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BOUNDING A DOMAIN CONTAINING ALL COMPACT INVARIANT SETS OF THE SYSTEM MODELLING THE RAYLEIGH-BÉNARD CONVECTION: THE SYMMETRY-BASED APPROACH

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Abstract

In our paper we study the localization problem of compact invariant sets of the system modelling the Rayleigh-Bénard convection. Our results are based on using the first order extremum conditions, quadratic localizing functions and the symmetrical prolongation constructed between the Lorenz system and this system.

Keywords: polynomial system; Lorenz system; localization; compact invariant sets; first-order extremum conditions

1 Introduction and some preliminaries

Localization problem of attractors of the Lorenz system [Lorenz, 1963] was examined in many papers, see eg. [Krishchenko and Starkov, 2006] with references therein because of its importance for studies of deterministic chaos of systems with complex dynamics and various applications in secure communications and other areas. In this paper we propose to investigate a location of all compact invariant sets of one 5-dimensional analog of the Lorenz system. The approach of this paper is based on using the first order extremum conditions and was described in details in our papers [2 – 4]. For our convenience, we recall some helpful results.

We consider a polynomial system

$$\dot{x} = f(x); \qquad (1)$$

$x \in \mathbf{R}^n$ is the state vector. Let $h(x)$ be a polynomial function such that h is not the first integral of (1). By $h|_B$ we denote the restriction of h on a set $B \subset \mathbf{R}^n$. By S_h we denote the set

$$S_h = \{x \in \mathbf{R}^n \mid \mathcal{L}_f h(x) = 0\},$$

where $\mathcal{L}_f h(x)$ is the Lie derivative.

Let us define

$$h_{\inf} = \inf\{h(x) \mid x \in S_h\};$$
$$h_{\sup} = \sup\{h(x) \mid x \in S_h\}.$$

The following assertion is useful in this paper [Krishchenko and Starkov, 2006].

Theorem 1 Each compact invariant set Γ of (1) is contained in the localization set

$$K(h) = \{x \in \mathbf{R}^n \mid h_{\inf} \le h(x) \le h_{\sup}\}.$$

The function h used in the formulation of this result is called localizing. A refinement of a localization bound is realized with help of

Theorem 2 Let $h_j(x), j = 1, 2$ be functions from $C^\infty(\mathbf{R}^n)$. Sets

$$K_1 = K(h_1);$$
$$K_2 = K_1 \cap K_{12},$$

with

$$K_{12} = \{x : h_{2,inf} \le h_2(x) \le h_{2,sup}\},$$
$$h_{2,sup} = \sup_{S(h_2)\cap K_1} h_2(x),$$
$$h_{2,inf} = \inf_{S(h_2)\cap K_1} h_2(x),$$

contain any compact invariant set of the system (1) and $K_1 \supseteq K_2$.

It is evident that if all compact invariant sets are located in sets N_1 and N_2, with $N_1; N_2 \subset \mathbf{R}^n$, then they are located in the set $N_1 \cap N_2$ as well.

2 The Rayleigh-Bénard convection - Lorenz model

The Rayleigh-Bénard convection - Lorenz model

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\ \dot{x}_2 &= -\sigma x_2 - \sigma y_2 \\ \dot{y}_1 &= r x_1 - x_1 z - y_1 \\ \dot{y}_2 &= -r x_2 + x_2 z - y_2 \\ \dot{z} &= x_1 y_1 - x_2 y_2 - bz \end{aligned} \tag{2}$$

was derived by Chen and Price developing the five-mode truncation scheme, with three modes used by Lorenz [Lorenz, 1963] to ensure the occurence of chaos and two other modes chosen to retain symmetry of the Rayleigh-Bénard convection in the truncation model, see [Chen and Price, 2006]. We recall that the Rayleigh-Bénard convection problem describing a fluid motion in a layer of fluid of uniform depth heated from below is based on the Boussinesq approximation applied to a coupled systems involving the Navier-Stokes equation, thermal conductivity equation, continuity equation and density approximation. Parameters of the system (2) have the same values as for the Lorenz system, [Lorenz, 1963]. The resulting system (2) resembles some important features in the behaviour of the Lorenz system dynamics. It is also can be seen from the careful algebraic analysis of right sides of both of systems. That is why it is not surprising that it is possible to transfer some localization results obtain earlier for the Lorenz system, see in [Krishchenko and Starkov, 2006], for the case of the system (2).

Suppose that a system $\mathcal{S}$ has an invariant set M and the set Ω contains all compact invariant sets of $\mathcal{S}$. Then the set $M \cap \Omega$ is a localization set for the system $\mathcal{S}|_M$. We note that in some cases one can propose a method of constructing a localization set for the system $\mathcal{S}$ if we know a localization set ω for the system $\mathcal{S}|_M$. It i clear that in this case the resulting localization set for $\mathcal{S}$ depends on the continuation method of $\mathcal{S}|_M$ up to $\mathcal{S}$.

It is found that the system (2) is a special continuation of the Lorenz system which can be called a mirror continuation. Based on this continuation we can associate localizing functions for the system (2) with localizing functions for the Lorenz system.

3 Symmetrical prolongations

Let us analyse the system (2). The change of variables $y_2 \to -y_2$ transforms this system into the system

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\ \dot{x}_2 &= -\sigma x_2 + \sigma y_2 \\ \dot{y}_1 &= r x_1 - x_1 z - y_1 \\ \dot{y}_2 &= r x_2 - x_2 z - y_2 \\ \dot{z} &= x_1 y_1 + x_2 y_2 - bz. \end{aligned} \tag{3}$$

Coordinate $3D$-planes

$$\begin{aligned} Ox_1y_1z &= \{x_2 = 0,\, y_2 = 0\}, \\ Ox_2y_2z &= \{x_1 = 0,\, y_1 = 0\} \end{aligned}$$

are invariant manifolds of system (3).

The restriction of the system (3) on the invariant plane Ox_1y_1z coinsides with the Lorenz system

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\ \dot{y}_1 &= r x_1 - x_1 z - y_1 \\ \dot{z} &= x_1 y_1 - bz. \end{aligned} \tag{4}$$

just as the restriction of the system (3) on the invariant plane Ox_2y_2z

$$\begin{aligned} \dot{x}_2 &= -\sigma x_2 + \sigma y_2 \\ \dot{y}_2 &= r x_2 - x_2 z - y_2 \\ \dot{z} &= x_2 y_2 - bz. \end{aligned}$$

The intersection of the indicated invariant $3D$-planes $Ox_1y_1z \cap Ox_2y_2z$ is the axis Oz. The restriction of the system (3) on Oz is the system

$$\dot{z} = -bz.$$

Let D_1 be the set of vector fields on $R^3 = \{(x_1, y_1, z)\}$ of the form

$$\phi(x_1, y_1, z)\frac{\partial}{\partial x_1} + \psi(x_1, y_1, z)\frac{\partial}{\partial y_1} + \xi(x_1, y_1, z)\frac{\partial}{\partial z},$$

where $\xi(x_1, y_1, z)$ is a polynomial in x_1, y_1,

$$\xi(x_1, y_1, z) = \sum_{i+j=0}^{s} c_{ij}(z) x_1^i y_1^j,$$

and let D be the set of vector fields on $R^5 = \{(x_1, x_2, y_1, y_2, z)\}$.

The map $symm : D_1 \to D$ is called a symmetrical prolongation of vector fields if

$$\begin{aligned} & symm(\phi(x_1, y_1, z)\frac{\partial}{\partial x_1} + \psi(x_1, y_1, z)\frac{\partial}{\partial y_1} + \\ & \xi(x_1, y_1, z)\frac{\partial}{\partial z}) = \phi(x_1, y_1, z)\frac{\partial}{\partial x_1} + \\ & \phi(x_2, y_2, z)\frac{\partial}{\partial x_2} + \psi(x_1, y_1, z)\frac{\partial}{\partial y_1} + \\ & \psi(x_2, y_2, z)\frac{\partial}{\partial y_2} + Symm(\xi(x_1, y_1, z))\frac{\partial}{\partial z}, \end{aligned}$$

where $Symm(\xi(x_1, y_1, z))$ is a symmetrical prolongation of a polynomial in x_1, y_1,

$$\begin{aligned} & Symm(\xi(x_1, y_1, z)) = Symm(\sum_{i+j=0}^{s} c_{ij}(z) x_1^i y_1^j) = \\ & c_{00}(z) + \sum_{i+j=1}^{s} c_{ij}(z)(x_1^i y_1^j + x_2^i y_2^j). \end{aligned}$$

Let

$$F = (-\sigma x_1 + \sigma y_1)\frac{\partial}{\partial x_1} + (-\sigma x_2 + \sigma y_2)\frac{\partial}{\partial x_2} + (rx_1 - x_1 z - y_1)\frac{\partial}{\partial y_1} + (rx_2 - x_2 z + y_2)\frac{\partial}{\partial y_2} + (x_1 y_1 + x_2 y_2 - bz)\frac{\partial}{\partial z}$$

be a vector field of system (3), and

$$L = (-\sigma x_1 + \sigma y_1)\frac{\partial}{\partial x_1} + (rx_1 - x_1 z - y_1)\frac{\partial}{\partial y_1} + (x_1 y_1 - bz)\frac{\partial}{\partial z}$$

be the vector field of system (4).

Then $L \in D_1$, $F = symm(L)$, and for any polynomial $p(x_1, y_1, z)$ in x_1, y_1,

$$\mathcal{L}_F Symm(\phi(x_1, y_1, z)) = Symm(\mathcal{L}_L \phi(x_1, y_1, z)).$$

At last, we notice that localization sets obtained below by using quadratic surfaces for the system (3) are the same as for the system (2) because corresponding localizing functions are even functions respecting the variables $y_1; y_2$.

4 Ellipsoidal localization

Let us take for the system (4) the localizing function [Krishchenko and Starkov, 2006]

$$h = x_1^2 + qy_1^2 + qz^2 - 2(\sigma + qr)z,$$

where $q > 0$. Then

$$\mathcal{L}_L h = 2(-\sigma x_1^2 - qy_1^2 - qbz^2 + (\sigma + qr)bz).$$

Let us take for the system (2) the localizing function

$$H = Symm(h) = x_1^2 + x_2^2 + q(y_1^2 + y_2^2 + z^2) - 2(\sigma + qr)z,$$

Then

$$\mathcal{L}_F H = 2(-\sigma(x_1^2 + x_2^2) - q(y_1^2 + y_2^2) - qbz^2 + (\sigma + qr)bz) = Symm(\mathcal{L}_L h).$$

and

$$S_H = \{(x_1, y_1, x_2, y_2, z) : \ \sigma(x_1^2 + x_2^2) + q(y_1^2 + y_2^2) + qbz^2 - (\sigma + qr)bz = 0\}$$

which is an ellipsoid because it is also expressed by

$$\sigma x_1^2 + \sigma x_2^2 + qy_1^2 + qy_2^2 + qb(z - \frac{\sigma + qr}{2q})^2 = R := \frac{(\sigma + qr)^2 b}{4q}.$$

Thus on S_H

$$|x_{1,2}| \le \sqrt{\frac{R}{\sigma}}; \quad |y_{1,2}| \le \sqrt{\frac{R}{q}};$$

$$|z - \frac{\sigma + qr}{2q}| \le \sqrt{\frac{R}{qb}} = \frac{\sigma + qr}{2q},$$

and under

$$0 \le z \le (\sigma + qr)/q$$

we get

$$qz^2 - 2(\sigma + qr)z \le 0,$$

therefore,

$$H_{\sup} \le [x_1^2 + x_2^2 + q(y_1^2 + y_2^2)] \mid_{S_H} \le 2\frac{R}{\sigma} + 2R = 2\frac{R}{\sigma}(\sigma + 1) = \frac{(\sigma + qr)^2 b}{2q\sigma}(\sigma + 1).$$

So each compact invariant set is contained in

$$K_q(H) := \{x_1^2 + x_2^2 + q(y_1^2 + y_2^2 + z^2) - 2(\sigma + qr)z \le \frac{(\sigma + qr)^2 b(\sigma + 1)}{2q\sigma}, q > 0\}.$$

The set $K_q(H)$ can be written as

$$\{x_1^2 + x_2^2 + q(y_1^2 + y_2^2) + q(z - \frac{\sigma + qr}{q})^2 \le \frac{(\sigma + qr)^2(b + b\sigma + 2\sigma)}{2\sigma q}, q > 0\}.$$

Since

$$x_1^2 + x_2^2 \le \frac{(\sigma + qr)^2(b + b\sigma + 2\sigma)}{2\sigma q}, q > 0,$$

we have that

$$x_1^2 + x_2^2 \le \min_{q>0} \frac{(\sigma + qr)^2(b + b\sigma + 2\sigma)}{2\sigma q} = 2r(b + b\sigma + 2\sigma).$$

As a result, we get bounds

$$| x_j | \leq \sqrt{2r(b+b\sigma+2\sigma)}, j=1;2. \quad (5)$$

By the same calculation we get bounds

$$| y_j | \leq r\sqrt{\frac{b}{2\sigma}+\frac{b}{2}+1},\ j=1;2, \quad (6)$$

$$|z-r| \leq r\sqrt{\frac{b}{2\sigma}+\frac{b}{2}+1}. \quad (7)$$

Now we have that each compact invariant set is contained in the set K defined as

$$K=\cap_{q>0}K_q(H)=$$
$$\cap_{q>0}\{q^2(y_1^2+y_2^2+z^2-2rz-\frac{b+b\sigma}{2\sigma}r^2)+$$
$$q(x_1^2+x_2^2-2\sigma z-b(1+\sigma)r)-b(1+\sigma)\frac{\sigma}{2}\leq 0\}$$

which leads to

$$K=\{(y_1^2+y_2^2+z^2-2rz-\tfrac{b+b\sigma}{2\sigma}r^2\leq 0,$$
$$x_1^2+x_2^2-2\sigma z-b(1+\sigma)r\leq$$
$$\leq\sqrt{2b(\sigma+\sigma^2)(2rz+\tfrac{b+b\sigma}{2\sigma}r^2-y_1^2-y_2^2-z^2)}\}.$$

5 Localization by cylindrical surfaces

5.1 The case I

Let us take for the system (4) the localizing function [Krishchenko and Starkov, 2006]

$$h_1=x_1^2-2\sigma z.$$

Then

$$\mathcal{L}_L h_1=-2\sigma x_1^2+2b\sigma z.$$

Therefore the set S_{h_1} is given by

$$z=b^{-1}x_1^2$$

and

$$h_1\,|_{S_{h_1}}=(1-2\sigma b^{-1})x_1^2.$$

Thus if $b=2\sigma$ then all compact invariant sets of the system (4) are located in

$$K(h_1)=\{(x_1,y_1,z):\ x_1^2-2\sigma z=0\}.$$

If $b>2\sigma$ then

$$h_{1\,\inf}=0,$$
$$h_{1\,\sup}=+\infty,$$

and all compact invariant sets are located in

$$K(h_1)=\{(x_1,y_1,z):\ x_1^2-2\sigma z\geq 0\}.$$

If $b<2\sigma$ then

$$h_{1\,\sup}=0,$$
$$h_{1\,\inf}=-\infty,$$

and all compact invariant sets are located in

$$K(h_1)=\{(x_1,y_1,z):\ x_1^2-2\sigma z\leq 0\}.$$

Now let us consider for the system (2) the localizing function

$$H_1=Symm(h_1)=x_1^2+x_2^2-2\sigma z.$$

Then

$$\mathcal{L}_F H_1=-2\sigma(x_1^2+x_2^2)+2b\sigma z=Symm(\mathcal{L}_L h_1).$$

Therefore the set S_{H_1} is given by

$$z=b^{-1}(x_1^2+x_2^2)$$

and

$$H_1\,|_{S_{H_1}}=(1-2\sigma b^{-1})(x_1^2+x_2^2).$$

Thus if $b=2\sigma$ then all compact invariant sets of the system (2) are located in

$$K(H_1)=\{(x_1,y_1,x_2,y_2,z):\ x_1^2+x_2^2-2\sigma z=0\}.$$

If $b>2\sigma$ then

$$H_{1\,\inf}=0,$$
$$H_{1\,\sup}=+\infty,$$

and all compact invariant sets are located in

$$K(H_1)=\{(x_1,y_1,x_2,y_2,z):\ x_1^2+x_2^2-2\sigma z\geq 0\}.$$

If $b < 2\sigma$ then

$$H_{1\,\sup} = 0,$$
$$H_{1\,\inf} = -\infty,$$

and all compact invariant sets are located in

$$K(H_1) = \{(x_1, y_1, x_2, y_2, z): \ x_1^2 + x_2^2 - 2\sigma z \le 0\}.$$

5.2 The case 2

Let us take for the system (4) the localizing function [Krishchenko and Starkov, 2006]

$$h_2 = (y_1^2 + z^2)/2 - rz.$$

Then

$$\mathcal{L}_L h_2 = -y_1^2 - bz^2 + brz.$$

Therefore the set S_h is given by

$$y_1^2 = -bz^2 + brz$$

and

$$h_2 \mid_{S_{h_2}} = (1-b)z^2/2 + r(b-2)z/2,$$

where $-bz^2 + brz \ge 0$. Thus

$$h_{2\,\inf} = -r^2/2.$$

If $0 < b \le 2$ then

$$h_{2\,\sup} = 0,$$

if $b > 2$ then

$$h_{2\,\sup} = r^2(b-2)^2/8(b-1),$$

and all compact invariant sets are located in

$$K(h_2) = \{(x_1, y_1, z): \ y_1^2 + z^2 - 2rz \le 2h_{2\,\sup}\}$$

because

$$-r^2 \le y_1^2 + z^2 - 2rz$$

for all y_1, z.

In the case $b > 2$

$$K(h_2) = \{(x_1, y_1, z): \ y_1^2 + (z-r)^2 \le \frac{r^2 b^2}{4(b-1)}\},$$

and therefore for all compact invariant sets we have the localization set

$$\omega_1 := \{(x_1, y_1, z): \ |y_1| \le \frac{rb}{2\sqrt{b-1}}, \quad |z - r| \le \frac{rb}{2\sqrt{b-1}}\}.$$

Let us take for the system (2) the localizing function

$$H_2 = Symm(h_2) = (y_1^2 + y_2^2 + z^2)/2 - rz.$$

Then

$$\mathcal{L}_F H_2 = -y_1^2 - y_2^2 - bz^2 + brz = Symm(\mathcal{L}_L h_2).$$

Therefore the set S_{H_2} is given by

$$y_1^2 + y_2^2 = -bz^2 + brz$$

and

$$H_2 \mid_{S_{H_2}} = (1-b)z^2/2 + r(b-2)z/2,$$

where $-bz^2 + brz \ge 0$. Thus $H_{2\,\inf} = -r^2/2$,

$$H_{2\,\sup} = 0 \text{ if } 0 < b \le 2;$$
$$H_{2\,\sup} = \frac{r^2(b-2)^2}{8(b-1)}, \text{ if } b > 2,$$

and all compact invariant sets are located in

$$K(H_2) = \{(x_1, y_1, x_2, y_2, z): \ y_1^2 + y_2^2 + z^2 - 2rz \le 2H_{2\,\sup}\}$$

because

$$-r^2 \le y_1^2 + y_2^2 + z^2 - 2rz$$

for all y_1, y_2, z.

In the case $b > 2$

$$K(H_2) = \{(x_1, y_1, x_2, y_2, z): \ y_1^2 + y_2^2 + (z-r)^2 \le \frac{r^2 b^2}{4(b-1)}\},$$

and therefore for all compact invariant sets we have the localization set

$$\omega_1 = \{(x_1, y_1, x_2, y_2, z): \\ |y_i| \le \frac{rb}{2\sqrt{b-1}}, i = 1, 2, \\ |z - r| \le \frac{rb}{2\sqrt{b-1}}\}.$$

6 Conclusion

In this paper we show how we can localize all compact invariant sets of the Rayleigh-Bnard convection - Lorenz model with help of localization sets of the Lorenz system which have been obtained by the authors in the earlier publication. With this goal we construct special symmetries called symmetrical prolongations allowing us to associate not only the Lorenz system with the Rayleigh-Bnard convection - Lorenz model but corresponding localizing functions applied here as well. Our approach essentially simplifies necessary computations for finding bounds of localization sets.

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References

Chen Z.-M., Price W.G. (2006) On the relation between Rayleigh- Benard convection and Lorenz system *Chaos, Solitons and Fractals*, **28**, pp. 571–578.

Krishchenko A.P. (1995) Localization of limit cycles. *Differential Equations*, **31**(11), pp. 1858–1865 (in Russian).

Krishchenko A.P. (1997) Estimations of Domains with Cycles. *Computers Math. Applications*, **34**(2-4), pp 325-332.

Krishchenko A.P., Starkov K.E. (2006) Localization of compact invariant sets of the Lorenz system. *Physics Letters A.*, **353**(6), pp. 383–388.

Lorenz E. (1963) Deterministic nonperiodic flow. *Journal of Atmospheric Science*, **20**, pp. 130–141.