STRONGLY AND WEAKLY MONOTONE LYAPUNOV FUNCTIONS AND GLOBAL OPTIMALITY CONDITIONS IN CONTROL PROBLEMS

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Abstract
In this report the Hamilton-Jacobi canonical sufficient conditions of global optimality are developed. These conditions are based on using sets of strongly monotone nonsmooth functions of Lyapunov type, which depend on initial data (t_0, x_0) (or final (t_1, x_1)). Being solutions of the corresponding Hamilton-Jacobi inequalities, these functions allow us to obtain lower bounds for the cost functionals and sufficient global optimality conditions in optimal control problems. Some applications of weakly monotone Lyapunov like functions are shortly discussed.

Key words
Monotone Lyapunov functions, Hamilton-Jacobi inequalities, sufficient global optimality conditions.

1 Introduction

The report is devoted to development of sufficient global optimality conditions based on the solutions to Hamilton-Jacobi inequalities for optimal control problems for the following dynamic system (S):

$$\dot{x} = f(t, x, u), \quad u(t) \in U.$$

Solutions $\varphi(t, x)$ to Hamilton-Jacobi inequalities and equation are interpreted as Lyapunov functions in the wide sense, called L-functions for short.

It is well-known that there are two types of L-functions, namely, strongly and weakly monotone ones [Clarke, Ledyaev, Stern and Wolenski, 1998]. For instance, we say that a function $\varphi : G \to R$, where $G \subset R \times R^n$, is *strongly increasing* on G if the function $t \to \varphi(t, x(t))$ does not decrease along all trajectories of system (S) which go through G, i.e., such that $\operatorname{graph} x(\cdot) \subset G$. If $\varphi(t, x(t))$ does not decrease along at least one trajectory passing through G, then φ is *weakly increasing* on G. Strongly and weakly decreasing functions are defined in a similar way. Note that it is not so important if L-function decreases or increases. A strongly increasing L-function, for example, becomes strongly decreasing just by sign reversal. The principal question is whether the function is monotone along all trajectories of the system or at least along one of them. We denote by Φ_+ the set of all strongly increasing on R^{n+1} L-functions.

The L-functions play a fundamental role in control theory, and good overviews of modern theory of generalized solutions to Hamilton-Jacobi equations and inequalities and their various applications are given in [Clarke, Ledyaev, Stern and Wolenski, 1998; Bardi and Capuzzo-Dolcetta, 1997; Subbotin, 1995].

The strongly monotone L-functions have rich applicability in optimal control theory. For instance, the sufficient conditions, proposed by Carathéodory in calculus of variations [Ioffe and Tikhomirov, 1979; Young, 1969], and by R. Bellman [Fleming and Rishel, 1975; Vinter, 2000] and V.F. Krotov [Krotov, 1996; Krotov and Gurman, 1973] in optimal control, involve strongly monotone functions exactly. They are called sometimes verification functions, Bellman functions, Krotov functions or K-functions, etc. The functions from Φ_+ also appear largely in numerous works on generalizations and refinements of these sufficient conditions (in addition to the already mentioned references, see surveys in [Dykhta, 2004; Krotov, 1996; Clarke, 1983; Gurman, 1997; Vinter, 2000; Dykhta and Samsonyuk, 2011]). In particular, these generalizations include canonical optimality theory supposed in [Dykhta, 2004; Dykhta, 1990; Milyutin, 2000; Milyutin and Osmolovskii, 1998].

Although many sufficient optimality conditions that use L-functions (and Hamilton-Jacobi inequalities) were conversed into necessary ones [Kotsiopoulos and Vinter, 1993; Clarke and Nour, 2005], for optimal control problems with general endpoint constraints and cost functionals such conversions have not yet obtained. Therefore, we suggest to extend the class of used L-functions by introducing functions $V(t, x; t_0, x_0)$ (or $V(t, x; t_1, x_1)$), which parametrically depend on initial data $(t_0, x_0) = (t_0, x(t_0))$ (or final data $(t_1, x_1) = (t_1, x(t_1))$). Such L-functions we call *positional* and functions of type $\varphi(t, x)$ are called *ordinary*. Recent researches [Dykha, 2010] suggest that such extension of the L-function class is rather useful, and probably it will admit to obtain not only sufficient, but also necessary optimality conditions. However, in this report we discuss just only sufficient optimality conditions based on a modification of canonical theory with positional L-functions.

The limits on the report make it impossible to dwell on applications of weakly monotone L-functions in optimal control. Nevertheless, let us notice that such L-functions play an important role in issues of invariance, controllability and attainability [Clarke, Ledyaev, Stern and Wolenski, 1998; Guseinov and Ushakov, 1990; Kurzhanskii and Filippova, 1993], in modern theory of dynamical programming [Clarke, Ledyaev, Stern and Wolenski, 1998; Clarke, Ledyaev and Subbotin, 1997; Subbotin, 1995; Bardi and Capuzzo-Dolcetta, 1997] and in numerical methods of control improvement [Dykhta, 2009]. These problems may be studied via positional L-functions as well.

2 Problem statement

Let us consider the following optimal control problem (P) in Mayer form with general (mixed) endpoint constraint and cost functional:

$$J(\sigma) = l(q) \to \min; \quad q \in Q, \qquad \dot{x} = f(t, x, u), \quad u(t) \in U. \tag{1}$$

Here, $q = (t_0, x(t_0); t_1, x(t_1))$ is an endpoint vector, a pair $\sigma = (x(t), u(t) \mid t \in \Delta)$ describes a process of system (1), the time interval $\Delta = [t_0, t_1]$ depends on σ, $x(\cdot)$ and $u(\cdot)$ are absolutely continuous and measurable bounded functions defined on Δ, $\dim x = n$, $\dim u = m$. Let us denote by Σ_f and Σ the set of all processes of the control system (1) and the set of processes that are feasible in (P) (that satisfy the endpoint constraint), respectively.

For the sake of simplicity, we suppose everywhere that functions $f(t, x, u)$, $l(q)$ are continuous with respect to all arguments, Q is a closed set. All additional assumptions will be specified as necessary.

Let $\bar{\sigma} = (\bar{x}(t), \bar{u}(t) \mid t \in \bar{\Delta} = [\bar{t}_0, \bar{t}_1]) \in \Sigma$ be an examined process with the endpoint vector $\bar{q} = \big(\bar{t}_0, \bar{x}(\bar{t}_0); \bar{t}_1, \bar{x}(\bar{t}_1)\big)$.

3 Positional L-functions and canonical sufficient global optimality conditions

Let us rigorously define a notion of positional strongly increasing L-function.

Definition 1. *We say that a continuous function $V(t, x; t_0, x_0) : R^{2n+2} \to R$ is a positional strongly increasing L-function, if the following hold:*

$$V(t_0, x_0; t_0, x_0) \geq 0, \quad \forall\, (t_0, x_0) \in R^{n+1}; \tag{2}$$

$\big(\forall (t_0, x_0) \in R^{n+1}\big)\big(\forall (t_*, x_*) \in [t_0, +\infty) \times R^n\big)$
$\big(\forall\, \sigma \in \Sigma_f, x(t_*) = x_*\big)$
the function $t \to V(t, x(t); t_0, x_0)$ does not decrease on $[t_*, t_1]$.

The set of all such functions we denote by $\mathcal{V}_+$.

Notice that any ordinary L-function $\varphi(t, x) \in \Phi_+$ is embedded in the set $\mathcal{V}_+$ by equality $V(t, x; t_0, x_0) = \varphi(t, x) - \varphi(t_0, x_0)$. Hence, $\Phi_+ \subset \mathcal{V}_+$.

Definition 2. *Denote by $\mathcal{R}$ the set of quadruples $q = (t_0, x_0; t_1, x_1)$ such that, for any $q \in \mathcal{R}$, there exists a process $\sigma \in \Sigma_f$ of system* (1) *with $x(t_0) = x_0$, $x(t_1) = x_1$. We call $\mathcal{R}$ the conjoined set of system* (1).

The following equality evidently holds:

$$\min(P) = \min\{l(q) \mid q \in \mathcal{R} \cap Q\} \tag{3}$$

Here, $\min(P) = \min\{J(\sigma) \mid \sigma \in \Sigma\}$. We shall use similar brief notations for the values of other optimization problems.

Let $\mathcal{V} \subset \mathcal{V}_+$ be a certain set of positional strongly increasing L-functions. Introduce the set

$$E(\mathcal{V}) = \{q \mid V(t_1, x_1; t_0, x_0) \geq 0 \quad \forall\, V \in \mathcal{V}\}$$

and consider the extremal problem $(EP(\mathcal{V}))$:

$$l(q) \to \inf, \quad q \in E(\mathcal{V}) \cap Q.$$

Theorem 1. *(a) Every set $\mathcal{V} \subset \mathcal{V}_+$ gives the estimate* $\min(P) \geq \inf(EP(\mathcal{V}))$.

(b) Suppose that there exists a set $\mathcal{V} \subset \mathcal{V}_+$ such that the vector $\bar{q}$ is a global minimum point in problem $(EP(\mathcal{V}))$, i.e., $J(\bar{\sigma}) = l(\bar{q}) = \min(EP(\mathcal{V}))$. Then $\bar{\sigma}$ is a global minimizer in (P).

Proof immediately follows from the evident inclusion $E(\mathcal{V}) \supset \mathcal{R}$ and equality (3).

Definition 3. *A set $\mathcal{V} \subset \mathcal{V}_+$ is said to be a lower support set for (P) if* $\min(P) = \inf(EP(\mathcal{V}))$.

It is easy to prove that if $\bar{\sigma}$ and $\mathcal{V}$ satisfy assertion (b) of Theorem 1, then every global optimal process $\widetilde{\sigma} \in \Sigma$ satisfies this proposition together with $\mathcal{V}$. Therefore, any lower support set $\mathcal{V}$ enables us to check optimality of any process.

In applications it is useful to take into account partial information about endpoint constraints. It may be realized by considering and estimating the set

$\mathcal{R}[Q] := \{q = (t_0, x_0; t_1, x_1) \in \mathcal{R} \mid (t_0, x_0) \in pr_{t_0x_0}Q, (t_1, x_1) \in pr_{t_1x_1}Q\} \supset \mathcal{R} \cap Q$ instead of $\mathcal{R}$.

Notice that the property of strong monotonicity of $V \in \mathcal{V}$ may be slightly weakened if the control system has an invariant set $S \subset R^{n+1}$. It means that $(t, x(t)) \in S$ on Δ along any $\sigma \in \Sigma$ [Clarke, Ledyaev, Stern and Wolenski, 1998; Vinter, 2000]. In this case, it is enough to demand strong monotonicity of V only along trajectories with $\operatorname{graph} x(\cdot) \in S$, and $(EP(\mathcal{V}))$ should be completed by the constraints $(t_0, x(t_0)) \in S$, $(t_1, x(t_1)) \in S$.

Since (t_1, x_1) and (t_0, x_0) are symmetrically included in problem (P), we can consider L-functions of the form $V(t, x; t_1, x_1)$ and reformulate the sufficient conditions with such functions.

In [Dykha, 2010; Dykhta and Samsonyuk, 2011] it is proved that canonical sufficient optimality conditions are more flexible and general than the sufficient conditions of Carathéodory and Krotov types. Namely, every resolving set of L-functions for any of these alternative approaches is a lower support set in the sense of Definition 3 as well.

We have obtained the canonical sufficient optimality conditions in a preliminary form without any differential assumptions on L-functions. Let us recall some infinitesimal tests for strong monotonicity [Subbotin, 1995; Clarke, Ledyaev, Stern and Wolenski, 1998; Vinter, 2000].

Introduce the Pontryagin function $H(t, x, \psi, u) = \psi \cdot f(t, x, u)$ and the lower Hamiltonian $h(t, x, \psi) = \inf\{H(t, x, \psi, u) \mid u \in U\}$.

Firstly, if V is smooth with respect to (t, x), then it is strongly increasing if and only if it satisfies (2) and the following inequality:

$$\begin{aligned} &\dot{V}(t, x, u; t_0, x_0) := V_t + H(t, x, V_x, u) \geq 0 \\ &\forall\, (t, x, u) \in [t_0, +\infty) \times R^n \times U, \\ &\forall\, (t_0, x_0) \in R^{n+1}, \end{aligned}$$

where $\nabla_{tx}V = (V_t, V_x)$ is the gradient of $V(\cdot, \cdot; t_0, x_0)$ at point (t, x).

Suppose that the following assumptions hold:

- (H1) The set U is compact;
- (H2) The function $f(t, \cdot, u)$ is locally Lipschitz continuous with respect to x uniformly in $(t, u) \in R \times U$;
- (H3) There exists a constant $c > 0$ such that $|f(t, x, u)| \leq c(1 + |x|)$ on $R^{n+1} \times U$;
- (H4) $f(t, x, U)$ is a convex set $\forall\, (t, x) \in R^{n+1}$.

When $V(\cdot, \cdot; t_0, x_0)$ is locally Lipschitz continuous, the function V is strongly increasing if it satisfies (2) and the following inequality holds at each point of differentiability of $V(\cdot, \cdot; t_0, x_0)$:

$$\begin{aligned} &\bar{h}(t, x, \nabla_{tx}V) := V_t + h(t, x, V_x) \geq 0 \\ &\forall\, (t, x) \in [t_0, +\infty) \times R^n, \forall\, (t_0, x_0) \in R^{n+1}. \end{aligned}$$

Finally, let $V(\cdot, \cdot; t_0, x_0)$ be just continuous. Then strong monotonicity of V is equivalent to the fact that V satisfies the proximal Hamilton-Jacobi inequality (together with (2)):

$$\begin{aligned} &\bar{h}(t, x, p) = p_t + h(t, x, p_x) \geq 0 \\ &\forall\, p = (p_t, p_x) \in \partial^P_{tx}V(t, x; t_0, x_0), \\ &\forall\, (t, x) \in [t_0, +\infty) \times R^n, \forall\, (t_0, x_0) \in R^{n+1}. \end{aligned} \tag{4}$$

Here, $\partial^P_{tx}V(t, x; t_0, x_0)$ is the partial proximal subdifferential of $V(\cdot, \cdot; t_0, x_0)$ at (t, x) [Clarke, Ledyaev, Stern and Wolenski, 1998; Vinter, 2000]. The inequality (4) should be verified only at points with $\partial^P_{tx}V \neq \emptyset$. Note that this proximal criteria ensures strong monotonicity even for a lower semicontinuous V.

4 Properties of lower support sets and optimal processes

Following [Milyutin and Osmolovskii, 1998, p. 117], we say that $\bar{\sigma}$ is a *point of almost global minimum* in (P) if $J(\bar{\sigma})$ is an isolated (from the left) point of $J(\Sigma)$.

We call problem (P) *degenerated at the point* $\bar{\sigma}$ (in the sense of almost global minimum at the point $\bar{\sigma}$) if the endpoint vector $\bar{q}$ affords a local minimum in the following finite-dimensional problem:

$$l(q) \to \min; \quad q \in Q.$$

If (P) is degenerated at the point $\bar{\sigma}$, then the singleton $\mathcal{V} = \{V \equiv 0\}$ is a trivial lower support set. This fact is obtained by using natural local variants of Theorem 1 and Definition 3. Hence, degenerated problems are not interesting.

Theorem 2. *Let (P) be not degenerated at the point $\bar{\sigma}$. If $\mathcal{V}$ is a lower support set and it is uniformly continuous at the point $\bar{\sigma}$, then the following properties hold:*

(a) $\forall\, \varepsilon > 0$ $\mathcal{V}_\varepsilon(\bar{\sigma}) := \{V \in \mathcal{V} \mid V(\bar{q}) \leq \varepsilon\} \neq \varnothing$ *and* $\inf\limits_{V \in \mathcal{V}} V(\bar{q}) = 0$*;*

(b) $\forall\, V \in \mathcal{V}_\varepsilon(\bar{\sigma})$ *the process $\bar{\sigma}$ is an ε-optimal in the following optimal control problem without endpoint constrains:*

$$\begin{gathered} \omega(V; \sigma) := V(t_1, x(t_1); t_0, x(t_0)) \to \min; \\ \sigma \in \Sigma_f; \end{gathered} \tag{5}$$

the ε-optimality of $\bar{\sigma}$ in (5) means that $\forall\, \sigma \in \Sigma_f$ $\omega(V; \sigma) \geq \omega(V; \bar{\sigma}) - \varepsilon$;

(c) if $V^j(\bar{q}) \to 0$ as $j \to \infty$ for a compositionally absolutely continuous sequence $\{V^j\} \subset \mathcal{V}$, then

$$\frac{d}{dt}V^j(t, \bar{x}(t); \bar{t}_0, \bar{x}(\bar{t}_0)) \to 0 \;\; in \;\; L_1(\bar{\Delta});$$

(d) the marginal function (the lower envelope of the set $\mathcal{V}$)

$$V_*(t, x; t_0, x_0) = \inf_{V \in \mathcal{V}} V(t, x; t_0, x_0)$$

satisfies the following statements:

1) V_* *is a strongly increasing* L*-function;*
2) *the singleton* $\{V_*\}$ *is a lower support set for* (P)*;*
3) *problems* $(EP(\mathcal{V}))$ *and* $(EP(V_*))$ *are equivalent to each other.*

From properties (a), (b), (c) one can obtain the connection between Approximate Pontryagin Maximum Principle [Mordukhovich, 2006] for control problems (5) and supergradients of functions $V \in \mathcal{V}_\varepsilon(\bar\sigma)$ evaluated along the trajectory $\bar x$. Analogous relation occurs for problem (P) and the function V_* under some additional compactness assumptions on the set $\mathcal{V}$.

The property (d) is especially important for applications of Theorem 1.

Firstly, let us remark that the similar property does not hold for a lower support set of ordinary L-functions (in the conventional variant of canonical optimality conditions [Dykhta, 2004; Dykhta, 1990; Milyutin and Osmolovskii, 1998]). Indeed, in the simplest example

$$\dot x = 0\cdot u, \quad |u| \le 1, \quad J = x(0)x(1) \to \min$$

the set $\Phi = \{\varphi_1 = x, \varphi_2 = -x\} \subset \Phi_+$ is a lower support set of ordinary L-functions. But the marginal function

$$\varphi_*(x) = \min\{\varphi_1 = x, \varphi_2 = -x\} = -|x| \in \Phi_+$$

is not lower support.

Secondly, the property (d) allows to interpret the search of a lower support set as a constructive method of finding a single lower support function V_*. Notice that V_* may have a very complicated structure, and direct search of V_* by solving Hamilton-Jacobi inequality (equation) may be a difficult problem.

Finally, the property (d) admits to construct an optimal (or suboptimal) feedback control of the form $u_*(t, x; t_0, x_0)$ even if no examined process is given a priori. This feedback control is obtained by marginal function V_* via well-known methods of dynamical programming [Bardi and Capuzzo-Dolcetta, 1997; Krasovskii and Subbotin, 1988; Subbotin, 1995; Clarke, Ledyaev, Stern and Wolenski, 1998; Clarke, Ledyaev and Subbotin, 1997]; the most general and natural ones are N.N. Krasovskii method of extremal aiming and its modifications allowing discontinuity of strategy u_*. We omit details of construction of this extremal (w.r.t. V_*) positional control. We just note that in this way sufficient optimality conditions from Theorem 1 become a solving method for (P) rather than a verification test for a given process $\bar\sigma$.

5 Examples

Positional strongly monotone L-functions may be found by different constructive methods that were developed in dynamic programming and applications of ordinary L-functions in optimal control theory [Krotov and Gurman, 1973; Krotov, 1996; Gurman, 1997; Dykhta, 2004; Subbotin, 1995; Bardi and Capuzzo-Dolcetta, 1997]. Here, we consider two examples to illustrate Theorem 1 and some methods of construction of lower support sets.

Example 1. Linear-quadratic optimal control problems traditionally attract attention of researchers since they have reach applications. Here, we give two variants of a nonstandard linear-quadratic example with a cost functional containing mixed endpoint term.

(A) *Variant with unbounded control*:

Let us write the considered problem in Mayer form:

$$J = y(T) + x^2(0) + 4x(0)x(T) + x^2(T) \to \inf;$$
$$\dot x = u, \quad \dot y = u^2, \quad y(0) = 0. \qquad (6)$$

By modifying the standard method of dynamical programming for linear-quadratic problems [Fleming and Rishel, 1975; Krotov, 1996; Krotov and Gurman, 1973] we can find the following positional linear-quadratic (w.r.t. state variable x) strongly increasing L-function

$$V(t, x, y; x_0) = \frac{1}{T+1-t}x^2 + \frac{4}{T+1-t}xx_0 + \frac{4}{T+1-t}x_0^2 - \frac{9}{T+1}x_0^2 + y.$$

The extremal feedback control

$$u_*(t, x; x_0) = -\frac{V_x}{2} = -\frac{x + 2x_0}{T+1-t}$$

is obtained via minimization of the total derivative $\dot V$ w.r.t. control.

Solving of endpoint problem $(EP(V))$ and further using of u_* show that V is a lower support function and:

1) if $T < 2$, then the unique global optimal process is $\bar\sigma = 0$ and $J(\bar\sigma) = 0$;
2) if $T = 2$, then there exist infinite number of global optimal processes $\widetilde\sigma(x_0)$ such that $\widetilde u(t, x_0) \equiv -x_0$, $\widetilde x(t, x_0) = x_0(1-t)$, $\widetilde y(t, x_0) = x_0^2 t$, $x_0 \in R$, and $J(\widetilde\sigma) = 0$;
3) if $T > 2$, then there is no optimal process and the cost functional J is unbounded from below.

(B) *Variant with bounded control* is obtained from (A) by adding the constraint $|u(t)| \le 1$. Obviously, in the case $T < 2$ the answer is the same as in variant (A); when $T = 2$ the processes $\widetilde\sigma(x_0)$, $|x_0| \le 1$, are optimal (other processes of collection $\widetilde\sigma(x_0)$ are not admissible).

The case $T > 2$ is the most interesting. Here, there are two global optimal processes

$$\sigma_1 = \big(x_1(t) = t - T/2,\ y_1(t) = t,\ u_1(t) \equiv 1\big),$$
$$\sigma_2 = \big(x_2(t) = -t + T/2,\ y_2(t) = t,\ u_2(t) \equiv -1\big).$$

They can be found by using the lower support set $\mathcal{V}$ consisting of the following collections of L-functions:

$$\begin{aligned} &V_{1,2}(t,x;x_0) = t \pm (x - x_0), \\ &V_3(y) = y, \quad V_4(t,y) = t - y; \end{aligned}$$

clearly, this collection allows to exactly describe the reachable sets of each equation from (6); and

$$V_5(t,x,y;x_0) = \begin{cases} -\dfrac{(x-x_0)^2}{t} + y, & |x - x_0| < t, \\ -2|x - x_0| + t + y, & |x - x_0| \geq t; \end{cases}$$

it is constructed as the lower envelope of a family of linear L-functions $V^\psi = \psi(x-x_0) + \frac{\psi^2}{4}t + y$, $|\psi| \leq 2$.

Notice that in both variants of the example the trivial extremal $\sigma = 0$ has the conjugate point $T = 2$. However, we did use no special test for checking Jacobi conjugate point condition.

Example 2. Consider the nonlinear control system

$$\begin{aligned} &\dot{x}_i = g_i(x_i)u_i, \quad i = \overline{1,n}, \\ &u = (u_i) \in \left\{ u \in R^n_+ \mid \sum_1^n u_i = 1 \right\}, \end{aligned} \tag{7}$$

where $x = (x_i) \in R^n_{++} := \{x > 0\}$ and the functions $g_i(x_i) > 0$ on R_+ are continuous. This system arises in some generalized economical models of optimal resource allocation [Danskin, 1967]. We will show that any problem in Mayer form in this system may be reduced to finite-dimensional one. This procedure is carried out by applying an infinite set of solutions to a Hamilton-Jacobi equation, which exactly describes the conjoined set of the system.

For this system

$$h(x,\psi) = \min_{u \in U} \sum_1^n \psi_i g_i(x_i) u_i = \min_{1 \leq i \leq n} \psi_i g_i(x_i)$$

and, for $\varphi \in C^1$, the Hamilton-Jacobi equation is

$$\begin{aligned} \bar{h}(x, \varphi_x, \varphi_t) = &\varphi_t(t,x) + \\ &\min_{1 \leq i \leq n} \varphi_{x_i}(t,x) g_i(x_i) = 0. \end{aligned} \tag{8}$$

To solve this equation we apply the method of separation of variables, assuming that

$$\begin{aligned} &\varphi(t,x) = \sum_1^n \varphi_i(t,x_i), \\ &\varphi_{ix_i}(t,x_i) g_i(x_i) = \alpha_i, \quad i = \overline{1,n}, \end{aligned}$$

where α_i are arbitrary constants. It is easy to check that this solution is

$$\begin{aligned} \varphi(t,x) &= \sum_1^n \alpha_i G_i(x_i) + (t_1 - t)m(\alpha) \\ &= \alpha \cdot G(x) + (t_1 - t)m(\alpha), \quad \alpha \in R^n, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &G_i(x_i) = \int \frac{dx_i}{g_i(x_i)}, \quad i = \overline{1,n}, \\ &G(x) = (G_i(x_i)), \quad m(\alpha) = \min_{u \in U} \langle \alpha, u \rangle. \end{aligned} \tag{10}$$

Formula (9) defines the n-parameterized set $\Phi = \{\varphi^\alpha(t,x) \mid \alpha \in R^n\}$ of solutions to equation (8). Thus every $\varphi \in \Phi$ is an ordinary strongly increasing L-function.

The set $E(\Phi)$ corresponding to Φ is described by the infinite system of inequalities:

$$\begin{aligned} E(\Phi) = \{&q = (x_0, x_1, t_1) \mid \\ &\alpha \cdot (G(x_1) - G(x_0)) \geq t_1 m(\alpha) \; \forall \, \alpha \in R^n\}. \end{aligned}$$

But $m(\alpha)$ coincides with the support function of the simplex U modulo sign (see (10)). Therefore, these inequalities turn into the inclusion

$$E(\Phi) = \{q \mid G(x_1) - G(x_0) \in t_1 U\}. \tag{11}$$

We know that this set is an outer estimate for the conjoined set $\mathcal{R}$, that is $E(\Phi) \supset \mathcal{R}$.

Let us show that, in fact, this estimate is exact and $E(\Phi) = \mathcal{R}$. We will use elements of the Goh's nonlinear transformation [Dykhta, 2004], that is quite useful for linear w.r.t. control problems with commutative vector fields. Supplement system (7) with the equation $\dot{y} = u$, $y(t_0) = y(0) = 0$. It is easy to verify that the vector-function

$$\eta(x,y) = G(x) - y$$

is a first integral of the completed system (and, consequently, its components may be interpreted as strongly increasing L-functions for this system). So, the equality

$$G(x(t)) - y(t) = G(x_0) \quad \forall \, t \in [0, t_1]$$

holds along any trajectory $(x(\cdot), y(\cdot))$ of the supplemented system.

It is obvious that we have alternatively obtained the exact description of the conjoined set $\mathcal{R}$ and established the accuracy of the estimating set (11).

6 Conclusion

In the report a new class of Lyapunov like functions called positional is introduced. In common with ordinary L-functions, the positional ones are found by corresponding Hamilton-Jacobi differential inequalities. In term of appropriate set of positional strongly monotone L-functions the canonical sufficient conditions of global optimality are formulated and analyzed applying to Mayer optimal control problem with general endpoint constraints and cost functional. Particularly, there is ascertained the possibility for passage from a resolving (lower support) set of positional strongly monotone L-functions to a single one that is a lower envelope of the given set. This feature allows us to suggest an approach to construction of optimal (or suboptimal) feedback control thought a natural modification of dynamical programming technique. The efficiency of proposed sufficient optimality conditions is illustrated on two examples.

Positional L-function may be useful in another issues of control theory too.

Acknowledgements

The research was supported by SB-UrB RAS (integrating project no. 85) and by RFBR (grant no. 11-01-00672-a).

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