ON A REFORMULATION OF THE MULTIPLE SCALES PERTURBATION METHOD FOR DIFFERENCE EQUATIONS

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Abstract
In the classical multiple scales perturbation method for ordinary difference equations ($O\Delta Es$) as developed in 1977 by Hoppensteadt and Miranker, difference equations are replaced at a certain moment in the perturbation procedure by ordinary differential equations ($ODEs$). Taking into account the possibly different behaviour of the solutions of an $O\Delta E$ and of the solutions of a nearby ODE, one can not always be sure that the constructed approximations by the Hoppensteadt-Miranker method indeed reflect the behaviour of the exact solutions of the $O\Delta Es$. For that reason an improved version of the multiple scales perturbation method for $O\Delta Es$ will be presented and formulated in this paper completely in terms of difference equations.

Key words
Difference equations, multiple scales perturbation method.

1 Introduction

Nowadays the multiple time-scales perturbation method for differential equations is a well-developed, well accepted, and very popular method to approximate solutions of weakly nonlinear differential equations. This method was developed in the period 1935-1962 by Krylov and Bogoliubov, Kuzmak, Kevorkian and Cole, Cochran, and Mahony. In the early 1970s Nayfeh popularized this method by writing many papers and books on this subject (see for instance [Nayfeh, 1973]). More recent books on this method and its historical development are for instance the books by [Nayfeh, 1973], [Holmes, 1995], [Kevorkian and Cole, 1996], [Murdock, 1991], and [Agarwal, Bohner, O'Regan and A.Peterson, 2002]. The development of the multiple scales perturbation method for ordinary difference equations ($O\Delta Es$) started in 1960 with the work of [Torng, 1960]. In this paper a second order $O\Delta E$ is reduced to a system of two first order $O\Delta Es$ by means of the method of variation of parameters for $O\Delta Es$. Then, nonlinear terms are expanded in discrete Fourier series, and a Krylov-Bogoliubov method (or equivalently, an averaging method) is applied to obtain the equations that describe the slow dynamics of the problem approximately. A similar method was presented in 1970 by Huston in [Huston, 1979]. From the results in [Huston, 1979] and in [Torng, 1960] it is clear that the solution of a weakly perturbed (non-) linear $O\Delta E$ behaves differently on different iteration scales. In 1977 Hoppensteadt and Miranker introduced in [Hoppensteadt and Miranker, 1977] the multiple scales perturbation method for $O\Delta Es$. For a problem with two significant iteration scales these authors assume that the solution x_n of an $O\Delta E$ for instance depends on n and $s = \epsilon n$ (that is, depends on a fast iteration scale and on a slow iteration scale). In the $O\Delta E$ x_{n+1} is then replaced by $x(n+1, \epsilon(n+1)) = x(n+1, s+\epsilon)$. In the perturbation scheme $x(n+1, s+\epsilon)$ is expanded in a Taylor polynomial, that is, $x(n+1, s+\epsilon) = x(n+1, s) + \epsilon \frac{\partial x}{\partial s}(n+1, s) + O(\epsilon^2)$, and to avoid unbounded terms (or secular terms) in the perturbation expansion for x_n one finally has to solve ordinary differential equations ($ODEs$) due to the derivatives in the Taylor expansions. So, in the perturbation procedure $O\Delta Es$ are replaced (partly) by $ODEs$. A slightly different multiple scales perturbation method for $O\Delta Es$ was introduced in [Subramanian and Krishnan] by Subramanian and Krishnan in 1979. In their approach the difference operator Δ is replaced by partial difference operators. For a problem with two significant iteration scales the authors of [Subramanian and Krishnan] introduced:

$$x_{n+1} - x_n = \Delta x(n, s) = \Delta_n(n, s) + \epsilon \Delta_s x(n, s), \quad (1)$$

where $\Delta_n x(n, s) = x(n+1, s) - x(n, s)$, and $\Delta_s x(n, s) = x(n, s+\epsilon) - x(n, s)$. This replacement is based on the two-timescales perturbation method for $ODEs$, where $x(t)$ is replaced by $\tilde{x}(t,\tau)$ with $\tau = \epsilon t$ and

$$\frac{dx(t)}{dt} = \frac{\partial \tilde{x}(t,\tau)}{\partial t} + \epsilon \frac{\partial \tilde{n}(t,\tau)}{\partial \tau}.$$

Nowadays the method of Hoppensteadt and Miranker is assumed to be the standard form of the multiple scales perturbation method for $O\Delta Es$ (see for instance [Holmes, 1995; Marathe and Chatterjee, 2006; Mickens, 1987; Remickens,1987]). Also recently this method was "rediscovered" by Luongo [Lungo, 1996] and by Maccari [Maccari, 1999]. It should be observed, however, that many results concerning $ODEs$ carry over quite easily to corresponding results for $O\Delta Es$, while other results are completely different from their continuous counterparts.

The reader is referred to [Agarwal,1992; Agarwal, Bohner, O'Regan and A.Peterson, 2002; Elaydi, 2005; Holmes, 1995; Kadalbajoo and Patidar, 2002; Kelly and Peterson, 1991; Mickens, 1987] for some further striking differences (and similarities) in the theory for $ODEs$ and for $O\Delta Es$. In the multiple scales perturbation method for $O\Delta Es$ as developed in [Hoppensteadt and Miranker, 1977] by Hoppensteadt and Miranker difference equations are replaced at a certain moment by differential equations. Taking into account the possibly different behaviour of the solutions of an $O\Delta E$ and of the solutions of an (nearby) ODE one can not always be sure that the constructed approximations by the Hoppensteadt-Miranker method indeed reflect the behaviour of the exact solutions of the $O\Delta E$. For that reason an improved version of the multiple scales perturbation method for $O\Delta Es$ will be presented and formulated in this paper completely in terms of difference equations.

This paper is organized as follows. In section 2 of this paper the multiple scales perturbation method for $O\Delta Es$ will be presented completely in terms of difference operators. How this method can be applied to a second order regularly perturbed, linear $O\Delta E$ will be shown in this paper, and how it can be applied to weakly nonlinear $O\Delta Es$ will be shown in a forthcoming paper [Van Horssen and Ter Brake]. The asymptotic validity of the constructed approximations on sufficiently long iteration scales will be also discussed in [Van Horssen and Ter Brake]. Finally, in section 3 of this paper some conclusions will be drawn, and some remarks on future research will be made.

2 The multiple scales perturbation method for $O\Delta Es$

In this section the multiple scales perturbation method for $O\Delta Es$ will be presented in a complete "difference operator" setting. Before introducing this method several operators have to be defined (and motivated). The well-known shift operator E, the difference operator Δ, and the identity operator I are defined as follows:

$$Ex_n = x_{n+1}, \quad \Delta x_n = x_{n+1} - x_n, \quad \text{and} \quad Ix_n = x_n. \tag{2}$$

The relationship between these operators easily follows from (2.1):

$$E = \Delta + I \quad \Leftrightarrow \quad \Delta = E - I. \tag{3}$$

The solution of a weakly perturbed $O\Delta E$ usually contains a rapidly changing part in n, and a slowly changing part in n. This is usually referred to as multiple scales behaviour. Consider the following functions:

$$\begin{aligned}
a_n = 3^n \quad & \Rightarrow \Delta a_n = 3^{n+1} - 3^n = (3-1)3^n \\
& \quad = 2a_n = O(a_n), \\
b_n = e^{\epsilon n} \quad & \Rightarrow \Delta b_n = e^{\epsilon(n+1)} - e^{\epsilon n} = O(\epsilon b_n), \\
c_n = (1+\epsilon)^n \quad & \Rightarrow \Delta c_n = (1+\epsilon)^{n+1} - (1+\epsilon)^n \\
& \quad = O(\epsilon c_n), \\
d_n = 3^n(1+\epsilon)^n & \Rightarrow \Delta d_n = 3^{n+1}(1+\epsilon)^{n+1} \\
& \quad -3^n(1+\epsilon)^n = (2+3\epsilon)d_n.
\end{aligned} \tag{4}$$

From (2.3) it is obvious that a_n only has a rapidly changing part in n, that b_n and c_n only have a slowly changing part in n, and that d_n has a rapidly changing part in n and a slowly one. To make this behaviour more clear in notation the following notations are proposed: $a_n = a(n)$, $b_n = b(\epsilon n), c_n = c(\epsilon n)$, and $d_n = d(n, \epsilon n)$. It should be observed that these notations are similar to the ones used in the multiple timescales perturbation method for $ODEs$. Now it is assumed that $x_n = x(n, \epsilon n)$. This assumption implies that the solution of the $O\Delta E$ depends on two variables. So, the $O\Delta E$ actually becomes a partial difference equation. For that reason also partial shift operators and partial difference operators have to be defined. The following definitions are proposed:

$$\begin{aligned}
E_1 x(n,\epsilon n) &= x(n+1,\epsilon n), \\
E_\epsilon x(n,\epsilon n) &= x(n,\epsilon(n+1)), \\
\Delta_1 x(n,\epsilon n) &= x(n+1,\epsilon n) - x(n,\epsilon n) \\
&= (E_1 - I)x(n,\epsilon n), \\
\Delta_\epsilon x(n,\epsilon n) &= x(n,\epsilon(n+1)) - x(n,\epsilon n) \\
&= (E_\epsilon - I)x(n,\epsilon n).
\end{aligned} \tag{5}$$

From (2.1), (2.2), and (2.4) it follows that (assuming $x_n = x(n, \epsilon n)$):

$$\begin{aligned}
\Delta x_n &= x_{n+1} - x_n = x(n+1,\epsilon(n+1)) - x(n,\epsilon n) \\
&= E_1 E_\epsilon x(n,\epsilon n) - Ix(n,\epsilon n) \\
&= (\Delta_1 + I)(\Delta_\epsilon + I)x(n,\epsilon n) - Ix(n,\epsilon n) \\
&= (\Delta_1 + \Delta_\epsilon + \Delta_1\Delta_\epsilon)x(n,\epsilon n).
\end{aligned}$$

And so, it follows that

$$\Delta = \Delta_1 + \Delta_\epsilon + \Delta_1\Delta_\epsilon, \quad \text{and} \quad E = E_1 E_\epsilon. \tag{6}$$

Furthermore, for the partial difference operators Δ_1 and Δ_ϵ it is assumed that (also based on (2.3)):

$$\begin{aligned} &\Delta_1 x(n,\epsilon n) = O(x(n,\epsilon n)), \quad \text{and} \\ &\Delta_\epsilon x(n,\epsilon n) = O(\epsilon x(n,\epsilon n)). \end{aligned} \tag{7}$$

From (2.5) it is obvious that in (1.1) the operator $\Delta_1\Delta_\epsilon$ is missing (see also [Subramanian and Krishnan]). When x_n depends on $m+1$ scales the given definitions can readily be generalized, yielding: (for $j = 0, 1, ..., m$)

$$\begin{aligned} &x_n = x(n, \epsilon n, \epsilon^2 n, ..., \epsilon^m n), \\ &E_{\epsilon^j} x(n, ..., \epsilon^m n = x(n, \epsilon n, ..., \epsilon^j(n+1), ..., \epsilon^m n), \\ &\Delta_{\epsilon^j} x(n, ...\epsilon^m n) = (E_{\epsilon^j} - I) x(n, .., \epsilon^m n), \\ &E = E_1 E_\epsilon E_{\epsilon^2} ... E_{\epsilon^m}, \\ &\Delta = (\Delta_1 + I)(\Delta_\epsilon + I)...(\Delta_{\epsilon^m} + I) - I, \\ &\Delta_{\epsilon^j} x(n, ..., \epsilon^m n) = O(\epsilon^j x(n, ..., \epsilon^m n)). \end{aligned} \tag{8}$$

Now it will be shown how these operators can be used. For that reason a simple example will be treated. Consider the weakly perturbed, linear, second order $O\Delta E$

$$x_{n+2} + \epsilon x_{n+1} + x_n = 0, \tag{9}$$

where ϵ is a small parameter with $0 < \epsilon \ll 1$. Using (2.1) and (2.2) it follows that (2.8) can be rewritten in:

$$\begin{aligned} &E^2 x_n + \epsilon E x_n + I x_n = 0 \quad \Leftrightarrow \\ &(\Delta + I)^2 x_n + \epsilon(\Delta + I) x_n + I x_n = 0 \Leftrightarrow \\ &\Delta^2 x_n + (\epsilon + 2)\Delta x_n + (2 + \epsilon) x_n = 0. \end{aligned} \tag{10}$$

Assuming that x_n depends on two scales (a fast scale n, and a slow scale ϵn) it follows that $x_n = x(n, \epsilon n)$ and that (2.8) or (2.9) becomes

$$\begin{aligned} &(\Delta_1 + \Delta_\epsilon + \Delta_1\Delta_\epsilon)^2 x(n,\epsilon n) \\ &+(\epsilon + 2)(\Delta_1 + \Delta_\epsilon + \Delta_1\Delta_\epsilon) x(n,\epsilon n) + \\ &+(2+\epsilon) x(n,\epsilon n) = 0 \quad \Leftrightarrow \\ &(\Delta_1^2 + 2\Delta_1 + 2) x(n,\epsilon n) + (2\Delta_1(\Delta_\epsilon + \Delta_1\Delta_\epsilon) \\ &+2(\Delta_\epsilon + \Delta_1\Delta_\epsilon) \\ &+\epsilon\Delta_1 + \epsilon) x(n,\epsilon n) + O(\epsilon^2 x(n,\epsilon n)) = 0 \quad \Leftrightarrow \\ &(\Delta_1^2 + 2\Delta_1 + 2) x(n,\epsilon n) + (2(\Delta_1 + I)(\Delta_\epsilon + \Delta_1\Delta_\epsilon) \\ &+\epsilon(\Delta_1 + I)) x(n,\epsilon n) \\ &+O(\epsilon^2 x(n,\epsilon n)) = 0. \end{aligned} \tag{11}$$

To construct an approximation for $x_n = x(n,\epsilon n)$ one now has to substitute into (2.10) a formal power series (in ϵ) for x_n, that is,

$$x(n,\epsilon n) = x_0(n,\epsilon n) + \epsilon x_1(n,\epsilon n) + \epsilon^2 x_2(n,\epsilon n) + ... \tag{12}$$

Then, by taking together those terms of equal powers in ϵ one obtains as $O(1)$-problem

$$\begin{aligned} &(\Delta_1^2 + 2\Delta_1 + 2) x_0(n,\epsilon n) = 0 \quad \Leftrightarrow \\ &x_0(n+2,\epsilon n) + x_0(n,\epsilon n) = 0, \end{aligned} \tag{13}$$

and as $O(\epsilon)$-problem

$$\begin{aligned} &\epsilon(\Delta_1^2 + 2\Delta_1 + 2) x_1(n,\epsilon n) \\ &+ \left(2(\Delta_1 + I)(\Delta_\epsilon + \Delta_1\Delta_\epsilon + \tfrac{\epsilon}{2})\right) x_0(n,\epsilon n) = 0, \end{aligned} \tag{14}$$

and so on. The $O(1)$-problem (2.12) can readily be solved, yielding

$$x_0(n,\epsilon n) = f_0(\epsilon n)\cos\left(\frac{n\pi}{2}\right) + g_0(\epsilon n)\sin\left(\frac{n\pi}{2}\right), \tag{15}$$

where $f_0(\epsilon n)$ and $g_0(\epsilon n)$ are still arbitrary functions, which can be used to avoid unbounded behaviour in $x_1(n,\epsilon n)$ on the $O(\frac{1}{\epsilon})$ iteration scale.
The $O(\epsilon)$-problem (2.13) now becomes:

$$\begin{aligned} &\epsilon(x_1(n+2,\epsilon n) + x_1(n,\epsilon n)) \\ &+2(x_0(n+2,\epsilon(n+1)) - x_0(n+2,\epsilon n)) \\ &+\epsilon x_0(n,\epsilon n) = 0 \\ \Leftrightarrow\ &\epsilon(x_1(n+2,\epsilon n) + x_1(n,\epsilon n)) \\ &= (2\Delta_\epsilon f_0(\epsilon n) - \epsilon g_0(\epsilon n))\cos(\tfrac{n\pi}{2}) + \\ &+(2\Delta_\epsilon g_0(\epsilon n) + \epsilon f_0(\epsilon n))\sin(\tfrac{n\pi}{2}). \end{aligned} \tag{16}$$

In the $O\Delta E$ (2.15) for $x_1(n,\epsilon n)$ it is obvious that the righthand side contains terms (i.e., $\cos(\frac{n\pi}{2})$ and $\sin(\frac{n\pi}{2})$), which are solutions of the homogeneous $O\Delta E$. Then, to avoid unbounded or secular behaviour in $x_1(n,\epsilon n)$ it follows that $f_0(\epsilon n)$ and $g_0(\epsilon n)$ have to satisfy:

$$\begin{aligned} &2\Delta_\epsilon f_0(\epsilon n) - \epsilon g_0(\epsilon n) = 0, \\ &2\Delta_\epsilon g_0(\epsilon n) + \epsilon f_0(\epsilon n) = 0. \end{aligned} \tag{17}$$

System (2.16) for $f_0(\epsilon n)$ and $g_0(\epsilon n)$ can readily be solved, yielding

$$\begin{aligned} &f_0(\epsilon n) = a_0(1 + \tfrac{\epsilon^2}{4})^{\frac{n}{2}}\cos(n\mu(\epsilon)) \\ &+b_0(1 + \tfrac{\epsilon^2}{4})^{\frac{n}{2}}\sin(n\mu(\epsilon)), \\ &g_0(\epsilon n) = -a_0(1 + \tfrac{\epsilon^2}{4})^{\frac{n}{2}}\sin(n\mu(\epsilon)) \\ &+b_0(1 + \tfrac{\epsilon^2}{4})^{\frac{n}{2}}\cos(n\mu(\epsilon)), \end{aligned} \tag{18}$$

where a_0 and b_0 are arbitrary constants, and where $\mu(\epsilon)$ is given by $\cos(\mu(\epsilon)) = (1+\frac{\epsilon^2}{4})^{-\frac{1}{2}}$, and $\sin(\mu(\epsilon)) = \frac{\epsilon}{2}(1+\frac{\epsilon^2}{4})^{-\frac{1}{2}}$. From these expressions $\mu(\epsilon)$ can be approximated by

$$\mu(\epsilon) = \frac{1}{2}\epsilon - \frac{1}{24}\epsilon^3 + O(\epsilon^5), \tag{19}$$

and from (2.15) $x_1(n, \epsilon n)$ can be determined, yielding

$$x_1(n, \epsilon n) = f_1(\epsilon n)\cos(\frac{n\pi}{2}) + g_1(\epsilon n)\sin(\frac{n\pi}{2}), \tag{20}$$

where $f_1(\epsilon n)$ and $g_1(\epsilon n)$ are still arbitrary functions which can be used to avoid secular terms in $x_2(n, \epsilon n)$. At this moment, however, we are not interested in the higher order approximations. For that reason we will take in (2.19) $f_1(\epsilon n)$ and $g_1(\epsilon n)$ equal to the constants a_1 and b_1 respectively. So far we have constructed an approximation for the solution of the $O\Delta E$ (2.8). In this case the approximation $x_0(n, \epsilon n)$ can be compared with the exact solution of the $O\Delta E$ (2.8). The exact solution is given by

$$x_n = a\, 1^n\ \cos(n\theta(\epsilon)) + b\, 1^n\ \sin(n\theta(\epsilon)), \tag{21}$$

where a and b are arbitrary constants, and where $\theta(\epsilon)$ is given by $\cos(\theta(\epsilon)) = -\frac{\epsilon}{2}$ and $\sin(\theta(\epsilon)) = (1-\frac{\epsilon^2}{4})^{\frac{1}{2}}$, and $\theta(\epsilon)$ can be approximated by $\theta(\epsilon) = \frac{\pi}{2} + \frac{\epsilon}{2} + \frac{\epsilon^3}{48} + O(\epsilon^5)$. The approximation $x_0(n, \epsilon n)$ is given by (2.14), (2.17), and (2.18). This approximation can be rewritten in the following form

$$\begin{aligned} x_0(n, \epsilon n) &= a_0\left(1+\tfrac{\epsilon^2}{4}\right)^{\frac{n}{2}}\cos\left(\tfrac{n\pi}{2} + n\mu(\epsilon)\right) \\ &+ b_0\left(1+\tfrac{\epsilon^2}{4}\right)^{\frac{n}{2}}\sin\left(\tfrac{n\pi}{2} + n\mu(\epsilon)\right). \end{aligned} \tag{22}$$

From (2.20) and (2.21) it can readily be deduced that the difference between the exact solution x_n and the approximation $x_0(n, \epsilon n)$ is of order ϵ for $n \sim \frac{1}{\epsilon}$. So, the constructed approximation is $O(\epsilon)$ accurate on an iteration scale of order $\frac{1}{\epsilon}$. Usually of course the exact solution of a weakly (non)linearly perturbed $O\Delta E$ will not be available. In a forthcoming paper [Van Horssen and Ter Brake] it will be shown how for such cases the asymptotic validity of an approximation can be obtained on a sufficiently long iteration scale.

3 Conclusions and remarks

In this paper an improved version of the multiple scales perturbation method for $O\Delta Es$ has been presented and formulated completely in terms of difference equations. It can be shown (see[Van Horssen and Ter Brake]) that this improved method can be applied to regularly perturbed $O\Delta Es$ and to singularly perturbed, linear $O\Delta Es$. The relative and/or absolute errors in the constructed approximations of the solutions of the $O\Delta Es$ can be determined, and it can be shown that these approximations are valid on long iteration scales.

How solutions of singularly perturbed, linear $O\Delta Es$ can be approximated will also be shown in [Van Horssen and Ter Brake]. Compared to the existing rescaling procedures for singularly perturbed $ODEs$ and $O\Delta ES$ (see for instance [Kevorkian and Cole, 1996; O'Malley, 1991; Sari and Zerizer, 2005; Verhulst, 2005]) also a slightly revised rescaling procedure will be presented in [Van Horssen and Ter Brake] to find the significant scalings for some singularly perturbed, linear $O\Delta Es$

It is to be expected that the presented perturbation method also can be applied successfully to weakly perturbed partial difference equations, and to singularly perturbed, weakly nonlinear $O\Delta Es$. Of course, these extensions will be interesting subjects for future research. Finally, it should be remarked that the presented perturbation method also can be used in the numerical analysis of certain classes of regularly or singularly perturbed differential equations to see whether the solutions of the discretized equations (i.e. the difference equations) have the same type of behaviour as the solutions of the differential equations or not.

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