ON THE DESTRUCTION OF ISLANDS OF STABILITY IN A TOKAMAK WITH ERGODIC MAGNETIC LIMITER USING KAM THEORY

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Abstract
Chaotic magnetic field lines play an important role in plasma confinement by tokamaks. They can either be generated in the plasma as results of natural instabilities or artificially produced by external conductors like ergodic magnetic limiter (EML). We consider a symplectic map for magnetic field lines in a tokamak with an ergodic limiter the nature of fixed points of EML map are studied in detail as perturbation parameter p and magnetic shear s are varied. The critical perturbation is determined for the surface with rotational-transform equal to the inverse golden mean. We describe the changes and the destruction of islands of stability for EML map. As the perturbation parameter increases the size of the island increases and then decreases abruptly. This decrease is due to the joining of an outer and inner chaotic domain. Critical value of the limiter strength p necessary to global chaos for the secondary islands as a function of the magnetic shear s is obtained.

Key words
Chaos, Nonlinear systems, Numerical methods.

1 Introduction

One of the areas in which chaotic dynamics has received most attention in recent years is plasma physics. In this paper we will focus on a specific type of plasmas, namely those generated magnetically confined in fusion machines, chaotic dynamics is one of the striking properties of the magnetic field lines in tokamaks and other fusion machines [Martin and Taylor, 1984; Portela, Viana and Caldas, 2003; Balescu, Vlad and Spineanu, 1998; Balescu, 1998]. We parametrize the field lines by using spatial ignorable coordinate (an azimuthal angle for axisymmetric configuration like in tokamaks). This parameter plays the role of time, so that magnetic field line equations can be viewed as canonical equations. One of the advantages of this approach is the possibility of describing field lines by means of a two dimentional Hamiltonian map. The equilibrium configurations are integrable systems whereas symmetry-breaking field perturbations such as EML[Martin and Taylor, 1984; Portela, Viana and Caldas, 2003] spoil their integrabilities, this may lead to chaotic behavior[Meiss, 1992; Lichtenberg and Lieberman, 1983; ott, 1993].

One of the applications of the chaotic magnetic field lines is to control the impurities released from the inner wall of the tokamaks. Another application is related to the control of disruptive instabilities, which are usually preceded by Mirnov oscillations.

For a periodic Hamiltonian system with two degrees of freedom the trajectory of the system winds around a torus which is called KAM surface. To study the motion on the torus the natural way is to look at the dynamics on the surface of section of the torus. The trajectories on the surface of section are called KAM curves.

For the perturbed Hamiltonian systems the possible stable fixed points are elliptic, near the elliptic point trajectories encircle it and do not approach it. These curves are known as main islands or simply islands.

The main purpose of this paper is to study the existence of the most noble KAM surface, changes and destruction of the islands of stability, in the EML map. In particular we describe a numerical procedure to find accurately the barrier transition to global chaos, and investigate validity of the chirikov overlapping criterion [Chirikov, 1979].

The rest of this paper is organized as follows: in the next section we summarized the theory of integrable and near integrable Hamiltonian systems in order to derive the EML map. Section III shows the destruction of the inverse golden mean KAM surface and finally in section V we'll discuss about the size of the main island with respect to perturbation parameter p for the EML map. The last section devoted to our conclusion.

2 The Ergodic magnetic limiter map

A tokamak Fig(1) contains plasma by the combined of two basic magnetic fields, a toroidal field $\overrightarrow{B_T}$ along the φ coordinate and poloidal field $\overrightarrow{B_p}$ along θ coordinate. The variables φ and θ are usually quoted as poloidal and toroidal angle. The equilibrium field $\overrightarrow{B_0} = \overrightarrow{B_T} + \overrightarrow{B_p}$ has a helical shape and lies on constant-pressure flux surface. The winding number or rotational-transform is the average poloidal angle swept by field line after one complete toroidal turn. Mathematically it is given by:

$$i(r) = 2\pi \frac{d\theta}{d\Phi} = \frac{2\pi}{q(r)} \qquad (1)$$

Where q(r) is safety factor of the flux surface.

We consider straight tokamak, neglect toroidal curvature and approximate it by a periodic cylindrical coordinate with coordinates r, θ and z = $R_0\Phi$ such that the tokamak

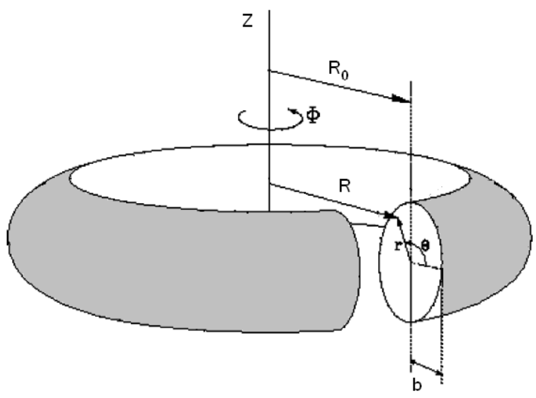


Figure 1. The local coordinates (r , θ , φ = Φ) in tokamak.

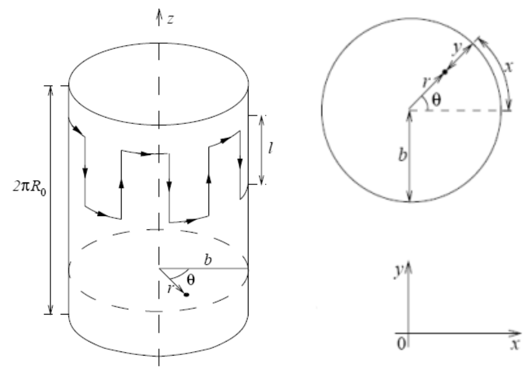


Figure 2. Periodic cylindrical approximation (straight tokamak) and rectangular coordinate system x and y.

wall is located at r = b [Martin and Taylor, 1984; Portela, Viana and Caldas, 2003].

In cylindrical approximation a flux surface is characterized by q(r) = const .In particular $q_b = q(r = b)$ is the magnetic shear:

$$s = \frac{2\pi b}{{q_b}^2} \frac{dq}{dr} \mid_{r=b} \tag{2}$$

The EML configuration we consider consists of a grid of current carrying wires in toroidal and poloidal directions(Fig(2)). There are m pairs of toroidal segments, equally spaced along poloidal direction, with adjacent segments conducting a current I in opposite senses. only the toroidal segments are relevant to the perturbing field of EML which produce islands structures among the flux surfaces and ergodic regions in which the flux surfaces are destroyed [Martin and Taylor, 1984].

We introduce a rectangular coordinate system (Fig(2)) x = bθ and y = b – r where x is periodic and is in the range $0 < x < 2\pi$. y is radial distance from the wall and y > 0 is inside and y < 0 is outside the tokamak.

The full derivation of EML map is done somewhere else [Martin and Taylor, 1984; Portela, Viana and Caldas, 2003], we just summarize the results.

Outside the limiter (l < z < $2\pi R_0$), the magnetic field line equation can be exactly integrated: to give the coordinates of (n + 1)th mapping point on the Poincare surface of section z = 0.

$$T_1 \begin{cases} x_{n+1} = x_n^* + \alpha + s y_n^* \\ y_{n+1} = y_n^* \end{cases} \tag{3}$$

Where x_n^* and y_n^* are auxiliary variables sampled at the EML edge z = l, and we define $\alpha = \frac{2\pi b}{q_b}$.

Inside the limiter the magnetic field line equations are near integrable

$$T_2 \begin{cases} x_n'^* = x_n' - p e^{-y_n'} \cos x_n' \\ y_n'^* = y_n' + \ln\left[\cos\left(x_n' - p e^{-y_n'} \cos x_n' \right)\right] \\ \qquad - \ln(\cos x_n') \end{cases} \tag{4}$$

The EML map can be written as the composition of two mapping $T = T_2 * T_1$. Dropping the primes of the variables for simplicity

$$T \begin{cases} x_{n+1} = x_n + s y_n + g(x_n, y_n) \\ y_{n+1} = y_n + h(x_n, y_n) \end{cases} \tag{5}$$

where

$$\begin{aligned} h(x,y) &= \ln\left[\frac{\cos(x - pe^{-y}\cos x)}{\cos x}\right] \\ g(x,y) &= -pe^{-y}\cos x + s h(x,y) \end{aligned} \tag{6}$$

The map (5)-(6) is exactly area preserving, for its Jacobian is equal to unity.

3 Destruction of the inverse golden mean surface

A map T^n denotes the quantity obtained upon inserting inside itself n-times. If after iterating the map n-times one returns to the same point, then the map has a periodic orbit of period n. on the other hand, irrational tori are densely filled-out upon repeated iteration. An important quantity is the rotation number, w, which is defined as [Morrison, 2000; Hudson, 2004]

$$w = \lim_{n \to \infty} \frac{\Delta\theta_n}{n} \tag{7}$$

Where θ is an arbitrary poloidal angle coordinate and $\Delta\theta_n = \theta_n - \theta_{n-1}$ is the angle in the nth iteration.

It is likely that any finite approximation to the rotational-transform limit will be a non-monotonic function of position in the chaotic region. In regular regions of space occupied predominantly with flux surfaces the situation is different. On KAM surfaces, the limit will converge to arbitrary accuracy by following a field line for a sufficient distance and efficient methods for evaluating the transform exist. Finally for the periodic field lines which close after a finite number of transits, the rotational-transform can be determined exactly after following the field line a finite distance.

For the surface with rotational-transform equal to the inverse golden mean γ^{-1} = [0,1,1,1,…] = 0.61803398874989 , The convergents are 1/2, 2/3, 3/5, …, 377/610, 610/987, 987/1597, 1597/2584, 2584/4181,… . The denominator of each convergent corresponds to the number of secondary islands which are located inside and outside the γ^{-1} surface respectively.

The rotational-transform of the main island for different values of the perturbation parameter p is determined numerically using formula (7). We found that the golden

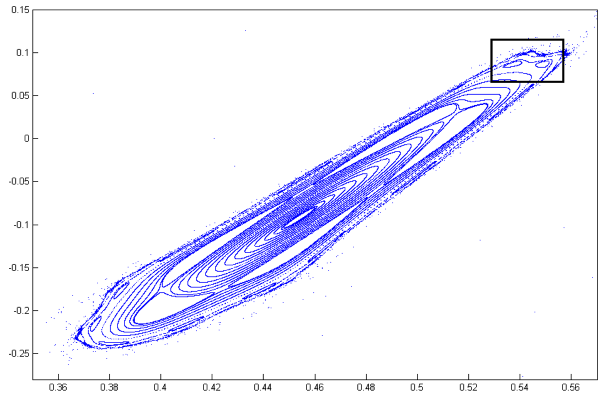


Figure 3. Poincare´ plot showing the golden mean surface for p = 0.550925 and s = 2π**.** While the golden mean surface is destructed, there still exist KAM surfaces. The region in the rectangle is shown in Fig(4) and Fig(5).

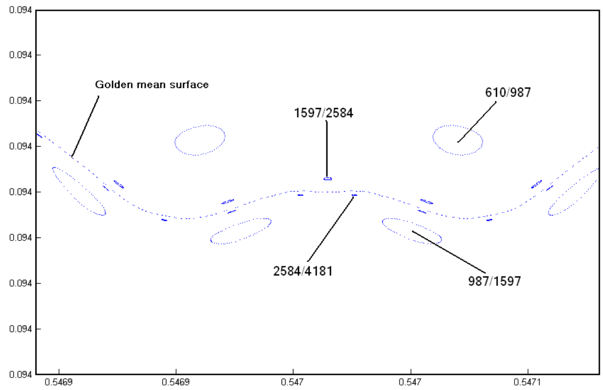


Figure 4. Poincare plot of the main island for s = 2π & p = 0.550920, before criticality (the rectangle area in Fig(3)).

mean surface exists for $0.519 \leq p \leq 0.550925$ for s = 2π. For less-than-critical perturbation p = 0.550920 and for-larger-than critical perturbation p = 0.550925 the detailed Poincare plots Fig(3), Fig(4) and Fig(5) confirm the breakup of the γ^{-1} surface.

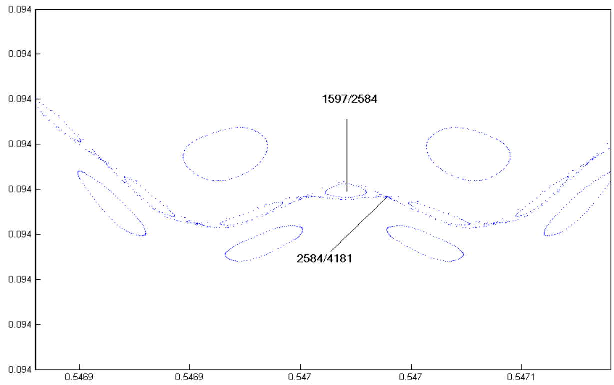


Figure 5. Poincare plot of the main island for s = 2π and p = 0.550925 , after criticality (the rectangle area in Fig(3)).

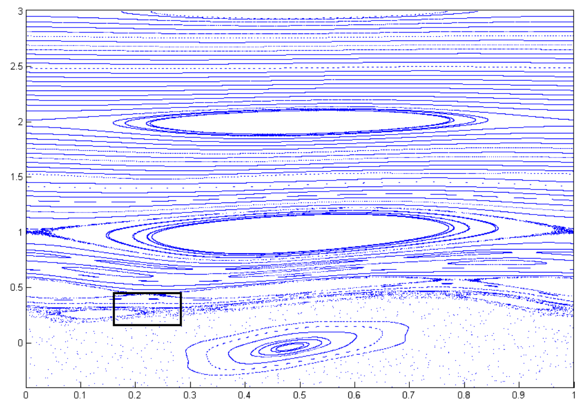


Figure 6. Poincare´ plot showing the golden mean surface, and the periodic orbits corresponding to its convergents, near criticality.

For the open KAM surfaces the γ^{-1} surface breaks up for approximately around p = 0.2251 for s = 2π which is illustrated in Fig(6), Fig(7) and Fig(8). As the figures for closed and open KAM show when the γ^{-1} surface is broken there still exist other noble surfaces.

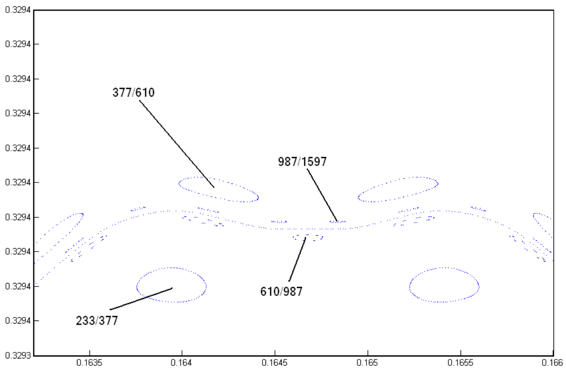


Figure 7. Poincare plot of the main island for s = 2π and p = 0.22540 before criticality (the rectangle area in Fig(3))

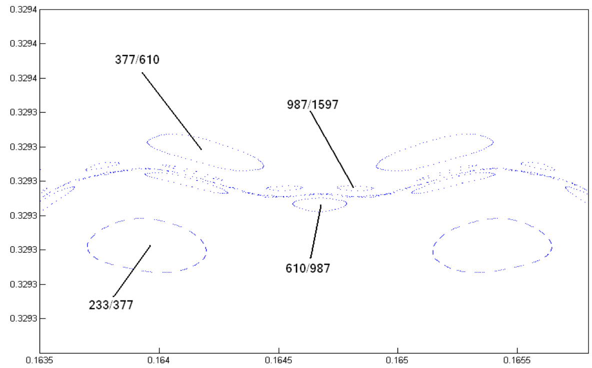


Figure 8. Poincare plot of the main island for s = 2π and p = 0.22550 after criticality (the rectangle area in Fig(3))

4 The width of the main island and evolution of the secondary islands to global chaos

The structure of an island is well known. It consists of invariant curves surrounding a stable periodic orbit and a hierarchy of secondary islands. The island is limited by a last KAM curve and beyond it there is a large chaotic sea, i.e. a large connected chaotic domain. However, as p increases, the last KAM curve is destroyed, i.e. it becomes a cantorus [Contopoulos, Harsoul, Voglis and Dvorak, 1999; Efthymiopoulos, Contopoulos, Voglis and Dvorak, 1997].

The size of an island, which enables us to estimate the size of the chaotic region, can be found if we know the last KAM curve around it. We measured the width of the main island which is the distance along x-axis from the center of the island. In Fig(9) the size of the main island n = 0 is shown for different values of p and s = 2π. As p increases, the size of the last KAM curve of the main island increases. But when this size becomes maximum the new last KAM curve is also destroyed and the size of the island decreases again (see the drops of the curve of Fig(9) at the resonances 1/21, 1/9, 1/8, …).

The set of the secondary islands moves outwards as p increases further and comes near the border of the main island. Eventually the last KAM curve changes and comes inside this set of islands, that are now left in the large 'chaotic sea'.

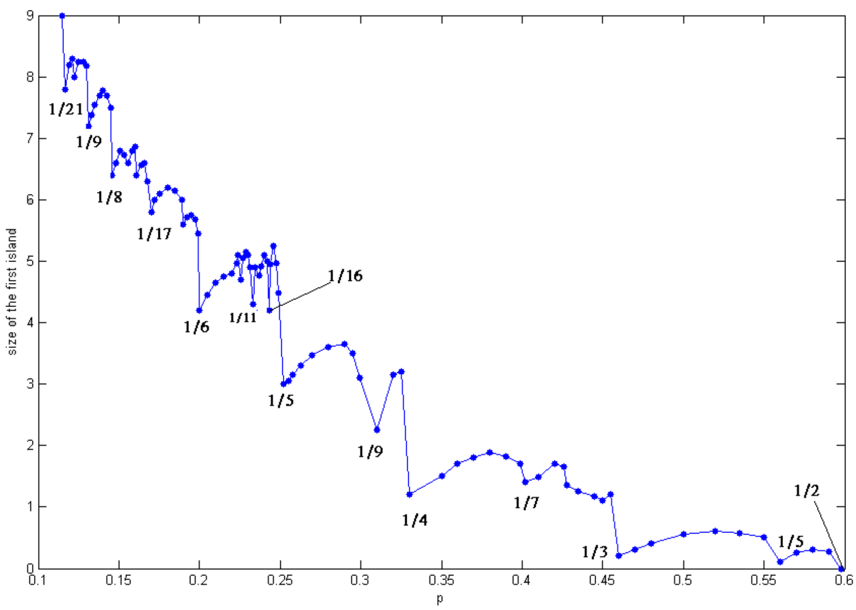
As Fig(9) shows last KAM curve does not exist for p approximately greater than 0.6, which is confirmed by Greene residue criterion[Mackay, 1992; Green,1979; Bountis and Helleman,1981; Marmi and stark, 1981]:

$$R = \frac{2 - \mathrm{TrM}}{4}$$

Where R is called residue and M is the linearized Jacobian matrix. If $0 \leq R \leq 1$ the eigenvalues of M are pure imaginary and complex conjugate of each other so that the fixed point is an elliptic point, or a center. Otherwise the eigenvalues are real numbers and the fixed point is a (hyperbolic) saddle point.

Fig(10) is the plot of R with respect to p for s = 2π which shows for p greater than 0.6, R is greater than 1 and therefore there is no last KAM for p > 0.6.

Figure 9. The width of the main island with respect to different values of p for s = 2π



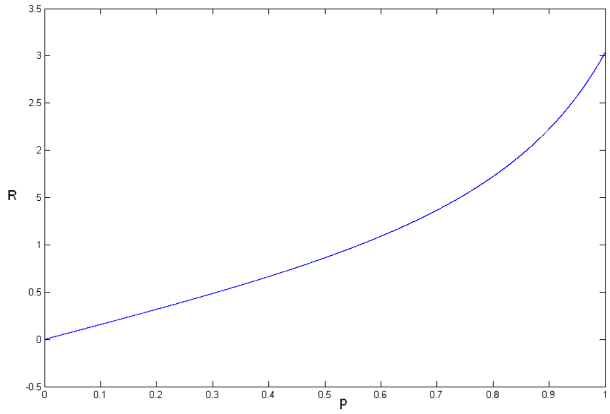


Figure 10. Residue with respect to perturbation parameter p for s = 2π.

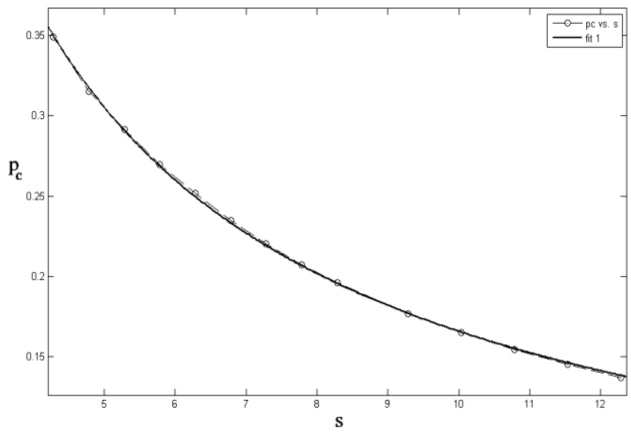


Figure 11. Critical value p_c with respect to s for secondary islands 1/5.

In Fig(9) we noticed that the dramatic reduction in the size of the main island occurs for 1/5 and 1/4 islands. Inside the last KAM curve there are small domains of chaos around each unstable periodic orbit, but these chaotic domains do not communicate with the outer chaotic sea, i.e. a large connected chaotic domain.

For p close to p_c the chaotic domain around each secondary island merges to chaotic sea. In Fig(11) and Fig(12) we plot the critical value of perturbation parameter p_c for different values of magnetic shear s for secondary islands 1/5 and 1/4 respectively. we select a point in a chaotic domain around each island and iterate the map 500,000 times. If these points enter the chaotic sea then p has its critical value p_c. Next we increase p by a small amount, say 10^{-4} , and repeat this procedure.

The full line is a least-squares power-law fit

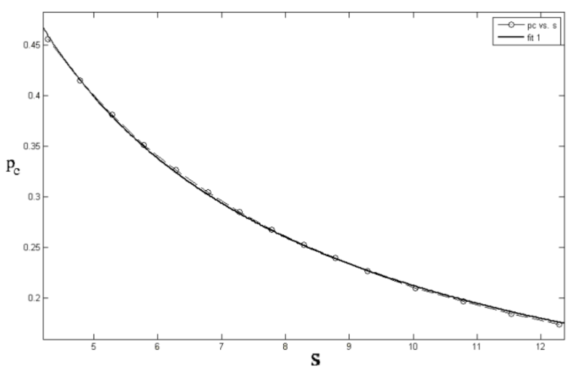
$$p_c(\mathrm{s}) \sim s^{-\sigma}$$

Where $\sigma \approx 0.9$.

5 Conclusion

For EML map we found that the open and closed last KAM torus has not the simplest noble rotation number. We found for a fixed value of magnetic shear the value of p beyond which there does not exist any KAM surface for the main island. This value was confirmed by the Greene residue theorem. A power law relation was derived between critical perturbation parameter and magnetic shear for the secondary islands.

Figure 12. Critical value p_c with respect to s for secondary islands 1/4.



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