ON VARIATION OF ENERGY AND AXIAL MOMENTUM IN ONE-DIMENSIONAL TRANSLATING CONTINUA

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1 Introduction

One-dimensional translating continua can be exemplified by a conveyor belt or a pipe conveying fluid. All, or a part, of the particles of such continua have an axial component of the velocity that is maintained by an external force. For a conveyor belt this force is provided by the pulleys, whereas the fluid flow through the pipe is forced by a pump. The necessarily present external forcing can destabilize vibration of the translating continua. Prediction of such instability is important for many engineering applications such as pipes conveying fluid, high speed magnetic tapes, conveyor belts, textile fibers, band saws, power transmission chains and other similar systems [Bolotin, 1963; Ziegler, 1968; Wickert and Mote, 1989; Païdoussis, 1998; Lee and Mote,1997; Païdoussis, 1998].

Most often, prediction of the critical velocity that would lead to the instability of a translating continuum is based on a linearized equation of the transverse motion. Such prediction is usually in good correspondence with experimental data. However, in a number of cases, the prediction based on the eigenvalue analysis of the linearized equation of motion contradicts the energy considerations. The most notorious example is a simply-supported pipe that conveys fluid at a constant speed [Païdoussis, 1998]. The eigenvalue analysis of such pipe predicts divergence at a certain flow speed, whereas the conventional energy analysis suggests that the energy of the system must be constant.

In this paper, to find the origin of the above-formulated contradiction, the energy equation is sought for that describes the energy variation in a one-dimensional continuum that translates axially at a constant speed. First, it is attempted to derive the energy equation from the linearized equation of the transverse motion of the continuum. It is shown that this approach delivers ambiguous results as multiple energy equations can be derived from the linearized equation of motion. It is then attempted to identify "the correct" energy equation by considering (additionally) the axial momentum of the continuum. As in the case of the energy equation, however, a number of equations can be obtained that describe variation of what seems to be the axial momentum of the continuum. It remains unclear how to choose the "correct" momentum and energy equations.

To reach to the origin of ambiguity of the energy and momentum equations, nonlinear considerations must be employed. In this paper, nonlinear vibrations of an inextensible conveyor belt are considered as an example. The energy density, the energy flux and the axial momentum are derived from the first principles in the same approximation as that lading to the linearized equation of motion. It is shown that the resulting expressions for the energy and axial momentum can not be guessed correctly based on the linearized equation of motion. It is also shown that the energy equation contains the work of a force that maintains translation of the belt.

The analysis presented in this paper allows to state that a number of physical situations resulting in the same linearized equation of motion of a translating continuum can correspond to significantly different expressions for the energy, momentum and the fluxes thereof.

2 Linearized equation of motion

Consider a generalized, uniform, one-dimensional continuum that either translates or conveys fluid in the positive x-direction. The velocity U of translation or convection is constant. It is assumed that the solid part of the continuum (pipe, cable, etc.) can be modelled as a tensioned Euler-Bernoulli beam and a uniform plug-flow can be employed for the fluid flow description [Bolotin, 1963; Ziegler, 1968; Wickert and Mote, 1989; Lee and Mote,1997; Païdoussis, 1998].

In accordance with [Païdoussis, 1998; Lee and Mote, 1997], the linearized equation of the transverse motion of such continuum is given as

$$EI\partial_{xxxx}w - \left(N - m_{\text{tr}}U^2\right)\partial_{xx}w + 2m_{\text{tr}}U\partial_{xt}w + \left(m_{\text{tr}} + m\right)\partial_{tt}w = 0, \tag{1}$$

where ∂_x and ∂_t designate partial derivatives with respect to the coordinate and time, respectively, $w(x,t)$ is the transverse deflection of the continuum, m and m_{tr} are the masses per unit length of the non-translating and translating parts of the continuum, EI is the bending stiffness, and N is the axial tension. This equation can be reduced to that of a translating tensioned beam by setting $m = 0$ and interpreting U and m_{tr} as the translation speed and the mass per unit length of the beam. A tensioned pipe convening fluid is obtained if U , m and m_{tr} are interpreted as the speed of the fluid flow through the pipe, the mass per unit length of the pipe and the mass per unit length of the fluid, respectively.

3 Energy equation

The differential form of the energy equation of a one-dimensional continuum relates the energy of the continuum per unit length to the energy flux through a cross section of the continuum. In conservative systems, the energy equation is one of the conservations laws that can be readily obtained employing the Lagrangian formalism. The translating continua, being inherently non-conservative, do not allow such a straightforward treatment.

A countable set of "energy equations" can be obtained for the translating continua, all being in correspondence with the linearized equation of motion. A simple way to obtain the energy equations is to multiply the equation of motion by $\partial w/\partial t$ and then rearrange the result into a form that resembles the energy variation law. In what follows three possible results of such rearrangement are given and discussed.

One candidate for the energy equation is given as

$$\partial_t e_1 + \partial_x F_1 = 0, \tag{2}$$

where

$$2e_1 = EI\left(\partial_{xx}w\right)^2 + N\left(\partial_x w\right)^2 + m\left(\partial_t w\right)^2 + m_{\text{tr}}\left(\left(\partial_t w\right)^2 - U^2\left(\partial_x w\right)^2\right), \tag{3}$$

$$F_1 = EI\left(\partial_{xxx}w\right)\left(\partial_t w\right) - EI\left(\partial_{xx}w\right)\left(\partial_{xt}w\right) - N\left(\partial_x w\right)\left(\partial_t w\right) + m_{\text{tr}}U\left(\partial_t w\right)\left(\partial_t w + u\partial_x w\right) \tag{4}$$

In Eqs. (2)-(4), $e_1(x,t)$ and $F_1(x,t)$ are the candidates for the energy per unit length and the energy flux through a cross-section x of the continuum (both excluding the constant terms associated with the translation).

It is relatively easy to criticize Eqs. (3)-(4) as the expression for the "energy" $e_1(x,t)$ in Eq. (3) is not positive definite. Therefore, $e_1(x,t)$ can not pretend to describe the true energy density of the continuum. Usually, $e_1(x,t)$ is referred to as pseudo-energy [Goldstein, 1980]. Note that Eqs. (2) -(4) can not be used for calculating the energy of translating continua based on the energy flux, see [Metrikine, Battjes and Kuiper, 2006].

Consider the next candidate for the energy equation. It is given as

$$\partial_t e_2 + \partial_x F_2 = 0, \tag{5}$$

where

$$2e_2 = EI\left(\partial_{xx}w\right)^2 + N\left(\partial_x w\right)^2 + m\left(\partial_t w\right)^2 + m_{\text{tr}}\left(\partial_t w + U\,\partial_x w\right)^2 + 2mU\left(\partial_t w\right)\left(\partial_x w\right), \tag{6}$$

$$F_2 = EI\left(\partial_{xxx}w\right)\left(\partial_t w\right) - EI\left(\partial_{xx}w\right)\left(\partial_{xt}w\right) - N\left(\partial_x w\right)\left(\partial_t w + U\partial_x w/2\right) - mU\left(\partial_t w\right)^2/2 + m_{\text{tr}}U\left(\partial_t w + U\partial_x w\right)^2/2 + EI\left(\left(\partial_{xxx}w\right)\left(\partial_x w\right) - \left(\partial_{xx}w\right)^2/2\right) \tag{7}$$

Obviously, Eqs. (6)-(7) do not describe the true energy density and the true energy flux in the continuum. For example, the last term in Eq. (6) is not positive-definite and a number of terms in Eq. (7) are not physically interpretable.

The most apparent drawback of the "energy equations" (2) and (5) is that these equations do not contain what is generally believed to be the true energy of the translating continuum. The next possible "energy equation" is free of this drawback and is given as

$$\partial_t e_3 + \partial_x F_3 = \partial_t R, \tag{8}$$

$$2e_3 = EI\left(\partial_{xx}w\right)^2 + N\left(\partial_x w\right)^2 + m\left(\partial_t w\right)^2 + m_{\text{tr}}\left(\partial_t w + U\partial_x w\right)^2, \tag{9}$$

$$F_3 = EI\left(\partial_{xxx}w\right)\left(\partial_t w\right) - EI\left(\partial_{xx}w\right)\left(\partial_{xt}w\right) \\ -N\left(\partial_x w\right)\left(\partial_x w\right) + m_{\mathrm{tr}}U\left(\partial_t w + U\partial_x w\right)\left(\partial_t w\right) \tag{10}$$

$$R = m_{\mathrm{tr}}U\left(\partial_t w + U\partial_x w\right)\left(\partial_x w\right) \tag{11}$$

Equations (8),(9) and (11) are fairly interpretable. The expression for e_3, Eq. (9), is generally believed to describe the energy per unit length of the translating continuum [Wickert and Mote, 1989; Lee and Mote,1997; Païdoussis, 1998]. Furthermore, Eq. (8) shows that the energy of a segment of the continuum can change not only due to the energy flux through the end cross-sections of this segment but also due to the work of an external force acting at each cross-section of the segment. The rate of this work is given by $\partial_t R$ on the right-hand side of Eq. (8). The presence of this term is favourable as it can help explain the bulk instability of long translating continua. The expression for R given by Eq. (11) is quite transparent as well. It gives the axial projection of the force on a differential element of the continuum associated with the momentum variation caused by the translation. Thus, the set of equations (8)-(11) could be considered physically interpretable (in contrast to the previous sets) if not for Eq. (10). The latter equation is supposed to describe the energy flux through a cross-section of the continuum. The last term in Eq. (10) is hard to interpret though. It is supposed to describe the non-constant part of the energy flux that is associated with the continuum translation. Correspondingly, instead of the last term in Eq. (10) one would rather expect $m_{\mathrm{tr}}U\left(\partial_t w + U\partial_x w\right)/2$ which is simply the flux of the non-constant part of the kinetic energy of translation.

Thus, though multiple possibilities exist to formulate a sort of energy equation based on the linearized equation of motion, none of them seem to be physically interpretable. To resolve this, a nonlinear formulation of the problem is necessary. An example of such formulation will be considered in Section 5 of this paper. Before that, however, a brief look will be taken in the next section at the balance of pseudo-momentum in the translating continuum.

4 Balance of pseudo-momentum

In the linear equation of motion of the continuum, Eq. (1) , the axial displacement does not appear explicitly but its existence is nevertheless implied [Paidoussis, 2005]. The forces associated with this mathematically absent but physically existent motion (every transverse motion generates an axial counterpart) can sometimes be found by using the balance of pseudo-momentum [Vesnitski, Kaplan and Utkin, 1983]. The rate of change of pseudo-momentum is normally inapplicable as the measure for the force exerted by transverse waves [Rowland and Pask, 1999; Denisov, 2000]. However, see [Pippard, 1992], it is applicable to mechanical systems with ideal constraints provided that the force is analyzed using the approach proposed by Rayleigh [Rayleigh, 1902]. In any case, the balance of pseudo-momentum is a useful tool to uncover energy sources hidden in the models.

As well as in the case of the “energy equations”, multiple “momentum equations” can be deduced from Eq. (1). Below, only one possibility is shown that is in correspondence with the most plausible energy equation derived in the previous section, namely with Eq. (8). This candidate for the momentum equation reads

$$\partial_t p + \partial_x T = \partial_x R, \tag{12}$$

where

$$p = -m_{\mathrm{tr}}\left(\partial_t w + U\partial_x w\right)\left(\partial_x w\right) \tag{13}$$

$$2T = -2EI\left(\partial_{xxx}w\right)\left(\partial_x w\right) + EI\left(\partial_{xx}w\right)^2 \\ +N\left(\partial_x w\right)^2 + m_{\mathrm{tr}}\left(\partial_t w + U\partial_x w\right)^2 + m\left(\partial_t w\right)^2 \tag{14}$$

and R is given by Eq. (11).

In Eq. (12), p is the pseudo-momentum per unit length and T is the pseudo-momentum flux, both in the axial direction. These are also often referred to as the wave momentum and wave pressure, respectively [Vesnitski, Kaplan and Utkin, 1983; Pippard, 1992].

The wave momentum p in Eq. (13) can be thought of as the necessary axial counterpart of the vertical momentum (they differ by the multiplier $\partial_x w$) that reflects the fact that each infinitesimal element of the transversely vibrating continuum has a small axial velocity (additional to the velocity of translation), unless the slope of this element relative to the undeformed axis is zero. Note that the wave momentum and the axial momentum of a continuum, as a rule, are not the same [Rowland and Pask, 1999].

The pseudo-momentum flux T given by Eq. (14) is supposed to represent the axial counterpart of the vertical force in the cross sections of the continuum. It is hard (if not impossible), however, to see whether this is the case indeed without digging into a non-linear formulation of the problem. An example of such formulation is considered in the next section.

5 Variation of the energy and momentum in inextensible translating cable

Consider a relatively simple example of an inextensible cable that has a negligible bending stiffness. As independent variables the time t and the arc length s will be used, the latter identifying a particular material point of the cable. Dependent variables will be the position of this point $\mathbf{r}(s,t)$ and the force in the corresponding cross-section. The condition of inextensibility entails that $\partial_s\mathbf{r}$ is a unit vector:

$$(\partial_s\mathbf{r})\cdot(\partial_s\mathbf{r})=1\,. \tag{15}$$

The equation of motion of a differential element of the cable can be readily derived and is given as

$$m_{\mathrm{tr}}\left(\partial_{tt}\mathbf{r}+2U\partial_{st}\mathbf{r}+U^2\partial_{ss}\mathbf{r}\right)-\partial_s\left(\sigma\,\partial_s\mathbf{r}\right)=0, \tag{16}$$

where $\sigma(s,t)\partial_s\mathbf{r}(s,t)$ is the force vector in a cross-section s.

The energy per unit length of the cable is clearly given as

$$e_{\mathrm{c}}(s,t)=\frac{1}{2}m_{\mathrm{tr}}\left(\partial_t\mathbf{r}+U\,\partial_s\mathbf{r}\right)^2, \tag{17}$$

which is simply the density of the kinetic energy of the cable as $\left(\partial_t\mathbf{r}+U\,\partial_{st}\mathbf{r}\right)$ is the velocity of a differential element of the cable. Note that as the cable is assumed inextensible, it does not possess the potential energy.

The energy flux trough a cross-section consists of the rate of work done by the force $(\sigma\,\partial_s\mathbf{r})$ and of the flux of the kinetic energy due to translation:

$$F_{\mathrm{c}}(s,t)=-\sigma(\partial_s\mathbf{r})\cdot(\partial_t\mathbf{r})+\frac{1}{2}m_{\mathrm{tr}}u\left(\partial_t\mathbf{r}+U\,\partial_s\mathbf{r}\right)^2. \tag{18}$$

The axial projection of the momentum can also be readily written as

$$p_{\mathrm{c}}(s,t)=m_{\mathrm{tr}}\left((\partial_t\mathbf{r})(\partial_s\mathbf{r})+U\right). \tag{19}$$

Let us now consider the small slope approximation in the $x-y$ plane. In this approximation the vector $\mathbf{r}(s,t)$ and $\sigma(s,t)$ can be written as [Broer, 1970]:

$$\mathbf{r}(s,t)=\left[x(s,t)\;\;y(s,t)\;\;0\right], \tag{20}$$

$$x(s,t)=s+\varepsilon^2u(s,t), \tag{21}$$

$$y(s,t)=\varepsilon w(s,t), \tag{22}$$

$$\sigma(s,t)=N+\varepsilon^2\tau(s,t). \tag{23}$$

where ε is a small dimensionless parameter.

Inserting equations (21)-(23) in Eq. (16) the following equations of motion in the vertical and horizontal directions are obtained:

$$m_{\mathrm{tr}}\left(\partial_{tt}w+2U\partial_{st}w+U^2\partial_{ss}w\right)-N\partial_{ss}w=0, \tag{24}$$

$$m_{\mathrm{tr}}\left(\partial_{tt}u+2U\partial_{st}u+U^2\partial_{ss}u\right)-N\partial_{ss}u-\partial_s\tau=0. \tag{25}$$

Obviously, Eq. (24) is a particular case of Eq. (1) that can be obtained by setting $EI=m=0$.

The condition of extensibility (15) reduces to

$$\partial_s u=-\frac{1}{2}(\partial_s w)^2. \tag{26}$$

The energy density and the energy flux, Eqs. (17) and (18), upon substitution of Eqs. (21)-(23) take the form:

$$e_{\mathrm{c}}=\frac{m_{\mathrm{tr}}}{2}\left(U^2+\varepsilon^2\left(\partial_t w+2U\left(\partial_t u+(\partial_t w)(\partial_s w)\right)\right)\right) \tag{27}$$

$$\begin{aligned}F_c=&\frac{m_{\mathrm{tr}}U}{2}\left(U^2+\varepsilon^2\left(\partial_t w+2U\left(\partial_t u+(\partial_t w)(\partial_s w)\right)\right)\right)\\&-N\varepsilon^2\left(\partial_t u+(\partial_t w)(\partial_s w)\right)\end{aligned} \tag{28}$$

Obviously, both e_{c} and F_{c} depend on the horizontal movement $u(s,t)$ of the cable.

Expressions (27), (28) can not be directly compared with those obtained in Section 3 based on the equation of motion in the vertical direction. To enable the comparison, the horizontal movement $u(s,t)$ should be expressed through the vertical movement $w(s,t)$ using the condition of inextensibility, Eq. (26). As follows from the latter condition

$$\partial_{st}u=\partial_{ts}u=-(\partial_s w)(\partial_{ts}w), \tag{29}$$

$$\partial_{ss}u=-(\partial_s w)(\partial_{ss}w). \tag{30}$$

Using the above equations and employing the equation of motion in the horizontal direction, Eq. (25), one can obtain the following equations for $\partial_t e_{\mathrm{c}}$ and $\partial_s F_{\mathrm{c}}$:

$$\begin{aligned}\varepsilon^{-2}\partial_t e_{\mathrm{c}}=&\frac{m_{\mathrm{tr}}}{2}\partial_t\left((\partial_t w+U\partial_s w)^2+(U\partial_s w)^2\right)\\&+U\partial_s\left(\tau-\frac{1}{2}(\partial_s w)^2\left(N-m_{\mathrm{tr}}U^2\right)\right)\end{aligned} \tag{31}$$

$$\varepsilon^{-2}\partial_s F_{\mathrm{c}} = \left(N - m_{\mathrm{tr}} U^2\right)\partial_t \left(\left(\partial_s w\right)^2\right) + \partial_s \left(\frac{1}{2} m_{\mathrm{tr}} U \left(\partial_t w\right)^2 - \left(N - m_{\mathrm{tr}} U^2\right)\left(\partial_s w\right)\left(\partial_t w\right)\right) \quad (32)$$

It can be readily seen that expressions (31) and (32) significantly differ from the corresponding particular cases of Eqs. (3),(6), (9) and Eqs. (4), (7), (10). For example, as follows from Eqs. (9) and (10) $\partial_t e_3$ and $\partial_x F_3$ in the case $EI = m = 0$ read

$$\partial_t e_3 = N\left(\partial_x w\right)^2 + m_{\mathrm{tr}}\left(\partial_t w + U\partial_x w\right)^2, \quad (33)$$

$$\partial_x F_3 = -N\left(\partial_x w\right)\left(\partial_x w\right) + m_{\mathrm{tr}} U\left(\partial_t w + U\partial_x w\right)\left(\partial_t w\right), \quad (34)$$

which is very different from equations (31) and (32). It is very unlikely and, most likely, impossible that expressions (31) and (32) can be guessed based on Eq. (24) alone. It is also clear that if the cable were assumed extensible, the rate of the energy density, Eq. (31) would be different though the linearized equation of the vertical motion would remain the same.

Thus, the following, most important, conclusion of this study can be drawn: the linearized equation of motion and the expressions for the energy flux in and the energy density of a translating continuum are not in one-to-one correspondence. Though a number of physical situations would result in the same linearized equation of transverse motion, the corresponding energy equations would all be different.

Let us now deduce the energy equation of the inextensible cable. Using equations (31) and (32) it can be found that

$$\partial_t e_{\mathrm{c}} + \partial_s F_{\mathrm{c}} = V\partial_s \tau \quad (35)$$

This equation shows that the energy of a segment of the cable can change due to the energy flux through the end cross-sections of this segment and due to the work of the axially varying stress in the cable. This work depends on the horizontal dynamics of the cable and can not be found without solving Eq. (25) subject to specific boundary conditions.

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