ON LYAPUNOV'S DIRECT METHOD IN CONTROL PROBLEMS FOR NONLINEAR SYSTEMS

Alexandr S. Andreev
Faculty of Mathematics and Information Technology
Ulyanovsk State University
Russia
andreevas@sv.ulsu.ru

Abstract

The new methods of solving stabilization problems of nonlinear systems are presented with use of the sign-constant Lyapunov functional and Lyapunov functions. This methods are results extension of [Andreev and Peregudova, 2005; Andreev and Rumyantsev, 2007; Andreev, 2009; Fu and Li, 2009; Pavlikov, 2007]. They allow us to obtain effective solutions of control problems with programmed motions of mechanical, including robotic systems. The motion control problem of the flipped mathematical pendulum is solved as an example.

Key words

nonlinear control system, Lyapunov functional, stabilization

1 Problem statement

Consider the control system, whose motion is described by a functional-differential equation with delay [El'sgolz and Norkin, 1971; Hale, 1977]

$$\dot{x}(t) = f(t, x_t, u), \tag{1}$$

where $x \in \mathbb{R}^n$, $C_H = \{\varphi \in C \colon \|\varphi\| < H,\ 0 < H < \infty\}$, $u \in \mathbb{R}^m$ is a control, $f \colon \mathbb{R}^+ \times C_H \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

Let $X = \{x \colon [t_0 - h, \infty) \to D_H\}$, $(t_0 \geqslant 0, D_H = \{x \in \mathbb{R}^n \colon |x| < H\})$ be a class of allowable programmed motions, i.e. absolutely continuous functions, that are provided by the controls $u \in U$, where U is a some set of functions $u = u(t, \varphi)$, that are piecewise continuous in domain $\mathbb{R}^+ \times C_H$.

Let $u \in U$ be the some control, $f_0 = f_0(t, \varphi) = f(t, \varphi, u^0)$ in the continuity domain $\mathbb{R}^+ \times D$, $D \subset C_H$ of the function $u = u^0(t, \varphi)$. Redefine $f_0(t, \varphi)$ to $F(t, \varphi)$ in the points of discontinuity and consider the functional-differential inclusion

$$\dot{x}(t) \in F(t, x_t). \tag{2}$$

Let $x^0(t) \in X$ be a some programmed motion, which is provided by the control $u = u^0(t, \varphi)$, so that the relation

$$\dot{x}^0(t) \in F(t, x^0_t)$$

is valid. Moreover, we assume that the solution $x = x^0(t)$ of the inclusion (2) is unique. One can define the set of limit inclusions

$$\dot{x}(t) \in F^*(t, x_t). \tag{3}$$

Let $x^0 \in X$ (or $x_t \in C_H$) be the some programmed motion, $u^0 \in U$ be the realizing control, $\{x^*(t)\}$ the limit solution (3) to $x^0(t)$.

2 Theorems about stabilization

The following theorems allow us to determine the conditions under which the control $u = u^0(t, \varphi)$ is stabilizing for the motion $x = x^0(t)$.

Theorem 1. *Suppose that there is a functional* $V \colon \mathbb{R}^+ \times C_{H_1} \to \mathbb{R}$ *such that*

1. $a_1(|\varphi(0) - x^0(t)|) \leqslant V(t, \varphi) \leqslant a_2(\|\varphi - x^0_t\|)$*;*
2. $\dot{V}(t, \varphi) \leqslant -W(t, \varphi) \leqslant 0$*;*
3. *for each limit pair* (F^*, W^*) *the set* $\{W^*(t, \varphi) = 0\}$ *does not contain the motions of (3), except for* $x = x^*(t)$*.*

Then the control $u^0(t, \varphi)$ *solves the problem of stabilizing* $x^0(t)$*.*

Theorem 2. *Suppose that there is a functional* $V = V(t, \varphi)$ *such that*

1. $0 \leqslant V(t,\varphi) \leqslant a_0(\|\varphi - x_t^0\|)$;
2. $\dot{V}(t,\varphi) \leqslant -W(t,\varphi) \leqslant 0$;
3. *for each limit pair* (F^*, W^*) *the set* $\{V^{-1}(t,c) = c = c_0 = const > 0\}$ *contains none of the solutions (3);*
4. *the family of the limit solutions* $\{x^*(t)\}$ *to* $x^0(t)$ *accordingly uniformly asymptotically stable relative to the family* $\{(F^*, W^*), V_\infty^{-1}(t,0)\}$.

Then the control $u^0(t,\varphi)$ *solves the problem of stabilizing* $x = x^0(t)$.

Theorem 3. *Suppose that there is a Lyapunov function* $V = V(t,x)$, $V \in C^1$ *such that*

1. $a_1(|x - x^0(t)|) \leqslant V(t,x) \leqslant a_2(|x - x^0(t)|)$;
2. $\dot{V}(t,\varphi) \leqslant 0$ *for each function* $\varphi \in C$ *such that* $V(t+s, \varphi(s) - x^0(t)) \leqslant V(t, \varphi(0) - x^0(0))$;
3. $|\dot{V}| \geqslant W \geqslant 0$ *for all* $(t,\varphi) \in \mathbb{R}^+ \times C_{H_1}$;
4. *for all* $c_0 > 0$ *and each pair* (V^*, W^*) *there does not exist continuous curves* $\upsilon\colon \mathbb{R} \to D_{H_1}$ *such that for all* $\tau \in \mathbb{R}$ *there is a* $\theta \in [\tau - h, \tau]$, *so the equalities*

$$V^*(\theta, \upsilon(\theta)) = c,\ W^*(\theta, \upsilon_\theta) = 0$$

are satisfied simultaneously.

Then the control $u = u^0(t,\varphi)$ *solves the problem of stabilizing* $x = x^0(t)$.

Assume that the controls $u = (u_1(t,\varphi), u_2(t,\varphi), \ldots, u_m(t,\varphi))$ are piecewise continuous and have a discontinuity at the surface $\{\psi_j(t,\varphi) = 0\}$ $\big(j = \overline{1,l}\big)$, where $\psi_j : \mathbb{R}^+ \times C_H \to \mathbb{R}$ are bounded, uniformly continuous functions.

Theorem 4. *Suppose that there is a Lyapunov functional* $V = V(t,\varphi)$ *such that*

1. $a_1(|\psi(t,\varphi)|) \leqslant V(t,\varphi) \leqslant a_2(\|\varphi - x_t^0\|)$;
2. $\dot{V}^*(t,\varphi) \leqslant a_3(|\psi(t,\varphi)|)$ *for each function* $\varphi \in C$ *such that* $V(t+s, \varphi(s) - x^0(t)) \leqslant V(t, \varphi(0) - x^0(0))$;
3. *the family* $\{x = x_t^*\}$ *is asymptotically stable relative to the set* $\{\psi^*(t,x) = 0\}$ *uniformly accordingly with respect to* $\{\dot{x}(t) \in F^*(t, x_t)\}$.

Then the control $u = u^0(t,\varphi)$ *is stabilizing for the motion* $x = x^0(t)$.

Theorem 5. *Assume that there is a Lyapunov functional* $V = V(t,\varphi)$ *such that*

1. *the conditions 1) and 3) of the theorem 4 are satisfied;*
2. *in the second condition of the theorem 4* $a_3 \in \mathcal{K}$.

Then, in addition to the conclusion of the theorem 4 each motion (2) reaches the surface $\{\psi(t,\varphi) = 0\}$ *for the finite time* T, $x_T \in \{\psi(T,\varphi) = 0\}$.

3 Example 1

Consider the one-dimensional control system

$$\dot{x} = u. \tag{4}$$

Determine a class of nonlinear controls of the form

$$\begin{gathered} u = -f(t, x(t - h(t))),\ f(t,0) = 0, \\ 0 \leqslant h(t) \leqslant h_0,\ h_0 > 0 \end{gathered} \tag{5}$$

that stabilize $x = 0$.

We assume that $f \in C^1$, its derivatives f'_t and f'_x are bounded, uniformly continuous with respect to $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $|f(t,x)| \geqslant a(|x|)$, $a \in \mathcal{K}$ is a monotonically increasing function.

Choose the Lyapunov function of the form

$$V = \frac{1}{2} f^2(t,x).$$

We have the equalities

$$\begin{gathered} f(t, x(t - h(t))) = f(t, x(t)) - \\ - \int\limits_{t-h(t)}^{t} f'_x(t, x(\tau)) \dot{x}(\tau)\, d\tau = \\ f(t, x(t)) - \int\limits_{t-h(t)}^{t} f'_x(t, x(\tau)) f(\tau, x(\tau - h(\tau)))\, d\tau. \end{gathered}$$

For any function $\varphi \in [-2h_0, 0]$ such that

$$f^2(t+s, \varphi(s)) \leqslant f^2(t, \varphi(0))$$

we find

$$\begin{gathered} \dot{V} \leqslant f'_t(t, \varphi(0)) f(t, \varphi(0)) - \\ f^2(t, \varphi(0)) + m f^2(t, \varphi(0)) h(t) \leqslant \\ \leqslant -\mu_0 f^2(t, \varphi(0)) \leqslant 0 \end{gathered}$$

if

$$\begin{gathered} f'_t(t, \varphi(0)) f(t, \varphi(0)) + \leqslant \\ \leqslant (1 - \mu_0 - m h_0)\, f^2(t, \varphi(0)). \end{gathered} \tag{6}$$

By the theorem 3 the control (5) solves the problem of stabilizing $x = 0$ of the system (4), if the condition (6) is met.

4 Example 2

Consider stabilization problem of the programmed motion

$$(\varphi_0(t), \dot\varphi_0(t)), |\dot\varphi_0(t)| \leqslant m = const, |\ddot\varphi_0(t)| \leqslant l = const$$

of the flipped mathematical pendulum

$$\ddot\varphi = \omega_0^2 \sin\varphi + u.$$

Let us assume that the control u is determined with delay by φ

$$u = -\mu \mathrm{sign}\Big(\dot\varphi(t) - \dot\varphi_0(t) + \\ + k \sin\frac{\varphi(t-h) - \varphi_0(t-h)}{2}\Big),\ (k > 0). \quad (7)$$

For the functional derivative

$$V = \frac{1}{2}\Big(\dot\varphi(t) - \dot\varphi_0(t) + \\ + k \sin\frac{\varphi(t-h) - \varphi_0(t-h)}{2}\Big)^2$$

we find the estimation

$$\dot V \leqslant -\mu_0 \Big|\dot\varphi(t) - \dot\varphi_0(t) + \\ + k \sin\frac{\varphi(t-h) - \varphi_0(t-h)}{2}\Big| \leqslant 0$$

if $\mu - \omega_0^2 - m \geqslant 2\mu_0 > 0$.

One can show that the motion $(\varphi_0(t), \dot\varphi_0(t))$ uniformly asymptotically stable relative to the motions $(\varphi(t), \dot\varphi(t))$ that lie on the set

$$\dot\varphi(t) \neq k \sin\frac{\varphi(t-h) - \varphi_0(t-h)}{2} = \dot\varphi_0(t)$$

where $h < 1/m$. By the theorem 5 the control (7) with delay $h < 1/m$ solves the stabilization problem of the given pendulum motion $(\varphi_0(t), \dot\varphi_0(t))$.

5 Acknowledgments

The research was financially supported by the Russian Foundation for Basic Research (project no. 11-01-00541a).

References

Hale, J. (1977). *Theory of functional Differential Equations*. Springer. New York.

El'sgolz, L. E., Norkin, S. B. (1971). *Vvedenie v teoriyu funktsional'no-differentsial'nykh uravnenii*. Mir. Moscow.

Andreev, A. S., Peregudova, O. A. (2005). K metodu sravneniya v zadachah ob asimptoticheskoi ustoichivosti. *Doklady Mathematics*, **400**(5), pp. 621–624.

Andreev, A. S., Rumyantsev, V. V. (2007). O stabilizatsii dvijeniya nestatsionarnoi upravlyaemoi sistemyi. *Doklady Mathematics*, **416**(5), pp. 627–629.

Andreev, A. S. (2009). Metod funkcionalov Lyapunova v zadache ob ustoichivosti funktsional'no-differentsial'nykh uravnenii. *Avtomatika i telemekhanika*, (9), pp. 4–55.

Fu, X., Li, X. (2009). Razumikhin-type theorems on exponential stability of impulsive infinite delay differential systems. *Journal of Computational and Applied Mathematics*, **224**(1), pp. 1–10.

Pavlikov, S. V. (2007). Stabilization of the Motions of Controlled Mechanical Systems by a Regulator with Delay. *Doklady Mathematics*, **412**(2), pp. 176–178.