ON KALMAN CANONICAL DECOMPOSITION OF LINEAR PERIODIC CONTINUOUS-TIME SYSTEMS WITH REAL COEFFICIENTS

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Abstract: In this note, structural decomposition of linear periodic continuous-time systems is discussed. A fundamental problem to decompose a state of a periodic system into controllable and uncontrollable parts is conjectured to be achieved by a continuously differentiable and periodic coordinate transformation with the same period of the system, however there is a counterexample to this conjecture. Hence we derive a condition for the existence of such a coordinate transformation. We also prove that, by relaxing a class of coordinate transformation, it is always possible to construct a periodic coordinate transformation with the double period of the periodic system.

Keywords: linear periodic systems, Kalman canonical decomposition

1. INTRODUCTION

In this note, we reconsider the Kalman canonical decomposition for linear periodic systems. In particular, we focus on the full information case and discuss the existence of a continuously differentiable and periodically time-varying coordinate transformation which decomposes a state into controllable and uncontrollable parts.

For linear time-varying systems which is not necessarily periodic, Kalman asserted that there exists a coordinate transformation which converts the coefficient matrices into special form valid at the fixed time instant (Kalman, 1962; R. E. Kalman and Narendra, 1962; Kalman, 1963; Weiss and Kalman, 1965). Youla (Youla, 1966) and Weiss (Weiss, 1968) then proved the existence of continuously differentiable and time-varying coordinate transformation which effects a structural decomposition for all time after some fixed time instant.

Applying the procedures by Youla (Youla, 1966) or Weiss (Weiss, 1968) into linear periodic systems, it is possible to construct a continuously differentiable and time-varying coordinate transformation, but the periodicity of the coordinate transformation is not obvious.

On the contraly, it was conjectured that it is always possible to construct a continuously differentiable and periodic coordinate transformation which transforms a state into controllabe and uncontrollable parts (see e.g. Bittanti and Bolzeron

(Bittanti and Bolzeron, 1985) and Nishimura and Kano (Nisimura and Kano, 1996)).

In this note, we firstly present a counterexample to this conjecture. Then we derive a condition for the existence of such a transformation. Comparing the derived condition with the former discussions, it is shown that the factorization of the controllability Gramian, which was supposed to be obviously possible (see e.g. (Bittanti and Bolzeron, 1985)), is not always possible indeed.

We also prove that, by relaxing a class of coordinate transformation, it is always possible to construct a periodic coordinate transformation with the double period of the given periodic system.

2. PRELIMINARIES

Consider a linear system

$$\dot{x} = A(t)x + B(t)u, \tag{1}$$

which is not necessarily periodic, where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is the input, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are supposed to be continuous, and $u(t)$ is supposed to be piecewise continuous and is denoted by $u \in \mathcal{U}$.

A state $x_0 \in \mathbb{R}^n$ of the system (1) is said to be controllable from time t if the state can be brought to the origin 0 at finite amount of time by a certain control function u. In other words, a state $x_0 \in \mathbb{R}^n$ of the system (1) is said to be controllable from time t if there exists a finite $s \geq t$ such that the integral equation

$$\Phi(s,t)x_0 + \int_t^s \Phi(s,\tau)B(\tau)u(\tau)d\tau = 0$$

has an admissible input $u \in \mathcal{U}$, where Φ denotes the state transition matrix for (1) with $u = 0$ for all $t \in \mathbb{R}$, i.e.

$$\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t)$$
$$\Phi(t,t) = I$$

The system (1) is said to be controllable, or (A, B) is said to be controllable, if all states $x_0 \in \mathbb{R}^n$ are controllable.

The set of states controllable from time t is denoted by

$$\mathcal{C}(t) := \left\{ \int_t^p \Phi(t,\tau)B(\tau)u(\tau)d\tau : p > t, u \in \mathcal{U} \right\},$$

which is said to be a controllable subspace at time t. $\mathcal{C}(t)$ satisfies the following properties (see Theorem 1 and Theorem 2 in (Weiss, 1968))

Lemma 1. (i) There exist a bounded scalar function $p(t) > 0$ such that

$$\mathcal{C}(t) = \operatorname{Im} W(t, t + p(t)),$$

where $\operatorname{Im} X$ denotes a image of a matrix X and W denotes the controllable Gramian defined by

$$W(t,s) := \int_t^s \Phi(t,\tau)B(\tau)B(\tau)'\Phi(t,\tau)'d\tau. \tag{2}$$

(ii) If $x_0 \in \mathcal{C}(s)$, $\Phi(t,s)x_0 \in \mathcal{C}(t)$ holds for $t \leq s$, i.e.

$$\Phi(t,s)\,\mathcal{C}(s) \subset \mathcal{C}(t) \quad \text{if} \quad s \geq t.$$

In this note we suppose that $A(t)$ and $B(t)$ is periodic with a period $T > 0$, which is said to be T-periodic for simplicity. Then $p(t)$ in Lemma 1 (i) can be chosen to be $p(t) = nT$, which is independent of time t. $\mathcal{C}(t)$ is Φ-invariant for linear periodic systems for both positive and negative directions in time.

Lemma 2. Suppose that $A(t)$ and $B(t)$ are continuous and T-periodic. Then
(i)

$$\mathcal{C}(t) = \operatorname{Im} W(t, t + nT)$$

(ii) $x_0 \in \mathcal{C}(t)$ iff $\Phi(s,t)x_0 \in \mathcal{C}(s)$ for all $s, t \in \mathbb{R}$, i.e.

$$\mathcal{C}(s) = \Phi(s,t)\,\mathcal{C}(t).$$

3. CONJECTURE OF KALMAN CANONICAL DECOMPOSITION FOR PERIODIC SYSTEMS

Consider a coordinate transformation $\xi = Z(t)x$ where $Z(t) \in \mathbb{R}^{n\times n}$ is continuously differentiable and invertible for all $t \in \mathbb{R}$, the system (1) is transformed to

$$\dot{\xi} = F(t)\xi + G(t)u \tag{3}$$

where

$$F(t) := (\dot{Z}(t) + Z(t)A(t))Z(t)^{-1} \tag{4}$$
$$G(t) := Z(t)B(t). \tag{5}$$

Suppose that $A(t)$ and $B(t)$ is T-periodic. Then, applying the procedures by Youla (Youla, 1966) or Weiss (Weiss, 1968), it is possible to construct a continuously differentiable and invertible matrix $Z(t) \in \mathbb{R}^{n\times n}$ such that the state of the system (3) is decomposed to controllable and uncontrollable parts, i.e. there exists a nonnegative integer $n_c \leq n$ such that $F(t)$ and $G(t)$ is decomposed to

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}$$
$$G(t) = \begin{bmatrix} G_1(t) \\ 0 \end{bmatrix}$$

where $F_{11}(t) \in \mathbb{R}^{n_c\times n_c}$, $F_{12}(t) \in \mathbb{R}^{n_c\times(n-n_c)}$, $F_{22}(t) \in \mathbb{R}^{(n-n_c)\times(n-n_c)}$, $G(t) \in \mathbb{R}^{n_c\times m}$, and (F_{11}, G_1) is controllable.

Although the periodicity of $Z(t)$ constructed by Youla (Youla, 1966) or Weiss (Weiss, 1968) is not obvious, it was conjectured that there is a continuously differentiable and T-periodic coordinate transformation matrix $Z(t) \in \mathbb{R}^{n\times n}$ (see e.g. (Bittanti and Bolzeron, 1985) and (Nisimura and Kano, 1996)).

Conjecture 1. Suppose that $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are continuous T-periodic. Then there exists a continuously differentiable and T-periodic matrix $Z(t) \in \mathbb{R}^{n\times n}$ which is invertible for all $t \in \mathbb{R}$ such that $F(t)$ defined by (4) and $G(t)$ defined by (5) satisfy the following block structure

$$F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} \tag{6}$$

$$G(t) = \begin{bmatrix} G_1(t) \\ 0 \end{bmatrix} \tag{7}$$

and (F_{11}, G_1) is controllable.

We note that, if there exits such a T-periodic and real-valued $Z(t)$, $F(t)$ and $G(t)$ become T-periodic and real-valued, and these properties are fundamental for analyzing the stabilizability by a real-valed state feedback.

4. A COUNTEREXAMPLE

In this section, we present a counterexample for Conjecture 1.

Let $\tilde{A} \in \mathbb{R}^{2\times 2}$ be a constant matrix and $\tilde{B}(t) \in \mathbb{R}^{2\times 1}$ be a continuous T-periodic matrix given by

$$\tilde{A} := \begin{bmatrix} 0 & \frac{\pi}{T} \\ -\frac{\pi}{T} & 0 \end{bmatrix} \tag{8}$$

$$\tilde{B}(t) := \begin{bmatrix} \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right) + \sin\left(\frac{\pi t}{T}\right)\right) \\ \sin\left(\frac{\pi t}{T}\right)\left(\cos\left(\frac{\pi t}{T}\right) - \sin\left(\frac{\pi t}{T}\right)\right) \end{bmatrix}. \tag{9}$$

Then the controllability Gramian over $[t, t+2T]$ is given by

$$\begin{aligned} &\tilde{W}(t, t+2T) \\ &= \int_t^{t+2T} e^{\tilde{A}(\tau-t)} \tilde{B}(\tau)\tilde{B}(\tau)' e^{\tilde{A}'(\tau-t)} d\tau \\ &= \begin{bmatrix} T\left(1+\sin\left(\frac{2\pi t}{T}\right)\right) & T\cos\left(\frac{2\pi t}{T}\right) \\ T\cos\left(\frac{2\pi t}{T}\right) & T\left(1-\sin\left(\frac{2\pi t}{T}\right)\right) \end{bmatrix} \end{aligned} \tag{10}$$

and satisfy

$$\operatorname{rank} \tilde{W}(t, t+2T) = 1$$

for all $t \in \mathbb{R}$, therefore $(\tilde{A}, \tilde{B})$ is uncontrollable.

Suppose that, followed by Conjecture 1, there exists a continuously differentiable, invertible and T-periodic matrix $Z(t) \in \mathbb{R}^{2\times 2}$ such that $\tilde{A}$ and $\tilde{B}(t)$ are transformed to $F(t)$ and $G(t)$ of the forms (6) and (7).

By the T-periodicity of $Z(t)$, the monodromy matrices in x-coordinate and in ξ-coordinate are similar, and the characteristic multipliers are invariant with respect to a coordinate transformation $\xi = Z(t)x$. Since $e^{\tilde{A}T} = -I$, the characteristic multipliers in x-coordinate are -1 with multiplicity 2, therefore they are also $-1 (< 0)$ with multiplicity 2 in ξ-coordinate.

On the contrary, it follows from (6) that the characteristic multipliers in ξ-coordinate are given by $\exp(\int_0^T F_{11}(\tau)d\tau)$ and $\exp(\int_0^T F_{22}(\tau)d\tau) (> 0)$.

Therefore we have a contradiction, which proves that a pair $\tilde{A}$ and $\tilde{B}(t)$ is a counterexample to Conjecture 1.

5. KALMAN CANONICAL DECOMPOSITION WITH THE SAME PERIOD OF SYSTEMS

In the previous section, we have shown that Conjecture 1 is not always satisfied for all linear periodic systems. On the other hand, it is well known that it is satisfied for linear time-invariant systems (see e.g. (Chui and Chen, 1989)).

In this section, we derive a necessary and sufficient condition such that a state of a linear periodic system is decomposed to controllable and uncontrollable parts by continuously differentiable and T-periodic coordinate transformation.

Theorem 1. Consider a linear periodic system described by (1) where $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are supposed to be continuous T-periodic. Let $\tilde{n}_c := \operatorname{rank} W(t, t+nT)$ where W is defined by (2). There exist a T-periodic matrix $Z(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and invertible for all $t \in \mathbb{R}$ such that

- $F(t)$ defined by (4) has a block structure of the form (6)
- $G(t)$ defined by (5) has a block structure of the form (7)
- (F_{11}, G_1) is controllable

iff there exists a T-periodic matrix $Q(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and orthogonal for all $t \in \mathbb{R}$ and a T-periodic matrix $E(t) \in \mathbb{R}^{\tilde{n}_c\times\tilde{n}_c}$ which is continuously differentiable and positive definite symmetric for all $t \in \mathbb{R}$ such that the controllability Gramian is factored by

$$W(t, t+nT) = Q(t)' \begin{bmatrix} E(t) & 0 \\ 0 & 0 \end{bmatrix} Q(t). \tag{11}$$

Moreover, if there exists such $Z(t)$, n_c which is a size of $F_{11}(t)$ is given by $n_c = \tilde{n}_c$.

Proof: Firstly we prove the necessity part. Let M denotes the controllability Gramian for (F, G)-pair, then we have

$$M(t,t+nT) = Z(t)W(t,t+nT)Z(t)'.$$

Let $\tilde{M}$ denotes the controllability Gramian for (F_{11}, G_{11})-pair. Since $F_{11}(t)$ and $G_1(t)$ is T-periodic, $\tilde{M}(t,t+nT)$ is also T-periodic. Since (F_{11}, G_1) is controllable, $\tilde{M}(t,t+nT)$ is positive definite symmetric for all $t \in \mathbb{R}$ (Chui and Chen, 1989). From the block structure of $F(t)$ and $G(t)$ given by (6) and (7), M and $\tilde{M}$ satisfy the following equation

$$M(t,t+nT) = \begin{bmatrix} \tilde{M}(t,t+nT) & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $W(t,t+nT)$ is factored by

$$W(t,t+nT) = Z(t)^{-1} \begin{bmatrix} \tilde{M}(t,t+nT) & 0 \\ 0 & 0 \end{bmatrix} (Z(t)')^{-1}$$

Since $Z(t)$ is invertible for all $t \in \mathbb{R}$, it is possible to apply the Gram-Schmidt's process for column vectors of $Z(t)^{-1}$ pointwise. There exist a T-periodic matrix $Q(t)$ which is continuously differentiable and orthogonal for all $t \in \mathbb{R}$ and an upper triangular T-periodic matrix $R(t)$ whose diagonal entries are positive for all $t \in \mathbb{R}$ such that

$$Z(t)^{-1} = Q(t)'R(t)$$

Decompose $R(t)$ followed by the block structure of $M(t,t+nT)$ and denote an upper left part of $R(t)$ by $R_{11}(t)$, and define

$$E(t) := R_{11}(t)\tilde{M}(t,t+nT)R_{11}(t)',$$

then $E(t)$ is T-periodic and positive definite symmetric for all $t \in \mathbb{R}$. It follows that $Q(t)$ and $E(t)$ satisfy (11).

Next we prove the sufficiency part. Factor $Q(t)$ followed by the factorization (11)

$$Q(t) =: \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}.$$

Define a coordinate transformation matrix by $Z(t) = Q(t)$ and consider a coordinate transformation $\xi = Q(t)x$. It follows from (11) that the controllability Gramian over $[t, t+nT]$ in ξ-coordinate is $Q(t)W(t,t+nT)Q(t)'$. From Lemma 2 (ii), the controllability subspace is invariant with respect to the state transition map

$$\operatorname{Im} \begin{bmatrix} E(s) & 0 \\ 0 & 0 \end{bmatrix} = \Theta(s,t) \operatorname{Im} \begin{bmatrix} E(t) & 0 \\ 0 & 0 \end{bmatrix}$$

where Θ denotes the state transition matrix in the ξ-coordinate $\Theta(s,t) := Q(s)\Phi(s,t)Q(t)^{-1}$. Since $E(t)$ is positive definite symmetric for all $t \in \mathbb{R}$, a lower left part of Θ is identically 0

$$\Theta(s,t) =: \begin{bmatrix} \Theta_{11}(s,t) & \Theta_{12}(s,t) \\ 0 & \Theta_{22}(s,t) \end{bmatrix}.$$

By its definition, $\Theta(s,t)$ is continuously differentiable and invertible for all $s, t \in \mathbb{R}$ and satisfies

$$\Theta(s+kT, t+kT) = \Theta(s,t) \tag{12}$$

for all $s, t \in \mathbb{R}$ and $k \in \mathbb{N}$. Define $F(t)$ by

$$F(s) := \frac{\partial \Theta(s,t)}{\partial s} \Theta(s,t)^{-1}, \tag{13}$$

then $F(t)$ has a block structure of the form (6), and the size of $F_{11}(t)$ and $E(t)$ is equivalent. We note that the right hand side of (13) is independent of t, therefore $F(t)$ is well-defined. It follows from

$$\begin{aligned} F(s+T) &= \frac{\partial \Theta(s+T,t+T)}{\partial s} \Theta(s+T,t+T)^{-1} \\ &= \frac{\partial \Theta(s,t)}{\partial s} \Theta(s,t)^{-1} \\ &= F(s) \end{aligned}$$

that $F(t)$ is continuous T-periodic, where we have used the equation $\Theta(s,t) = \Theta(s,t+T)\Theta(t+T,t)$ in the second identity and have used (12) in the third identity.

Denote the B-matrix in the ξ-coordinate by $G(t) := Q(t)B(t)$, $G(t)$ is continuous T-periodic. It can be shown that $G(t)$ has a block structure of the form (7). Indeed, multiplying (11) by $Q(t)$ from the left and $Q(t)'$ from the right, it follows that

$$\begin{aligned} &Q_2(t)W(t,t+nT)Q_2(t)' \\ &= \int_t^{t+nT} \Theta_{22}(t,\tau)Q_2(\tau)B(\tau)B(\tau)'Q_2(\tau)'\Theta(t,\tau)'d\tau \\ &= 0. \end{aligned}$$

Since the integrand is positive semidefinite symmetric and continuous, it is equivalent to 0 for all $\tau \in [t, t+nT]$. Moreover, since $\Theta_{22}(s,t)$ is invertible for all $s, t \in \mathbb{R}$ and $Q(t)$ and $B(t)$ is T-periodic, it follows that

$$Q_2(t)B(t) = 0$$

for all $t \in \mathbb{R}$.

Multiplying (11) by $Q(t)$ form the left and $Q(t)'$ from the right, it follows that

$$\begin{aligned} &Q_1(t)W(t,t+nT)Q_1(t)' \\ &= \int_t^{t+nT} \Theta_{11}(t,\tau)G_1(\tau)G_1(\tau)'\Theta_{11}(t,\tau)'G_1(t)'d\tau \\ &= E(t), \end{aligned}$$

therefore (F_{11}, G_1) is controllable. (QED)

A linear time-invariant system can be regarded as T-periodic system, and the controllability Gramian $W(t,t+nT)$ is independent of time t. Hence there are constant matrices Q and E satisfying (11) for all $t \in \mathbb{R}$, and, as well known, Conjecture 1 is always satisfied for linear time-invariant systems.

For general linear periodic systems, there exist T-periodic functions $Q(t)$ and $E(t)$ satisfying (11) for all $t \in \mathbb{R}$ iff there exist continuously differentiable, bounded and T-periodic bases in $\mathcal{C}(t)$.

Since $W(t, t+nT)$ is continuous, bounded and T-periodic and a rank of $W(t, t+nT)$ is independent of t, there exist bounded and T-periodic bases in $\mathcal{C}(t)$. By the Φ-invariance of $\mathcal{C}(t)$, there exist continuously differentiable and bounded bases for $\mathcal{C}(t)$. However the existence of bases satisfying all properties is not obvious.

Hence it is not obvious that W is always factored by (11), while it was supposed to be obvious in the former discussions (see e.g. (Bittanti and Bolzeron, 1985) and (Nisimura and Kano, 1996)). A counterexample in Section 4 proves that it is not always possible indeed.

6. EXAMPLE FOR THEOREM 1

In this section, we demonstrate the statement of Theorem 1 by a counterexample in Section 4.

Suppose that there exists a T-periodic coordinate transformation $Z(t)$ which transforms $(\tilde{A}, \tilde{B})$-pair into the block structure of the forms (6) and (7). Since the eigenvalues of $\tilde{W}(t, t+2T)$ in (10) are given by 0 and $2T$, followed by Theorem 1, there exist a continuously differentiable, orthogonal and T-periodic matrix $Q(t) \in \mathbb{R}^{2\times 2}$ such that

$$\tilde{W}(t, t+2T) = Q(t)' \begin{bmatrix} 2T & 0 \\ 0 & 0 \end{bmatrix} Q(t), \qquad (14)$$

which corresponds to (11). We note that first column vector of $Q(t)'$, which is denoted by $w(t)$, is a eigenvector of $W(t, t+2T)$ for the eigenvalue $2T$. On the other hand,

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 1+\sin\left(\frac{2\pi t}{T}\right) \\ \cos\left(\frac{2\pi t}{T}\right) \end{bmatrix}$$

is also a eigenvector of $\tilde{W}(t, t+2T)$ for the eigenvalue $2T$. Since $v_1(t)$ and $v_2(t)$ has a common zero at $t = \frac{3T}{4}$, there exists a function $g(t) \in \mathbb{R}$ which has a singular point at $t = \frac{3T}{4}$ and satisfies

$$w(t) = v(t)g(t).$$

Then $g(t)$ satisfies the following on $[0, T]$.
(i) $g(t)$ is continuously differentiable except for $t = \frac{3T}{4}$ and T-periodic, therefore it follows that

$$g(0) = g(T) > 0$$

or

$$g(0) = g(T) < 0.$$

(ii) $g(t) \neq 0$ for all $t \in [0, T]$.
(ii) $g(t)$ has a 1-st order pole at $t = \frac{3T}{4}$, therefore it follows that

$$\lim_{t\to\frac{3T}{4}-} g(t) = -\infty \quad \& \quad \lim_{t\to\frac{3T}{4}+} g(t) = +\infty$$

or

$$\lim_{t\to\frac{3T}{4}-} g(t) = +\infty \quad \& \quad \lim_{t\to\frac{3T}{4}+} g(t) = -\infty.$$

We note that $v_1(t)$ has a 2-nd order zero at $t = \frac{3T}{4}$ and $v_2(t)$ has a 1-st order zero at $t = \frac{3T}{4}$ (see Figure 1), therefore $g(t)$ has a 1-st order pole at $t = \frac{3T}{4}$.

It is clear that those properties are not simultaneously satisfied for each cases. Hence there is no T-periodic coordinate transformation $Z(t)$ which transforms $(\tilde{A}, \tilde{B})$-pair into the block triangular structure of (6) and (7), as shown in Theorem 1.

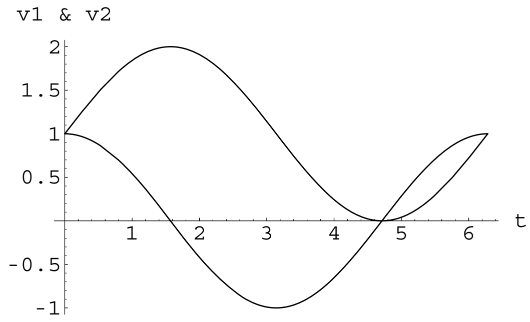

Fig. 1. Elements of an eigenvector $v(t)$ $(T = 2\pi)$

7. KALMAN CANONICAL DECOMPOSITION WITH THE DOUBLE PERIOD OF SYSTEMS

In this section, we prove that, by relaxing a class of periodic coordinate transformation, it is always possible to construct a $2T$-periodic coordinate transformation which decompose a state of a linear periodic system into controllable and uncontrollable parts.

Theorem 2. Consider a linear periodic system described by (1) where $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are supposed to be continuous T-periodic. Let $\tilde{n}_c :=$ rank $W(t, t+nT)$ where W is defined by (2). Then there exist a $2T$-periodic matrix $Z(t) \in \mathbb{R}^{n\times n}$ which is continuously differentiable and invertible for all $t \in \mathbb{R}$ such that

- $F(t)$ defined by (4) has a block structure of the form (6)
- $G(t)$ defined by (5) has a block structure of the form (7)
- (F_{11}, G_1) is controllable

and n_c which is a size of $F_{11}(t)$ is given by $n_c = \tilde{n}_c$.

Proof: Let the characteristic multiplier of $W(t, t+nT)$ be factored by

$$\det(\lambda I - W(t, t+nT)) = p(\lambda, t)\lambda^{n-\tilde{n}_c}$$

where $p(\lambda, t)$ is polynomical of $\tilde{n}_c$-th order polynomial of λ and satisfies $p(0, t) \neq 0$ for all t. Since $W(t, t+nT)$ is T-periodic, it follows from Theorem 3 and Remark 3 of (Sibuya, 1965) such that there exist $2T$-periodic continuously differentiable matrix $V(t) \in \mathbb{R}^{n\times n}, E_1(t) \in \mathbb{R}^{\tilde{n}_c\times\tilde{n}_c}, E_2(t) \in \mathbb{R}^{(n-\tilde{n}_c)\times(n-\tilde{n}_c)}$ such that

$$W(t, t+nT) = V(t) \begin{bmatrix} E_1(t) & 0 \\ 0 & E_2(t) \end{bmatrix} V(t)^{-1},$$

where $V(t)$ is invertible for all t and $E_1(t)$ and $E_2(t)$ satisfy

$$\det(\lambda I - E_1(t)) = p(\lambda, t)$$
$$\det(\lambda I - E_2(t)) = \lambda^{n-\tilde{n}_c}.$$

Applying the Gram-Schmidt's process on column vectors of $V(t)$ pointwise, $V(t)$ is factored by

$$V(t) = Q(t)'R(t)$$
$$R(t) =: \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ 0 & R_{22}(t) \end{bmatrix}$$

where $Q(t)$ is $2T$-periodic continuously differentiable orthogonal matrix and $R(t)$ is $2T$-periodic continuously differentiable upper triangular matrix. $Q(t)'W(t, t+nT)Q(t)$ is upper triangular and symmetric, an upper right part of $Q(t)'W(t, t+nT)Q(t)$ is 0, i.e.

$$Q(t)W(t, t+nT)Q(t)' = \begin{bmatrix} R_{11}(t)E_1(t)R_{11}(t)^{-1} & 0 \\ 0 & R_{22}(t)E_2(t)R_{22}(t)^{-1} \end{bmatrix}.$$

It follows from

$$\det(\lambda I - R_{22}(t)E_2(t)R_{22}(t)^{-1}) = \lambda^{n-\tilde{n}_c}$$

that all eigenvalues of $R_{22}(t)E_2(t)R_{22}(t)^{-1}$ are 0. Since $R_{22}(t)E_2(t)R_{22}(t)^{-1}$ is symmetric,

$$R_{22}(t)E_2(t)R_{22}(t)^{-1} = 0.$$

Hence $W(t, t+nT)$ is factored by

$$W(t, t+nT) = Q(t)' \begin{bmatrix} E(t) & 0 \\ 0 & 0 \end{bmatrix} Q(t) \qquad (15)$$

where $E(t) := R_{11}E_1(t)R_{11}^{-1}$ is $2T$-periodic continuously differentiable and positive definite symmetric for all t. Let $Z(t) = Q(t)$ and applying the sufficiency part of Theorem 1, we have the assertion. (QED)

8. EXAMPLE FOR THEOREM 2

In this section, we demonstrate the statement of Theorem 2 by a counterexample in Section 4.

$\tilde{W}(t, t+2T)$ in (10) is factored by

$$\tilde{W}(t, t+2T) = \tilde{Q}(t)' \begin{bmatrix} 2T & 0 \\ 0 & 0 \end{bmatrix} \tilde{Q}(t)$$
$$\tilde{Q}(t) = \begin{bmatrix} \tilde{Q}_{11}(t) & \tilde{Q}_{12}(t) \\ \tilde{Q}_{12}(t) & -\tilde{Q}_{22}(t) \end{bmatrix}$$
$$\tilde{Q}_{11}(t) = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi t}{T}\right) + \sin\left(\frac{\pi t}{T}\right)\right)$$
$$\tilde{Q}_{12}(t) = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi t}{T}\right) - \sin\left(\frac{\pi t}{T}\right)\right)$$

which corresponds to (15). Substitute $A(t) = \tilde{A}$, $B(t) = \tilde{B}(t)$, $Z(t) = \tilde{Q}(t)$ into (4) and (6), then $\tilde{A}$ and $\tilde{B}(t)$ are transformed to

$$(\dot{Z}(t) + Z(t)\tilde{A})Z(t)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$Z(t)\tilde{B}(t) = \begin{bmatrix} \sqrt{2}\sin\left(\frac{\pi t}{T}\right) \\ 0 \end{bmatrix}$$

as shown in Theorem 2.

9. CONCLUSION

In this note, we discussed the problem of transforming a linear periodic system into a Kalman canonical decomposition using a continuously differentiable periodic coordinate transformation . It was conjectured that it is always possible to construct a transformation with the same period of the system, however we showed that there is a counterexample to this conjecture. Then we derive a necessary and sufficient condition for the existence of such a transformation. We also prove that, by relaxing a class of coordinate transformation, it is always possible to construct a periodic coordinate transformation with the double period of the system.

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