$

and $\tilde{L}(1)=L$, which is the Laplacian matrix. Then, (14) can be written as

$$zX(z)=X(z)-z^{-D}\tilde{L}(z)X(z),$$

where $X(z)=[X_1(z),X_2(z),...,X_n(z)]^T$. The characteristic equation is

$$\det((z-1)I+z^{-D}\tilde{L}(z))=0.$$

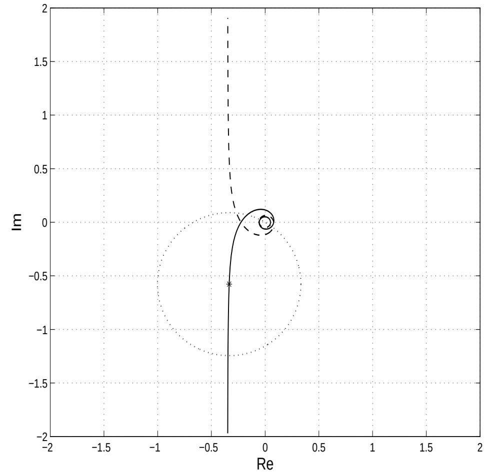

Fig. 3. Nyquist plot of $G(\omega)$.

Define $p(z)=\det((z-1)I+z^{-D}\tilde{L}(z))$. Then, we will prove that all the zeros of $p(z)$ have modulus less than unity except for a zero at $z=1$ in the following.

Let $z=1$, $p(1)=\det(1^{-D}\tilde{L}(1))=\det(L)$. Since $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ has a globally reachable node, 0 is a simple eigenvalue of L ([5]), i.e., $\det(L)=0$ and $\mathrm{rank}(L)=n-1$. Thus, $p(z)$ indeed has only one zero at $z=1$.

Now, we prove that the zeros of $f(z)=p(z)/(z-1)$ have modulus less than unity. Obviously, $f(z)=\det(I+\frac{z^{-D}}{z-1}\tilde{L}(z))$. Based on the general Nyquist stability criterion ([1]), the zeros of $f(z)$ have modulus less than unity, if the eigenloci of $\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}\tilde{L}(\mathrm{j}\omega)$, i.e., $\lambda(\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}\tilde{L}(\mathrm{j}\omega))$, does not enclose the point $(-1,\mathrm{j}0)$ for $\omega\in[-\pi,\pi]$.

Similar to [3], we can use the Greshgorin disk theorem to estimate the matrix eigenvalue. Then, we have

$$\begin{aligned}&\lambda(\tfrac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}\tilde{L}(\mathrm{j}\omega))\\ \in\ &\textstyle\bigcup_{i\in\mathcal{I}}\Big\{\zeta:\zeta\in C,|\zeta-(\sum_{v_j\in N_i}a_{ij})\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\leq\\ &|(\textstyle\sum_{v_j\in N_i}a_{ij})\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\Big\}\\ \subseteq\ &\Big\{\zeta:\zeta\in C,|\zeta-K_{\max}\tfrac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\leq|K_{\max}\tfrac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\Big\}\end{aligned} \tag{15}$$

for all $\omega\in[-\pi,\pi]$, where $K_{\max}=\max_{i\in\mathcal{I}}\sum_{v_j\in N_i}a_{ij}$.

Now, define

$$\begin{aligned}G(\omega) &= K_{\max}\tfrac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}\\ &= -K_{\max}\tfrac{\sin(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})}-\mathrm{j}K_{\max}\tfrac{\cos(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})},\end{aligned} \tag{16}$$

and the Nyquist plot of $G(\omega)$ for $\omega\in[-\pi,\pi]$ is illustrated in Fig. 3. Note that $G(\omega)$ is the center of the disc $\{\zeta:\zeta\in C,|\zeta-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\leq|K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\}$. So, $\lambda(\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}\tilde{L}(\mathrm{j}\omega))$ does not enclose the point $(-1,\mathrm{j}0)$ for $\omega\in[-\pi,\pi]$ as long as we prove that $(-a,\mathrm{j}0)$ with $a\geq 1$ dose not in the disc $\{\zeta:\zeta\in C,|\zeta-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\leq|K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|\}$ for all $\omega\in[-\pi,\pi]$, i.e., $|-a+\mathrm{j}0-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|>|K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|$ holds for all $\omega\in[-\pi,\pi]$ with $a\geq 1$.

From (15),

$$\begin{aligned}
& |-a+\mathrm{j}0-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|^2-|K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|^2 \\
= & ((-a+K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})})^2+(K_{\max}\frac{\cos(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})})^2) \\
& -((-K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})})^2+(-K_{\max}\frac{\cos(\frac{2D+1}{2}\omega)}{2\sin(\frac{\omega}{2})})^2) \\
= & a(a-K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})}).
\end{aligned}$$

Because $\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \leq 2D+1$ holds for $\omega \in [-\pi,\pi]$ by Lemma 3, using (6) we obtain

$$\begin{aligned}
K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} & \leq K_{\max}(2D+1) \\
& < 1 \\
& \leq a.
\end{aligned}$$

Thus, $|a+\mathrm{j}0-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|^2-|K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|^2 > 0$, i.e., $|a+\mathrm{j}0-K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}| > |K_{\max}\frac{\mathrm{e}^{-\mathrm{j}\omega D}}{\mathrm{e}^{\mathrm{j}\omega}-1}|$ holds for $\omega \in [-\pi,\pi]$ with $a \geq 1$.

Now, we have proved that the zeros of $p(z)$ have modulus less than unity except for a zero at $z = 1$. By Lemma 1, system (5) asymptotically converges to a steady state $\lim_{k\to\infty} x_i(k) = x_{i0}, i \in \mathcal{I}$, and $x_0 = [x_{10}, x_{20}, \cdots, x_{n0}]^T \in \{x : Lx = 0\}$. Since $\mathrm{rank}(L) = n-1$ and $L[1,1,\cdots,1]^T = 0$ based on the definition of the Laplacian matrix L, the solutions of $Lx_0 = 0$ can be expressed as $x_0 = c[1,1,...,1]^T$, where c is a constant. Therefore, the system (5) asymptotically solves a consensus problem. Theorem 1 is proved.

V. Conclusion

In this paper, we propose a consensus protocol to solve the consensus problem of the multi-agent system with diverse communication delays. Using the frequency domain analysis and matrix theory, sufficient conditions, which are on the agent's own input delay, weights of edges to neighbors, and the interconnection topology of the network, are obtained for the system asymptotically converging to a consensus. The results also illustrate that the communication delays don't influence the system converging to a consensus, but have an impact on the dynamical performance of the multi-agent system.

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