DYNAMIC SYSTEMS CONTROL WITH SYMMETRIZATION OF PHASE LIMITATIONS

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Abstract
In aircraft control one of the most urgent problems is providing desired motion regimes. In the general case it is usually impossible to realize the desired motion precisely, due to the various indeterminacies. So feasible accuracy in conditions of various disturbances is considered. The geometric method based on presentation of phase limitations as symmetric polyhedrons, which determine allowable variations of real aircraft motion trajectories, is presented in this work.

Key words
No more than six key words must be given.

1 Introduction

While controlling aircrafts one of the most relevant problems is the problem of the desired (programmed) motion regimes realization. While solving this problem we encounter many obstacles including first of all the existence of various indeterminacies, which prevent us from gaining the full information about the aircraft's model. For example: structural and parametric indeterminacies of aircrafts simulator; external and internal indeterminacies, determined by external disturbances and by disturbances in indicators channels. In the general case precise realization of desired motion is usually impossible, so the current problem can be appropriately formulated as a realization of desired motion regimes with feasible accuracy. This? realization accuracy can be defined as providing of some phase limitations, which desired motion trajectories must fulfill. Many works [1]-[5] are dedicated to the problem of program motion regimes synthesis. However, suggested methods generally disregard the above-mentioned indeterminacies and the existence of different variable motion constraint, including phase limitations. At the same time there are different methods, which can be used for solving newly formulated problem: Pontryagin principle of maximum; dynamic programming method; Lyapunov functions method. In the works [6]-[10] the usage of those methods is shown. But also in this case the problem cannot be fully solved because of: high model dimension, complexity of formulated correlations and consideration difficulty of various limitations. The method for solving problems of this type is presented in this work. This method allows direct consideration of various limitations (first of all - phase limitations), to analyze the solvability regarding the desired law of control and to implement control by given phase limitations.

2 Statement of the Task

Let's operate with linear models of aircrafts (in general case they can be taken from the linearization of arbitrary aircrafts models regarding programmed trajectories). We will consider models in the state space. Let's present an equation as:

$$\dot{x} = Ax + Bu + Dv \tag{1}$$

Where x, u, v-$n \times 1, m \times 1, r \times 1$state, control and disturbance vectors;A, B, D-$n \times n, n \times m, n \times r$ matrices accordingly. The vectors u, v satisfy limitations:

$$u \in U, v \in V \tag{2}$$

where $U \subseteq R^m, V \subseteq R^r$ - some given sets in the Euclidean spaces R^m, R^r. The state vector x must satisfy the phase limitations of following type:

$$x = x(t) \in Q(t), t \geq t_0 \tag{3}$$

where $Q = Q(t) \subset R^n$ - some given set in the space state R^n. Let's regard Q as a convex polyhedron in R^n and it is described by the following equations:

$$\left.\begin{array}{l} Q = \{x \in R^n : \psi_i(x,t) \leq 0, i \in \overline{1,\chi}\} \\ \psi_i(x,t) = (\alpha^i, x) - q_i(t) \end{array}\right\} \tag{4}$$

where $q_i(t), \alpha^i$ - given scalar function and $n \times 1$ vector. From here on we will consider stationary case only, when the system (1) is stationary and, besides, $\alpha^i \equiv const, q_i(t) = q_i \equiv const$. Let's regard Q as a polyhedron symmetrical with respect to the grid origin $0 \in R^n$. Besides, let's consider polyhedrons with a general number of sides in R^n equals $2n$. For the system (1) it is needed to choose such a control law that

$$\left. \begin{array}{c} u = Ky \\ y = Cx \end{array} \right\} \tag{5}$$

where y-$l \times 1$ output vector;K, C - $m \times l, l \times n$ regulator and output (measuring) matrices, that with consideration of limitations (2) the phase limitations (3) will be fulfilled.

3 Forming the Sufficient Conditions of Solvability

For solving the stated task we will use general equations of author's above-formulated Phase Limitations Variations Method (PLVM), which was offered in works [11], [12]. In the general case the given equations look as follows:

$$\left. \begin{array}{l} (\nabla_x \psi_i, f(x,u,v)\frac{\partial \psi_i}{\partial t} \leq 0 \\ \forall x \in \Gamma Q \bigcap \Gamma Q_i, i \in \overline{1,\chi}, t \geq t_0 \end{array} \right\} \tag{6}$$

Where $\nabla_x \psi_i$ - gradient of the function ψ_i;$f(\cdot) = Ax + Bu + Dv$;$(X,Y) = \sum_{i=1}^{n} X_i Y_i$ - scalar product of vectors $X, Y \in R^n$; Q - border of polyhedron Q; Q_i - surface, forming i-side of polyhedron $Q(\psi_i(x,t) = 0), x \in R^n$. For this case of phase limitations regarding [13] inequalities (6) are reformed to (here $v \equiv 0$):

$$(\widetilde{A}^T \alpha^i, {M_\nu}^i) - \dot{q} < 0, t \geq t_0, i \in \overline{1,\chi}, \nu \in \overline{1,N_i} \tag{7}$$

Where $\widetilde{A} = A + BKC$; ${M_\nu}^i$ - arbitrary apex of i-side $Q \bigcap Q_i$ of polyhedron Q; N_i- number of apexes on i-side. If Q is a polyhedron with $2n$ sides (then $\chi = 2n$), symmetrical to $0 \in R^n$, then correlations (7) equal the following:

$$(\widetilde{A}^T \alpha^i, {M_\nu}^i) - \dot{q} < 0, t \geq t_0, i \in \overline{1,n}, \nu \in \overline{1,N_i} \tag{8}$$

Where $N_i = N = 2^{n-1}\ \forall i \in \overline{1,n}$, since for opposing sides of Q inequalities are equal. At the same time numerations of sides remains random. For above-stated stationary case inequalities (8) become

$$(\widetilde{A}^T \alpha^i, {M_\nu}^i) < 0, t \geq t_0, i \in \overline{1,n}, \nu \in \overline{1,N}, N = 2^{n-1} \tag{9}$$

As the side numeration is random, then we will choose random apex $M \in Q$, which is intersection of sides, and we will renumber this sides from 1 to n. Considering this, let's formulate synthesis task as following: for system (1) it is needed to synthesize such control law (5), that regarding limitations (2) for arbitrary apex $M = \bigcap_{i=1}^{n} \Gamma Q \cap \Gamma Q_i$ inequalities (9) will be fulfilled.

4 Task Reduction to Equivalent Form

Let's convert the polyhedron Q into a more handy form - rectangular parallelepiped, which sides are parallel to the coordinate hyper planes of space R^n (or symmetry axis are parallel or are congruent with coordinate axis of space R^n, which is the same). This transformation can always be realized, if Q satisfies the above-stated conditions. For this satisfaction we will turn to the new basis in space R^n, where Q has the desired form. This in turn can be realized in the following way: Let

$$z = Tx \tag{10}$$

Where z - $n \times 1$ new variable; T - $n \times n$ nonsingular transformation matrix, with help of which Q is converted to desired rectangular parallelepiped $\overline{Q}$. Regarding (10) the state equation of the system (1) will reduce to:

$$\left. \begin{array}{l} \dot{z} = \overline{A}z + \overline{B}u + \overline{D}v, z(t_0) = z_0, t \geq t_0 \\ \overline{A} = TAT^{-1}, \overline{B} = TB, \overline{D} = TD \end{array} \right\} \tag{11}$$

Let's consider equations (9). For i-side of Q we will get

$$(s, M_\nu^i) < 0, \nu \in \overline{1,N} \tag{12}$$

Then for fulfillment of (12) it is necessary and sufficient, that

$$s \in K_i \tag{13}$$

Where K_i - cone with an apex in grid origin and with N sides, which belong to hyper planes $H_\nu^i = \{s \in R^n : (s, M_\nu^i) = 0\}$ accordingly. It is obvious, that inequalities (9) fulfill then and only then, when

$$\widetilde{A}^T \alpha^i \in {}_i, i \in \overline{1,n} \tag{14}$$

K_i is assigned by the following correlation:

$$K_i = \{x \in R^n : m_i x_i + \sum_{j=1, j \neq i}^{n} (\pm m_j) x_j \leq 0\} \tag{15}$$

where i-coordinates of all apexes $M_\nu^i \in \Gamma\overline{Q_i} \bigcap \Gamma\overline{Q}$ are the same and equal to m_i, and all other apexes coordinates on the side $\Gamma Q_i \bigcap \Gamma Q$ can possess the values $\pm m_j, j \in \overline{1,n}$1. For distinctness we will choose an apex M, which has

$$m_i < 0, \forall i \in \overline{1,n} \tag{16}$$

Then K_i is assigned by inequality

$$x_i \geq \frac{1}{m_i} \cdot \sum_{j=1, j\neq i}^{n} (\pm m_j) x_j \qquad (17)$$

It is easy to see, that K_i is a symmetric figure. At the same time the symmetry axis of a cone K_i is a positive axle $0x_i$. From this, regarding the suggestion about choosing M (19), we will come to: $\alpha^i = [\underbrace{0\ldots0}_{i-1}\ -1\ \underbrace{0\ldots0}_{n-i}]^T$. Then Then

$$\widetilde{A}^T \alpha^i = -\widetilde{a}_i,\ i \in \overline{1,n} \qquad (18)$$

where

$$\widetilde{a}_i = \overline{a}_i + \overline{C}^T K^T \overline{b}_i \qquad (19)$$

and $\overline{a}_i,\ \overline{b}_i$ - $n \times 1,\ m \times 1$ i-row vectors of $\overline{A},\ \overline{B}$ matrices. Finally, regarding (17), we will come to the following correlation

$$-(\overline{a}_i + \overline{C}^T K^T \overline{b}_i) \in K_i,\ i \in \overline{1,n} \qquad (20)$$

Which is sufficient condition of the phase limitations (3) fulfillment. Then the problem of synthesis can be formulated in the following way: for the system (1) it is needed to synthesize such a matrix of regulator K, which must provide the fulfillment of correlations (20).

5 Solving the Synthesis Problem

Let's present correlation (20) as

$$\overline{K}^T \overline{b}_i \in \widetilde{a}_i + K_i,\ i \in \overline{1,n} \qquad (21)$$

where $\overline{K}^T = -C^T K^T$ Let $\overline{b}_i = [\overline{b}_{i1}\ \overline{b}_{i2} \ldots \overline{b}_{im}]^T,\ i \in \overline{1,n}$. Then (21) can be presented as following:

$$\sum_{j=1}^{m} \overline{b}_{ij} \overline{k}_j \in \overline{a}_i + K_i,\ i \in \overline{1,n} \qquad (22)$$

where $\overline{k}_j$ is a $n \times 1$ column vector of matrix $overline K^T$. If $\varphi \in K_i$ is a random element of cone K_i, then it can be described as:

$$\varphi^i = P_i \cdot s^i \qquad (23)$$

Where P_i will be named as cone operator, and s^i - some vector, which coordinates will be positive random values. It is easy to see, that as cone operator P_i we can use a matrix with columns, which were formed from vectors, which are cone sides of K_i themselves. Then $P_i = [p_1^i\ p_2^i \ldots p_N^i]$ - $n \times N$ matrix, $i \in \overline{1,n}$; p_ν^i - cone sides, $\nu \in \overline{1,N}$; $s^i = [s_1^i\ s_2^i \ldots s_N^i],\ N \times 1$ vector, $s_\nu^i \geq 0\ \forall i \in \overline{1,n},\ \nu \in \overline{1,N}$ Considering (23) correlations (22) can be transformed into

$$\sum_{j=1}^{m} \overline{b}_{ij} \overline{k}_j = \overline{a}_i + P_i s^i,\ i \in \overline{1,n} \qquad (24)$$

The solvability of equations (24) concerning $\overline{K}$ depends on the choice of positive vectors $s^i,\ i \in \overline{1,n}$. Let's consider the existence of $s^i > 0,\ i \in \overline{1,n}$, which determine the solvability of (27). For this we will reduce equations (24) to the standard form, i.e. when unknown variables in the left part of equation (matrix $\overline{K}$ coefficients) are brought to one column vector. Let $\overline{k}_j = [\overline{k}_{1j}\ \overline{k}_{2j} \ldots \overline{k}_{nj}]^T,\ j \in \overline{1,m}$. Let's form the following vector:

$$\widetilde{k} = [\overline{k}_{11} \ldots \overline{k}_{1m}\ \overline{k}_{21} \ldots \overline{k}_{2m} \ldots \overline{k}_{n1} \ldots \overline{k}_{nm}]^T - (nm) \times 1 \qquad (25)$$

Then considering this variables order we will convert equation system (27) into:

$$\widetilde{B} \cdot \widetilde{k} = a + p(s) \qquad (26)$$

Where $\widetilde{B} = \begin{bmatrix} \overline{B} & 0 & 0 & 0 \\ 0 & \overline{B} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \overline{B} \end{bmatrix}; n^2 \times (n \cdot m)$
$a = [(\overline{a}^1)^T\ (\overline{a}^2)^T \ldots (\overline{a}^n)^T]^T$-$n^2 \times 1$
$\overline{a}^i = [\overline{a}_{1i}\ \overline{a}_{2i} \ldots \overline{a}_{ni}]^T,\ j \in \overline{1,m}$ - i-column vector of matrix $\overline{A}$; $p = [(p_1^1,\ s^1) \ldots (p_1^n,\ s^n)\ (p_2^1,\ s^1) \ldots (p_2^n,\ s^n) \ldots (p_n^1,\ s^1) \ldots (p_n^n,\ s^n)]^T$, where $p_\xi^i = [p_{\xi 1}^i\ p_{\xi 2}^i \ldots p_{\xi N}^i]^T$ - ξ-row vector of matrix P_i. Equation (26) is solvable then and only then, when the following condition is fulfilled:

$$a + p(s) \in L(\widetilde{B}) \qquad (27)$$

Where $L(\widetilde{B})$ is a subspace in R^{n^2} formed by column vectors of matrix $\widetilde{B}$. Condition (27) will be fulfilled then and only then, when the distance $\rho(a+p, L)$ between vector $(a+p(s))$ and subspace $L(\widetilde{B})$ equals null, i.e.

$$\rho(a + p(s),\ L(\widetilde{B}) = 0 \qquad (28)$$

Let $rank\widetilde{B} = \widetilde{m}$. Then in $\widetilde{B}$ $\widetilde{m}$ of linearly independent columns can be picked out: $b_1,\ b_2, \ldots,\ b_{\widetilde{m}}$, which form the basis of a subspace $L(\widetilde{B})$. Let's form the matrix $\widehat{B} = [b_1,\ b_2, \ldots,\ b_{\widetilde{m}}]$. The distance $\rho(\cdot)$ is defined as

$$\rho(\cdot) = \|\widehat{a} - \widehat{a}^0\| \tag{29}$$

Where $\widehat{a} = a + p(s)$; $\widehat{a}^0$ - orthogonal projection of vector $\widehat{a}$ onto subspace $L(\widetilde{B}$, $\|\cdot\|$ - Euclid norm in R^{n^2}. It is known, [14], that

$$\widehat{a}^0 = \widehat{B}(\widehat{B}^T\widehat{B})^{-1}\widehat{B}^T\widehat{a} = F \cdot \widehat{a} \tag{30}$$

Considering (29) we will have

$$\rho(\cdot) = \|\widehat{a} - \widehat{a}^0\| = \|(E - F)\widehat{a}\| \tag{31}$$

E - $n^2 \times n^2$ unitary matrix. Then for the fulfillment of condition (31) the fulfillment of the following correlation is needed:

$$\min_{s^i,\, i\in\overline{1,n}} \|(E-F)(a+p(s^1,\, s^2, \dots,\, s^n))\|^2 = 0 \tag{32}$$

That is equal to (28). Correlation (32) is fulfilled with positive vectors $s^i \geq 0,\ i \in \overline{1,n}$ then and only then, when the conditions (20)-(22) are fair. I.e., if (20)-(22) are fair, then there are always desired positive vectors $s^i,\ i \in \overline{1,n}$, for which (32) is fulfilled. The problem of minimization of quadratic function $\rho^2(\cdot)$ can be reduced to the system of algebraic equation of the following type:

$$\nabla_{s^i}\rho^2(\cdot) = 0,\ i \in \overline{1,n} \tag{33}$$

Where $\nabla_{s^i}\rho^2$ - vector s^i gradient of function ρ^2. For system (36) already known algorithms of quadratic programming can be used. As is easy to see, that the problem of synthesis can be severely simplified, if in (26) we will use not the full cone operator P_i, but a "truncated" operator $\widehat{P}_i$, which corresponds to some "truncated" cone $\widehat{K}_i$, which has a more simple form than K_i, and for which

$$\varphi = \widehat{P}_i\widehat{s}^i \in \widehat{K}_i \subset K_i,\ i \in \overline{1,n} \tag{34}$$

where the "truncated" vector $\widehat{s}^i$ has lower dimension than s^i.

6 Synthesis Regarding Nonlinearity and Indeterminacies

Let's consider a more general case, when the object mode looks like:

$$\dot{x} = \varphi(x) + Bu + Dv \tag{35}$$

where $\varphi(x)$-$n \times 1$ vector-function, which takes on limited values on the polyhedron Q.

$$\varphi(x) = Ax + f(x) \tag{36}$$

where

$$f(x) \in \Phi \forall x \in \Gamma Q \tag{37}$$

Φ-some given set, which contains the value area of the function $f(x)$ on polyhedron Q. Then

$$\left.\begin{array}{l} \dot{x} = Ax + Bu + w, \\ w = f(x) + Dv. \end{array}\right\} \tag{38}$$

For the system (38) the equations (6) become

$$\left.\begin{array}{l} (\widetilde{A}^T\alpha^i, M_\nu^i) \leq w_i, t \geq t_0, i \in \overline{1,n}, \nu \in \overline{1,N} \\ w_i = f_i(x) + (d_i, v) \end{array}\right\} \tag{39}$$

where w_i, f_i - i-components of vector-functions w and $f(x), d_i$ - i-row-vector of matrix D.
Analogously to the derivation of correlation (14), it can be shown, that inequalities (39) are equal to the following condition

$$\widetilde{A}^T\alpha^i \in \frac{w_i}{m_i}e_i + K_i, i \in \overline{1,n} \tag{40}$$

where $e_i = [0 \dots 0\ 1\ 0 \dots 0]^T$ and which with taking into consideration (18)-(20) comes to

$$\overline{k}^T\overline{b}_i \in \overline{a}_i\frac{w_i}{m_i}e_i + K_i, i \in \overline{1,n} \tag{41}$$

Let's designate $s = [(s^1)^T\ (s^2)^T \dots (s^n)^T]^T$

$$p = \begin{bmatrix} p_1^1 & 0 & \dots & 0 \\ 0 & p_1^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_1^m \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ p_m^1 & 0 & \dots & 0 \\ 0 & p_m^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n^n \end{bmatrix}; n^2 \times (N \cdot n) \text{ matrix,}$$

$\widetilde{N} = N \cdot n, p_i$ - $n^2 \times 1$-column-vector, 0-1 $\times$ N-null vector;
$\Delta = [\frac{w_1}{m_1}e_1^T\ \frac{w_2}{m_2}e_2^T\ \dots\ \frac{w_n}{m_n}e_n^T]^T$ - $n^2 \times 1$ vector. Then similarly to (25), the condition (41) comes to equation

$$\widetilde{B}\widetilde{k} = \Delta + a + Ps. \tag{42}$$

For solvability of (42) the existence of such a vector $s > 0$ for which

$$\Delta + a + Ps \in L(\widetilde{B}). \tag{43}$$

is necessary and sufficient. Let's use the condition (43) for the synthesis problem's solvability check and solving. Let

$$P = P_0 + \overline{P}, \Delta = \Delta^0 + \overline{\Delta}, a = a^0 + \overline{a}, \tag{44}$$

where $a^0, \Delta^0 \in L(\widetilde{B}); \overline{a}, \overline{\Delta} \perp L(\widetilde{B}); P_0, \overline{P}$ - matrices, with columns, that belongs to $L(\widetilde{B})$ and are orthogonal to $L(\widetilde{B})$ accordingly. And according to (30)

$$P_0 = F \cdot P, \Delta^0 = F \cdot \Delta, a^0 = \Delta \cdot a, \tag{45}$$

Equation (42) will come to

$$\widetilde{B}\widetilde{k} = (\Delta^0 + a^0 + P_0 s) + (\overline{\Delta} + \overline{a} + \overline{P}s) \tag{46}$$

For solvability of (46) it is necessary and sufficient, that

$$\overline{\Delta} + \overline{a} + \overline{P}s = 0, s \geq 0. \tag{47}$$

Besides (47) for the existence of a fixed matrix ¡K ¿ the following correlation must fulfill

$$\Delta^0 + P_0 s = \sigma \equiv const, s \geq 0. \tag{48}$$

Then from the equation

$$\widetilde{B}\widetilde{k} = a^0 + \sigma \tag{49}$$

the required matrix $\widetilde{k}$ is derived. Since because of indeterminacy vector Δ can take on different values, i.e. $\Delta \in \Omega$ (Ω - given set, determined with the consideration of sets V (2), Φ (37) and with correlation (38)), then equations (47), (48) must fulfill for $\forall \Delta \in \Omega$.

7 Conclusion

The method, presented in this work, allows solving the problem of regulator synthesis regarding limitations on motion trajectories. Those limitations are symmetrical phase polyhedrons. In this case it is possible to simplify the correlations, which the system must satisfy, and to reduce their quantity. New correlations form connection between properties of phase polyhedron and properties of system under consideration, which define the solvability condition of the synthesis problem. These correlations are algebraic equations, which can be solved by the known vectorial methods. The examined case of control by state vector can be generalized in case of the control by output. These obtained correlations can be severely simplified with the usage of "truncated" cone operators. The presented method allows solving the problem of robust control of dynamic objects efficiently enough.

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