EXPONENTIAL DISSIPATIVITY OF DIFFUSION PROCESSES WITH MARKOVIAN SWITCHING AND ROBUST SIMULTANEOUS STABILIZATION*

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1. System description. Consider a nonlinear affine in control system described by the following stochastic differential equations with Markovian switching:

$$dx_t = [a(x_t, r_t) + B(x_t, r_t)u_t]dt + \sum_{l=1}^{N} \gamma_l [f_l(x_t, r_t) + G_l(x_t, r_t)u_t]dw_l, \tag{1.1}$$

$$z_t = c(x_t, r_t), \quad t \geq t_0, \tag{1.2}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the input vector, $z_t \in \mathbb{R}^k$ is the output vector, r_t is a homogenous Markov chain whose state space is a set of integers $\mathbb{N} = \{1, 2, ..., \nu\}$ and transition matrix $P(\tau) = [P_{ij}(\tau)]_1^\nu = [\text{Prob}\{r(t+\tau) = j \mid r(t) = i\}]_1^\nu = \exp(\Pi\tau), \quad 0 \leq t \leq t+\tau, \quad \Pi = [\pi_{ij}]_1^\nu$ with $\pi_{ij} \geq 0, \; j \neq i, \pi_{ii} = -\sum_{j\neq i}^{\nu} \pi_{ij}, \; w_t = [w_{1t} w_{2t} \dots w_{Nt}]$ is standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with the natural filtration $\mathcal{F}_t, \; t \geq t_0$ generated by w up to time t; the initial conditions $x_{t_0} = x_0$ and $r_{t_0} = i_0$ are deterministic, w_t and r_t are independent and u_t is $\mathcal{F}_t$-adapted.

Denote $\phi(x_t, r_t, u_t) = a(x_t, r_t) + B(x_t, r_t)u_t, \; \Psi(x_t, r_t, u_t) = [f_1(x_t, r_t) + G_1(x_t, r_t)u_t \dots f_N(x_t, r_t) + G_N(x_t, r_t)u_t]$. To ensure the existence and uniqueness of the solution of (1.1) and the existence of a trivial solution we assume that for all $i \in \mathbb{N}, \; \phi(0, i, 0) \equiv 0, \Psi(0, i, 0) \equiv 0, \; \phi$ and Ψ are continuous in u, locally Lipschitz and have linear growth [1].

Let $C^2(\mathbb{R}^n \times \mathbb{N}; \mathbb{R})$ denote the set of non-negative functions $V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ which are continuously twice differentiable in x and consider an operator L on $C^2(\mathbb{R}^n \times \mathbb{N}; \mathbb{R})$, which for $V \in C^2(\mathbb{R}^n \times \mathbb{N}; \mathbb{R})$ and for $\mathcal{F}_t$-measurable u defines $L_u V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ by

$$L_u V(x, i) = V_x(x, i)[a(x, i) + B(x, i)u] + \sum_{j=1}^{\nu} \pi_{ij} V(x, j) +$$

$$+ \frac{1}{2} \sum_{l=1}^{N} [f_l(x, i) + G_l(x, i)u]' V_{xx}(x, i) [f_l(x, i) + G_l(x, i)u], \tag{1.3}$$

where as usual $V_x(x, i) = \left[\frac{\partial V(x,i)}{\partial x_1} \dots \frac{\partial V(x,i)}{\partial x_n} \right]$ and $V_{xx}(x, i) = \left[\frac{\partial^2 V}{\partial x_j \partial x_k} \right]_{n\times n}$. This operator represents the differential generator of Markov process $[x_t \; r_t]$ in the hybrid state space $\mathbb{R}^n \times \mathbb{N}$ [1].

2. Dissipative diffusion processes with Markovian switching. Denote $\mathcal{L}^2_{\mathcal{F}}([s, T], \mathbb{R}^m)$ the set of all $\mathcal{F}_t$-adapted input processes such that

$$\| u \|^2_{\mathcal{L}^2([s,T])} \triangleq \mathrm{E} \int_s^T \| u_t \|^2 \, dt < \infty, \; s \geq 0.$$

Following to concept by Willems [2] consider a function $W : \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{N} \to \mathbb{R}$ associated with the system (1.1), (1.2). This function is called the supply rate on $[s, \infty)$ if it has the following property: for any $u \in \mathcal{L}^2_{\mathcal{F}}([s, T], \mathbb{R}^m)$ the output (1.2) of the system (1.1) with deterministic initial conditions $x_s = x_0, \; r_s = i_0$ is such that

$$\mathrm{E} \int_s^T \mid W(u_t, r_t, z_t) \mid dt < \infty, \; \forall T \geq s \geq 0.$$

Definition 2.1. *System (1.1), (1.2) with supply rate W is said to be exponentially dissipative on $[t_0, \infty), \; t_0 \geq 0$, if there exists a nonnegative continuous function $V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$ called the storage*

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function and function $\mu(x,i) > 0,\ x \neq 0,\ \mu(0,i) = 0$, *such that for all* $t \geq t_0 \geq 0,\ x_{t_0} = x \in \mathbb{R}^n,\ r_{t_0} = i \in \mathbb{N}$

$$\mathrm{E}_{x,i}[V(x_t, r_t) + \int_{t_0}^{t} \mu(x_t, r_t)dt] - V(x,i) \leq \mathrm{E}_{x,i} \int_{t_0}^{t} W(u_\tau, r_\tau, z_\tau)d\tau. \tag{2.1}$$

The above inequality following to [2] can be called the exponential dissipation inequality.

DEFINITION 2.2. *The available storage* $V_a(x,i)$ *of the system (1.1), (1.2) with supply rate* $W(u,i,z)$ *is the function defined for* $t \geq t_0$ *by*

$$V_a(x,i) = \sup_{u \in \mathcal{L}^2_{\mathcal{F}}([t_0,t],\mathbb{R}^m)} \sup_{t \geq t_0} \mathrm{E}_{x,i} \int_{t_0}^{t} [-W(u_\tau, r_\tau, z_\tau) + \mu(x_\tau, r_\tau)]d\tau).$$

As in the deterministic case the available storage plays important role in determining whether or not the system is dissipative. This is shown in the following theorem.

THEOREM 2.3. *The available storage* $V_a(x,i)$ *is finite for all* $x \in \mathbb{R}^n, i \in \mathbb{N}$ *if and only if the system (1.1), (1.2) is exponentially dissipative on* $[t_0, \infty),\ t_0 \geq 0$. *Moreover, for any possible storage function* $V\ 0 \leq V_a(x,i) \leq V(x,i)\ \ \forall x \in \mathbb{R}^n,\ i \in \mathbb{N}$. *and* V_a *is itself a possible storage function.*

Next we consider the supple rate in a special form

$$W(u,i,z) = z'Q(i)z + 2z'S(i)u + u'R(i)u, \tag{2.2}$$

where $Q(i) = Q'(i), S(i)$ and $R(i) = R'(i)$ $(i \in \mathbb{N})$ are matrices of compatible dimensions. The following assumption allows to use the differential generator and to obtain some more constructive results.

ASSUMPTION 1. *The storage function if it exists belongs to* $C^2(\mathbb{R}^n \times \mathbb{N}; \mathbb{R})$

THEOREM 2.4. *A necessary and sufficient condition for system (1.1), (1.2) with a supply rate* $W(\cdot,\cdot,\cdot)$ *to be dissipative on* $[t_0,\infty)$ *is that there exists nonnegative function* $V \in C^2(\mathbb{R}^n \times \mathbb{N}; \mathbb{R})$ *and functions* $q : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^{n_1 \times m}$ *and* $v : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^{n_1 \times m}$ *for some integer* $n_1 > 0$, *such that*

$$-\mu(x,i) + c'(x,i)Q(i)c(x,i) - V_x(x,i)a(x,i) - \sum_{j=1}^{\nu} \pi_{ij}V(x,j) - \frac{1}{2}\sum_{l=1}^{N} \gamma_l^2 f_l'(x,i)V_{xx}(x,i)f_l(x,i) = q'(x,i)q(x,i), \tag{2.3}$$

$$2c'(x,i)S(x,i) - V_x(x,i)B(x,i) - \sum_{l=1}^{N} \gamma_l^2 f_l'(x,i)V_{xx}(x,i)G_l(x,i) = 2v'(x,i)q(x,i), \tag{2.4}$$

$$R(i) - \frac{1}{2}\sum_{l=1}^{N} \gamma_l^2 G_l'(x,i)V_{xx}(x,i)G_l(x,i) = v'(x,i)v(x,i). \tag{2.5}$$

The equations (2.3) – (2.5) represent a nonlinear generalization of stochastic Lur'e equations arising in theory of absolute stochastic stability [3, 4].

3. Dissipativity and stabilization. In theory of deterministic systems an important role plays the supply rate in the form of inner product of input and output variables:

$$W(u,i,z) = z'u. \tag{3.1}$$

The dissipative system with supple rate (3.1) is said to be passive [3, 4]. The definition below gives a possible extension of the passivity notion to stochastic systems [5].

DEFINITION 3.1. *System (1.1), (1.2) is said to be passive on* $[t_0,\infty),\ t_0 \geq 0$ *if it is dissipative on* $[t_0,\infty),\ t_0 \geq 0$, *with supply rate (3.1) and the storage function satisfies* $V(0,i) = 0$ *for all* $i \in \mathbb{N}$.

Analyzing Lur'e equation (2.3) – (2.5) it is easy to see that notion of passivity makes sense only for particular case of system (1.1), (1.2) with $G_l(x,i) \equiv 0$, in other case the equation (2.5) is unsolvable.

The exponentially dissipative system (1.1), (1.2) has the following property of stabilizability by output feedback.

THEOREM 3.2. *Let the system (1.1), (1.2) exponentially dissipative with storage function* $V(x,i)$, *satisfying the inequality*

$$\lambda_1 \parallel x \parallel^2 \leq V(x,i) \leq \lambda_2 \parallel x, \parallel^2,\ \lambda_1, \lambda_2 > 0, \tag{3.2}$$

the supply rate has the form

$$W = z'Qz + z'Su + u'Ru$$

and $\mu(x, i) = x'Mx,\ M = M' > 0,\ i \in \mathbb{N}$. *Let* $\varphi : \mathbb{R}^m \to \mathbb{R}^p$ *is a function satisfying the condition*

$$z'Qz - z'S\varphi(z) + \varphi'(z)R\varphi(z) \le 0\ \forall z \ne 0\ \ \varphi(0) = 0. \tag{3.3}$$

Then the output feedback control

$$u = -\varphi(z) \tag{3.4}$$

provides exponential stability in the mean square of the trivial solution $x_t \equiv 0$ *of the system (1.1).*

This theorem allows to use the exponential dissipativity instead passivity in the process of synthesis of stabilizing control for stochastic systems.

4. Application to robust simultaneous stabilization of uncertain system. Let a set of deterministic nonlinear systems described by the following differential equations:

$$\dot{x}_t = a_i(x_t) + B_i(x_t)u_t + \sum_{l=1}^{N} \sigma_l(t)(f_{li}(x_t) + G_{li}(x_t)u_t), \tag{4.1}$$

$$z_t = c_i(x_t),\ t \ge t_0,\ i = 1, \dots, \nu, \tag{4.2}$$

where $\sigma_l(t),\ t \ge 0, l = 1, \dots, N$ are uncertain parameters such that

$$\mid \sigma_l(t) \mid \le \delta_l,\ t \ge 0,\ l = 1, \dots, N, \tag{4.3}$$

other notations are the same as above. Consider the following *robust simultaneous stabilization problem*: determine the output feedback control law

$$u = -\upsilon(z),\ \upsilon(0) = 0 \tag{4.4}$$

such that all the closed loop systems from the set (4.1), (4.2) are asymptotically stable for all $\sigma_l(t)$, satisfying (4.3). As a generalization of the results [6] we obtain the following theorem.

THEOREM 4.1. *Assume that for the system (1.1), (1.2) with control law (4.4) and with* $a(x, i) = \hat{a}(x, i) + \alpha I$ *there exists a quadratic Lyapunov function*

$$V(x) = x'Px, \tag{4.5}$$

such that $LV(x) \le 0$ *and*

$$\alpha - \frac{1}{2}\sum_{l=1}^{N} \frac{\delta_l^2}{\gamma_l^2} > 0. \tag{4.6}$$

Then this control law solves the robust simultaneous stabilization problem for the system (4.1), (4.2).

COROLLARY 4.2. *Suppose that (4.4) is robust stabilizing control and moreover the system (1.1), (1.2) with this control and with* $a(x, i) = \hat{a}(x, i) + \alpha I$ *is exponentially dissipative with positive definite storage function (4.5), supply rate*

$$W = z'Qz + z'Su + u'Ru$$

and $\mu(x, i) = x'Mx,\ M = M' > 0,\ i \in \mathbb{N}$. *If* $\varphi : \mathbb{R}^m \to \mathbb{R}^p$ *is the function satisfying condition (3.3), then*

$$u = -(\varphi(z) + \upsilon(z)) \tag{4.7}$$

is the robust stabilizing control for the system (4.1), (4.2).

These results mean that we can use all known and new techniques of solution of stochastic simultaneous stabilization problem for solving the robust simultaneous stabilization problem. Moreover if the closed loop system with $a(x, i) = \hat{a}(x, i) + \alpha I$ is exponentially dissipative with storage function and supply rate above, then the set of systems (4.1), (4.2) does not lose the robustness property under control variations satisfying (3.3).

As a particular case consider the set of linear systems with uncertain parameters

$$\dot{x} = A_i x_t + B_i u_t + \sum_{l=1}^{N} \sigma_l(t)(A_{li} x_t + B_{li} u_t), \tag{4.8}$$

$$z_t = C x_t,\ t \geq 0,\ i = 1, \ldots, \nu. \tag{4.9}$$

Define the gain matrix F such that all the closed loop systems from the set (4.8), (4.9) with output feedback

$$u(t) = -v(z) = -F z(t) \tag{4.10}$$

are asymptotically stable for parameters uncertainties satisfying (4.3) and for variations (4.7) of the control law (4.10), satisfying (3.3).

After assignation of the noise intensities and parameter of stability margin α according to (4.6) the solution of this problem is reduced to finding of the pair of matrices $H = H' > 0$ F, satisfying the system of matrix quadratic inequalities

$$(A_{\alpha i} - B_i F C)' H + H(A_{\alpha i} - B_i F C) + M - C'QC + \sum_{j=1}^{N} \gamma_j^2 (A_{ij} - \tag{4.11}$$

$$B_{ij} F C)' H (A_{ij} - B_{ij} F C) + (H B_i - \sum_{j=1}^{N} \gamma_j^2 (A_{ij} - B_{ij} F C)' H B_{ij} - C'S)(R -$$

$$\sum_{j=1}^{N} \gamma_j^2 B_{ij}' H B_{ij})^{-1} (H B_i - \sum_{l=1}^{N} \gamma_l^2 (A_{ij} - B_{ij} F C)' H B_{ij} - C'S)' \leq 0,$$

where $A_{\alpha i} = (A_i + \alpha I)$. The solution of (4.11) even in particular cases is NP-hard problem [8]. In this paper developing the results of [9, 10] we propose a two step procedure: first based on a convergent iteration algorithm we obtain the gain matrix F of robust stabilizing control, then based on solution of linear matrix inequalities which follow from (4.11) with given F we check the inequalities (3.3). A numerical example is given.

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