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ON ONE VARIANT OF THE COMPARISON METHOD FOR CONSERVATIVE SYSTEMS

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Abstract
The paper discusses some results related to application of differential corollaries in qualitative investigation of mechanical systems. Examples of definite systems, for which this approach is efficient, are given.

Key words
Comparison method, differential corollaries.

1 ON DIFFERENTIAL COROLLARIES

Let us consider the system of differential equations of motion:

$$\dot{x}_i = X_i(t, x_1, x_2, \ldots, x_n), \;\; i = 1, 2, \ldots, n. \quad (1)$$

Definition 1. The system (1) assumes a truncated chain of differential corollaries when the following set of equalities is valid:

$$\dot{V}_k \equiv \sum_{i=1}^{n} \frac{\partial V_k}{\partial x_i} X_i + \frac{\partial V_k}{\partial t} = -\varphi_{k+1}(t,x) V_{k+1}(t,x),$$
$$k = 0, 1, \ldots, g-1, \quad (2)$$

where $V_k(t,x),\ (k = 0, 1, \ldots, g-1),\ \varphi_l(t,x),\ (l = 1, 2, \ldots, g)$ and $V_g = 0$ are some functions.

Let equations (1) have the chain of differential corollaries (2). Let us complement (2) with the following system of equations:

$$\dot{\psi_m} \equiv \sum_{i=1}^{n} \frac{\partial \psi_m}{\partial x_i} X_i + \frac{\partial \psi_m}{\partial t} = \varphi_m(x) \psi_{m-1}(t,x),$$
$$\psi_0 = 1, \;\; m = 1, 2, \ldots, g, \quad (3)$$

and then put system (1) in correspondence to the system of differential equations

$$\dot{V}_k = -\varphi_{k+1} V_{k+1}, \;\; \dot{\psi}_k = \varphi_k \psi_{k-1},$$
$$k = 0, 1, \ldots, g-1, \; \psi_0 = 1. \quad (4)$$

We will consider the latter system in the capacity of the comparison system. The following statement is valid.

Theorem 1.
If system (4) has the first integral $W(V_0, \ldots, V_g, \psi_1, \ldots, \psi_{g-1})$, then system (1) has the first integral $W(V_0\ (t,x),\ \ldots,\ V_g(t,x),$ $\psi_1(t,x), \ldots, \psi_{g-1}(t,x))$, where functions ψ_l satisfy equations (3), and V_k satisfies, respectively, (2).

Validity of this statement is almost obvious. According to (2), (3), the derivative of W computed due to differential equations does not depend on definite forms of functions V_k, ψ_l, φ_i.

Proposition 1.
If the chain of differential corollaries (2) is truncated:

$$\dot{V}_{g-1} = \sum_{i=1}^{n} \frac{\partial V_{g-1}}{\partial x_i} X_i + \frac{\partial V_{g-1}}{\partial t} = 0, \;\; V_g = 0,$$

then the comparison system (4) takes the form:

$$\dot{V}_0 = -\varphi_1 V_1, \; \dot{V}_1 = -\varphi_2 V_2, \ldots, \dot{V}_{g-1} =$$
$$-\varphi_{g-1} V_g = 0, \; \dot{\psi}_1 = \varphi_1,$$
$$\dot{\psi}_2 = \varphi_2 \psi_1, \ldots, \dot{\psi}_{g-1} = \varphi_{g-1} \psi_{g-2} \quad (5)$$

and assumes the two first integrals:

$$V_{g-1} = c_1, \;\; K = V_0 + \psi_1 V_1 + \ldots$$
$$+ \psi_{(g-1)} V_{(g-1)}(x) = c_2. \quad (6)$$

Proof. The statement that V_{g-1} is the first integral follows directly from the last equation in the first group of comparison equations (5). In order to prove that K is the first integral, it is sufficient to compute the derivative of K due to equations (5). Indeed,

$$\dot{K} = \dot{V}_0 + \dot{\psi}_1 V_1 + \psi_1 \dot{V}_1 + \dot{\psi}_2 V_2 + \psi_2 \dot{V}_2 + \ldots$$

$$+\psi_{(g-2)}\dot{V}_{(g-2)} + \psi_{(g-1)}\dot{V}_{(g-1)} + \dot{\psi}_{(g-1)}V_{(g-1)}.$$

Having substituted the values of derivatives $\dot{V}_j$ $(j = 1, \ldots, g-1)$, $\dot{\psi}_k$ $(k = 1, \ldots, g-1)$ from (5) into the latter equation, we have

$$\dot{K} = -\varphi_1 V_1 + \varphi_1 V_1 - \psi_1\varphi_2 V_2 + \psi_1\varphi_2 V_2 + \ldots$$
$$-\psi_{g-2}\varphi_{g-1}V_{g-1} + \psi_{g-2}\varphi_{g-1}V_{g-1} = 0,$$

i.e. $\dot{K} = 0$. This completes the proof.

Corollary 1. If system (1) assumes a truncated (efficient) chain of differential corollaries, then, besides the integral $V_{(g-1)}$, it possesses also the first integral

$$K(t,x) = V_0 + \psi_1(t,x)V_1(t,x) + \psi_2(t,x)V_2(t,x)$$
$$+ \ldots + \psi_{(g-1)}(t,x)V_{(g-1)}(t,x). \qquad (7)$$

Here functions $\psi_l(x)$, $l = 1, 2, \ldots, g-1$ satisfy partial differential equations (3).

Validity of the latter statement follows from Theorem 1 and Proposition 1.

Note, the construction with truncated chains of equalities has been used in other way by Laplace (see [Laplace, 1799]) also for finding first integrals.

Corollary 2. If in equations (2) all the functions φ_i are dependent only on t $(\varphi_i = \varphi_i(t))$, then K is the non-autonomous integral

$$K(t,x) = V_0 + \psi_1(t)V_1(t,x) + \psi_2(t)V_2(t,x)$$
$$+ \ldots + \psi_{g-1}(t)V_{g-1}(t,x),$$

where

$$\psi_1(t) = \int_0^t \varphi_1(\xi)d\xi;\ \psi_2(t) = \int_0^t \psi_1(\xi)\varphi_2(\xi)d\xi; \ldots;$$

$$\psi_{g-1}(t) = \int_0^t \psi_{g-2}(\xi)\varphi_{g-1}(\xi)d\xi.$$

The latter expressions for $\psi_i(t)$ are obtained by direct sequential integration of the system of differential equations (3), in which all the functions are considered to be dependent only on t.

When additional conditions are imposed on the form of the chain of differential corollaries, it is possible to obtain (with the aid of these corollaries) considerably more information on the initial system. For example, if all $\varphi_k(t,x)$ $(k = 1, \ldots, g)$ in the truncated chain of differential corollaries of system (2) are similar, i.e.

$$\dot{V}_k \equiv \sum_{i=1}^{n} \frac{\partial V_k}{\partial x_i} X_i + \frac{\partial V_k}{\partial t} = -\varphi(t,x)V_{k+1}(t,x),$$
$$k = 0, 1, \ldots, g-1,\ \ V_g = 0, \qquad (8)$$

then Theorem 2 is valid. If the system of differential equations (1) assumes the truncated chain of differential corollaries (8), then it has the following first integrals:

$$K(x,t) = V_0(x,t) + \psi(x,t)V_1(x,t) + \frac{1}{2}\psi^2(x,t)V_2(x,t)$$

$$+ \ldots + \frac{1}{(g-1)!}\psi^{(g-1)}(x,t)V_{(g-1)}(x,t),$$

$$\frac{\partial K}{\partial \psi} = V_1(x,t) + \psi(x,t)V_2(x,t) + \frac{1}{2}\psi^2(x,t)V_3(x,t)$$

$$+ \ldots + \frac{1}{(g-2)!}\psi^{g-2}(x,t)V_{g-2}(x,t),$$

....

$$\frac{\partial^{(g-2)} K}{\partial \psi^{(g-2)}} = V_{g-2}(x,t) + \psi(x,t)V_{g-1}(x,t),$$

where

$$\dot{\psi} \equiv \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} X_i + \frac{\partial \psi}{\partial t} = \varphi(x,t). \qquad (9)$$

Proof. Let us put the following comparison system

$$\dot{V}_0 = -\varphi V_1,\ \ \dot{V}_1 = -\varphi V_2, \ldots, \dot{V}_{g-1} = -\varphi V_g = 0,$$

$$\dot{\psi} = \varphi.$$

in correspondence to system (1) and show that it has the first integral

$$K(V,\psi) = V_0 + \psi V_1 + \frac{1}{2}\psi^2 V_2 + \ldots$$
$$+ \frac{1}{(g-1)!}\psi^{(g-1)}V_{(g-1)}. \qquad (10)$$

When computing the derivative of $K(V,\psi)$ due to the comparison system, we obtain the following identity:

$$\frac{dK}{dt} = \dot{V}_0 + \psi\dot{V}_1 + \dot{\psi}V_1 + \psi\dot{\psi}V_2 + \frac{\psi^2}{2}\dot{V}_2 + \ldots$$

$$+\frac{\psi^{g-2}}{(g-2)!}\dot{\psi}V_{g-1}+\frac{1}{(g-1)!}\psi^{g-1}\dot{V_{g-1}} = -\varphi V_1+\varphi V_1-$$

$$\psi\varphi V_2+\psi\varphi V_2-\frac{1}{2}\psi^2\varphi V_3+\ldots-\frac{1}{(g-1)!}\psi^{g-1}\varphi V_g \equiv 0.$$

This identity shows that $K(V,\psi)$ is indeed the first integral of the comparison system. Now, according to Theorem 1, we obtain that $K(x,t)$ is the first integral of system (1).

Next, compute the partial derivative of $K(V,\psi)$ with respect to ψ :

$$\frac{\partial K}{\partial \psi} = W_1 = V_1 + \psi V_2 + \ldots + \frac{\psi^{(g-2)}}{(g-2)!}V_{(g-1)},$$

and compute the derivative of the latter expression due to the comparison system. As a result, we have:

$$\begin{aligned}
&\frac{d}{dt}(\frac{\partial K}{\partial \psi}) = \dot{V}_1 + \dot{\psi}V_2 + \psi\dot{V}_2 + \ldots\\
&+\frac{\psi^{g-3}}{(g-3)!}\varphi V_{g-1} - \frac{\psi^{g-2}}{(g-2)!}\varphi V_g =\\
&-\varphi V_2 + \varphi V_2 - \psi\varphi V_3 + \psi\varphi V_3 - \frac{1}{2}\psi^2\varphi V_3\\
&+\ldots+\frac{1}{(g-3)!}\psi^{g-3}\varphi V_{g-1}\\
&-\frac{1}{(g-2)!}\psi^{g-2}\varphi V_g \equiv 0.
\end{aligned}$$

Since the derivative due to the comparison system has appeared to be identically zero, it is possible to state that the first partial derivative of K with respect to ψ is the first integral of the comparison system. Consequently, the function

$$\begin{aligned}
&\frac{\partial K}{\partial \psi} = W_1 = V_1(x,t) + \psi(x,t)V_2(x,t)\\
&+\ldots+\frac{\psi^{(g-2)}(x,t)}{(g-2)!}V_{(g-1)}(x,t),
\end{aligned}$$

due to Theorem 1 is the first integral of system (1).

When proceeding on similarly, it can easily be shown that all the rest of the partial derivatives of K (10) with respect ψ are first integrals of system (1). This completes the proof.

Now let us use the above technique for the analysis of some problems of mechanics.

Consider the problem of motion of a rigid body, which is described by Euler's equations (an extended case). Let equations of motion write:

$$A\dot{p} = (B-C)qr + \varphi\gamma_1, \;\; B\dot{q} = (C-A)rp + \varphi\gamma_2,$$

$$C\dot{r} = (A-B)pq + \varphi\gamma_3,$$

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \;\; \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \;\; \dot{\gamma}_3 = q\gamma_1 - p\gamma_2,$$

where $\varphi = \varphi(p,q,r,\gamma_1,\gamma_2,\gamma_3)$ is a smooth function of its arguments. The equations assume the following truncated chain of differential corollaries:

$$\frac{1}{2}\frac{d}{dt}(A^2p^2+B^2q^2+C^2r^2) = \varphi(Ap\gamma_1+Bq\gamma_2+Cr\gamma_3),$$

$$\frac{d}{dt}(Ap\gamma_1 + Bq\gamma_2 + C\gamma_3) = \varphi(\gamma_1^2+\gamma_2^2+\gamma_3^2),$$

$$\frac{d}{dt}(\gamma_1^2+\gamma_2^2+\gamma_3^2) = 0.$$

In this case (it can easily be shown), all the conditions of the theorem 2 are satisfied. Consequently, Euler's differential equations have the first integrals

$$\begin{aligned}
&K = \frac{1}{2}(A^2p^2+B^2q^2+C^2r^2) - \psi(Ap\gamma_1\\
&+Bq\gamma_2 + C\gamma_3) + \frac{1}{2}\psi^2(\gamma_1^2+\gamma_2^2+\gamma_3^2) = h,\\
&\frac{\partial K}{\partial \psi} = (Ap\gamma_1 + Bq\gamma_2 + C\gamma_3) - \psi(\gamma_1^2+\gamma_2^2\\
&+\gamma_3^2) = c_1, \; \frac{\partial^2 K}{\partial \psi^2} = (\gamma_1^2+\gamma_2^2+\gamma_3^2) = 1, \quad (11)
\end{aligned}$$

where the function $\psi(p,q,r,\gamma_1,\gamma_2,\gamma_3)$ is defined by the corresponding partial differential equation. While using Routh-Lyapunov's technique [Irtegov, 1989], let us find the invariant manifolds of steady motions (IMSMs), which correspond to the complete linear bundle of the latter integrals in the problem under scrutiny:

$$\begin{aligned}
&2K = A^2p^2 + B^2q^2 + C^2r^2 - 2\psi(Ap\gamma_1 + Bq\gamma_2\\
&+Cr\gamma_3) + \psi^2(\gamma_1^2+\gamma_2^2+\gamma_3^2) - \lambda_1[\psi - (Ap\gamma_1\\
&+Bq\gamma_2 + Cr\gamma_3)] + \lambda_2(\gamma_1^2+\gamma_2^2+\gamma_3^2),
\end{aligned}$$

where λ_1 and λ_2 are some constants (if it is necessary, these may be considered also as constants of some problem's first integrals). Now write down the stationary conditions for the family of first integrals K with respect to the phase variables:

$$\frac{\partial K}{\partial p} = A(Ap-(\psi+\lambda_1)\gamma_1) - \frac{\partial \psi}{\partial p}[(Ap\gamma_1+Bq\gamma_2+Cr\gamma_3)$$

$$-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0,$$

$$\frac{\partial K}{\partial q}=B(Bq-(\psi+\lambda_1)\gamma_2)-\frac{\partial\psi}{\partial q}[(Ap\gamma_1+Bq\gamma_2+Cr\gamma_3)$$

$$-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0,$$

$$\frac{\partial K}{\partial r}=C(Cr-(\psi+\lambda_1)\gamma_3)-\frac{\partial\psi}{\partial r}[(Ap\gamma_1+Bq\gamma_2+Cr\gamma_3)$$

$$-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0,$$

$$\frac{\partial K}{\partial\gamma_1}=-(\psi+\lambda_1)Ap+(\psi^2+\lambda_2)\gamma_1-\frac{\partial\psi}{\partial\gamma_1}[(Ap\gamma_1+Bq\gamma_2$$

$$+Cr\gamma_3)-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0,$$

$$\frac{\partial K}{\partial\gamma_2}=-(\psi+\lambda_1)Bq+(\psi^2+\lambda_2)\gamma_2-\frac{\partial\psi}{\partial\gamma_2}[(Ap\gamma_1+Bq\gamma_2$$

$$+Cr\gamma_3)-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0,$$

$$\frac{\partial K}{\partial\gamma_3}=-(\psi+\lambda_1)Cr+(\psi^2+\lambda_2)\gamma_3-\frac{\partial\psi}{\partial\gamma_3}[(Ap\gamma_1+Bq\gamma_2$$

$$+Cr\gamma_3)-\psi(\gamma_1^2+\gamma_2^2+\gamma_3^2)+\lambda_1]=0.$$

It can readily be seen, the coefficient of the partial derivatives of ψ with respect to the problem's variables is the first integral of equations of motion. Under the respective choice of λ_1 the given terms turn zero. The determinant of the rest of the terms for the respective pairs of linear equations write:

$$\Delta=(\lambda_2-\lambda_1^2-2\lambda_1\psi).$$

Obviously, Δ turns zero only for the constant ψ:

$$\psi=\frac{\lambda_2-\lambda_1^2}{2\lambda_1}$$

In this case, the system has the two-parameter family of IMSMs:

$$Ap-\frac{\lambda_2+\lambda_1^2}{2\lambda_1}\gamma_1=0,\;\; Bq-\frac{\lambda_2+\lambda_1^2}{2\lambda_1}\gamma_2=0,$$

$$Cr-\frac{\lambda_2+\lambda_1^2}{2\lambda_1}\gamma_3=0.$$

Such a family of IMSMs takes place also for the classical Euler's top in the Greenhill's case [Appel, 1960].

Consider now the IMSMs, which correspond to the family of integrals K for $\lambda_1=0$. The stationary conditions here write:

$$\begin{aligned}
\frac{\partial K}{\partial p}&=A(Ap-\psi\gamma_1)-\frac{\partial\psi}{\partial p}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0,\\
\frac{\partial K}{\partial q}&=B(Bq-\psi\gamma_2)-\frac{\partial\psi}{\partial q}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0,\\
\frac{\partial K}{\partial r}&=C(Cr-\psi\gamma_3)-\frac{\partial\psi}{\partial r}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0,\\
\frac{\partial K}{\partial\gamma_1}&=-\psi(Ap-\psi\gamma_1)-\frac{\partial\psi}{\partial\gamma_1}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0,\\
\frac{\partial K}{\partial\gamma_2}&=-\psi(Bq-\psi\gamma_2)-\frac{\partial\psi}{\partial\gamma_2}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0,\\
\frac{\partial K}{\partial\gamma_3}&=-\psi(Cr-\psi\gamma_3)-\frac{\partial\psi}{\partial\gamma_3}[(Ap\gamma_1+Bq\gamma_2\\
&\quad+Cr\gamma_3)-\psi]=0.
\end{aligned}$$

The expression $Ap\gamma_1+Bq\gamma_2+Cr\gamma_3-\psi=c$ is the first integral. When the constant of this integral turns zero, the stationary equations define the following IMSM for the initial differential equations:

$$Ap-\psi\gamma_1=0,\;\; Bq-\psi\gamma_2=0,\;\; Cr-\psi\gamma_3=0. \quad (12)$$

There is the following relation between ψ and φ:

$$\begin{aligned}
&\varphi(p,q,r,\gamma_1,\gamma_2,\gamma_3)=\\
&\frac{1}{(1-\frac{\partial\psi}{\partial p}\frac{\gamma_3}{A}+\frac{\partial\psi}{\partial q}\frac{\gamma_3}{B}+\frac{\partial\psi}{\partial r}\frac{\gamma_3}{C})}\\
&\Big\{(\frac{\partial\psi}{\partial p}\frac{(B-C)}{A}qr+\frac{\partial\psi}{\partial q}\frac{(C-A)}{B}rp\\
&+\frac{\partial\psi}{\partial r}\frac{(A-B)}{A}pq)+(\frac{\partial\psi}{\partial\gamma_1}(r\gamma_2-q\gamma_3)\\
&+\frac{\partial\psi}{\partial\gamma_2}(p\gamma_3-r\gamma_1)+\frac{\partial\psi}{\partial\gamma_3}(q\gamma_1-p\gamma_2))\Big\}.
\end{aligned}$$

The latter partial differential equation may be used, for example, for defining φ when $\psi(p, q, r, \gamma_1, \gamma_2, \gamma_3)$ is given. Let us use the first integral K for obtaining sufficient conditions of stability of the IMSMs found. To this end, introduce deviations from IMSMs (12)

$$y_1 = Ap - \psi\gamma_1,\;\; y_2 = Bq - \psi\gamma_2,\;\; y_3 = Cr - \psi\gamma_3,$$

and – with the aid of these expressions – exclude $\gamma_1, \gamma_2, \gamma_3$ from integral K. As a result, we have:

$$\begin{aligned} &2K = [(y_1 + \psi\gamma_1)^2 + (y_2 + \psi\gamma_2)^2 \\ &+(y_3 + \psi\gamma_3)^2] - 2\psi[\gamma_1(y_1 + \psi\gamma_1) + \gamma_2(y_2 + \psi\gamma_2) \\ &+\gamma_3(y_3 + \psi\gamma_3)] + \psi^2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) = \\ &(y_1^2 + y_2^2 + y_3^2). \end{aligned}$$

Since the integral is a sign-definite quadratic form of the variables y_1, y_2, y_3 for any bounded functions $\psi(p, q, r, \gamma_1, \gamma_2, \gamma_3)$, hence according to Zubov's theorem [Zubov, 1989], the IMSM is stable.

2 CYCLIC CHAINS

Consider the following linear system of equations:

$$\dot{x} = ax + by,\;\; \dot{y} = bx - ay. \tag{13}$$

It can easily be verified that the system assumes the following cyclic chain of differential corollaries:

$$\frac{dV}{dt} = W,\;\; \frac{dW}{dt} = 2(a^2 + b^2)V,$$

where $2V = x^2 + y^2,\; W = ax^2 + 2bxy - ay^2$. This system may be considered as a comparison system for the system (13).

It can easily be shown that the system of differential equations

$$\dot{V} = AW,\;\; \dot{W} = BV,\;\; A = const,\;\; B = const$$

has the first integral

$$V^2 - \frac{A}{B}W^2 = const.$$

This first integral writes:

$$\Omega = (a^2{+}b^2)(x^2{+}y^2)^2{-}(ax^2{+}2bxy{-}ay^2)^2 = const.$$

The latter expression is factorized:

$$\begin{aligned} &\Omega = (\sqrt{(a^2 + b^2)}(x^2 + y^2) \\ &-(ax^2 + 2bxy - ay^2)) \\ &(\sqrt{(a^2 + b^2)}(x^2 + y^2) + (ax^2 + 2bxy - ay^2)). \end{aligned}$$

It can easily be verified that the two quadratic forms obtained in this case are the particular integrals of differential equations (13), i.e. these define the invariant manifolds of system (13). So, the cyclic chain also allows to find first integrals, while reducing the solution of the problem to integration of systems of differential equations, which may turn out to be simpler than the initial system of equations.

3 THE SECOND ORDER SYSTEM

Theorem 3. If a system of second order differential equations has the differential corollary of the form

$$\dot{V} = \varphi(t)V(\dot{x}, x, t),$$

where $V(\dot{x}, x, t)$ is the function linear with respect to velocities

$$V(\dot{x}, x, t) = \sum_{i=1}^{n} b_i(x)\dot{x}_i,$$

then, as a result of replacement of the independent variable, the system is reduced to the form

$$\ddot{x}_i = \sum_{j=1}^{n}\sum_{l=1}^{n} b_j(x)\dot{x}_j\dot{x}_l$$

and assumes the first integral

$$V(\dot{x}, x, t) = \sum_{i=1}^{n} b_i(x)\dot{x}_i.$$

Proof. Compute the derivative of the function $V = \sum_{i=1}^{n} b_i(x)\dot{x}_i$ due to the system of equations

$$\ddot{x}_i = f_i(x, \dot{x})\;\; i = 1, 2, \dots, n.$$

Using the differential corollary, we have:

$$\dot{V} = \sum_{i=1}^{n} b_i(x)\ddot{x}_i{+}\sum_{i=1}^{n}\sum_{i=1}^{n}\frac{\partial b_i}{\partial x_l}\dot{x}_i\dot{x}_l = \varphi(t)\sum_{i=1}^{n} b_i(x)\dot{x}_i$$

or

$$\sum_{i=1}^{n}[b_i(x)f_i + (\sum_{i=1}^{n}\frac{\partial b_i}{\partial x_l}\dot{x}_l - \varphi(t)b_i(x))\dot{x}_i] = 0.$$

Hence

$$f_i = \varphi(t)\dot{x}_i - \sum_{i=1}^{n}\frac{\partial \ln b_i}{\partial x_l}\dot{x}_l\dot{x}_i.$$

Therefore, the initial system writes:

$$\ddot{x}_i - \varphi(t)\dot{x}_i = -\sum_{i=1}^{n} \frac{\partial lnb_i}{\partial x_l}\dot{x}_l\dot{x}_i. \qquad (14)$$

Replace the independent variable

$$d\tau = e^{\int_0^t \varphi(\xi)d\xi}dt.$$

As a result of simple computations, system (14) writes:

$$\frac{d^2x_i}{d\tau^2} = -\sum_{l=1}^{n} \frac{\partial lnb_i}{\partial x_l}\frac{dx_l}{d\tau}\frac{dx_i}{d\tau}, \;\; i = 1, ..., n.$$

Obviously, the latter system assumes the first integral

$$V = \sum_{i=1}^{n} b_i(x)\frac{dx_i}{d\tau}.$$

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