ABSOLUTE OBSERVATION STABILITY FOR EVOLUTIONARY VARITIONAL INEQUALITIES

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Abstract
We derive absolute observation stability and instability results for controlled evolutionary inequalities which are based on frequency-domain characteristics of the linear part of the inequalities. The uncertainty parts of the inequalities (nonlinearities which represent external forces and constitutive laws) are described by certain local and integral quadratic constraints. Other terms in the considered evolutionary inequalities represent contact-type properties of a mechanical system with dry friction.

Key words
Absolute observation stability, evolutionary variational inequalities, frequency-domain conditions

1 Introduction

Suppose that Y_0 is a real Hilbert space. We denote by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ the scalar product resp. the norm on Y_0. Let $A : \mathcal{D}(A) \to Y_0$ be the generator of a C_0-semigroup on Y_0 and define the set $Y_1 := \mathcal{D}(A)$. Here $\mathcal{D}(A)$ is the domain of A, which is dense in Y_0 since A is a generator. We denote with $\rho(A)$ the resolvent set of A. The spectrum of A, which is the complement of $\rho\,(A)$, is denoted by $\sigma(A)$. If we define with an arbitrary but fixed $\beta \in \rho\,(A) \cap \mathbb{R}$ for any $y, \eta \in Y_1$, the value

$$(y,\eta)_1 := ((\beta I - A)y,\ (\beta I - A)\eta)_0\ , \tag{1.1}$$

then the set Y_1 equipped with this scalar product $(\cdot,\cdot)_1$ and the corresponding norm $\|\cdot\|_1$ becomes a Hilbert space (different numbers β give different but equivalent norms). Denote by Y_{-1} the Hilbert space which is the completion of Y_0 with respect to the norm $\|y\|_{-1} := \|(\beta I - A)^{-1}y\|_0$ and which has the corresponding scalar product

$$(y,\eta)_{-1} := \left((\beta I - A)^{-1}y,\ (\beta I - A)^{-1}\eta\right)_0\ , \quad \forall\, y,\eta \in Y_{-1}. \tag{1.2}$$

Thus, we get the inclusions $Y_1 \subset Y_0 \subset Y_{-1}$, which are dense with continuous embedding, i.e. for $\alpha = 1, 0$ $Y_\alpha \subset Y_{\alpha-1}$, is dense and $\|y\|_{\alpha-1} \le c\|y\|_\alpha\ ,\ \forall\ y \in Y_\alpha$. Sometimes [Banks, Gilliam and Shubov, 1997; Banks and Ito, 1988] the introduced triple of spaces (Y_1, Y_0, Y_{-1}) is called a *Gelfand triple*. The pair (Y_1, Y_{-1}) is also called *Hilbert rigging* of the *pivot space* Y_0, Y_1 is an *interpolation space* of Y_0, and Y_{-1} is an *extrapolation space* of Y_0. Since for any $y \in Y_0$ and $z \in Y_1$ we have

$$|(y,z)_0| = |(\beta I - A)^{-1}y, ((\beta I - A)z)_0| \le \|y\|_{-1}\|z\|_1\ , \tag{1.3}$$

we can extend $(\cdot, z)_0$ by continuity onto Y_{-1} obtaining the inequality

$$|(y,z)_0| \le \|y\|_{-1}\|z\|_1\ , \quad \forall\, y \in Y_{-1}, \forall\, z \in Y_1.$$

Let us denote this extension also by $(\cdot,\cdot)_{-1,1}$ and call it *duality pairing* on $Y_{-1} \times Y_1$. The operator A has a unique extension to an operator in $\mathcal{L}(Y_0, Y_{-1})$ which we denote by the same symbol. Suppose now that $T > 0$ is arbitrary and define the norm for Bochner measurable functions in $L^2(0,T;Y_j)$ $\quad (j = 1, 0, -1)$ through

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{1.4}$$

Let $\mathcal{L}_T$ be the space of functions such that $y \in L^2(0,T;Y_1)$ and $\dot{y} \in L^2(0,T;Y_{-1})$, where the time derivative $\dot{y}$ is understood in the sense of distributions with values in a Hilbert space. The space $\mathcal{L}_T$ equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2\right)^{1/2} \tag{1.5}$$

is a Hilbert space and will be used for the description of solutions to evolutionary systems.

2 Evolutionary variational inequalities

Suppose that $T > 0$ is arbitrary and consider for a.a. $t \in [0,T]$ the observed and controlled evolutionary variational inequality

$$\begin{aligned}
&(\dot{y} - Ay - B\xi - f(t), \eta - y)_{-1,1} && (2.1)\\
&+ \psi(\eta) - \psi(y) \geq 0\,, \qquad \forall\, \eta \in Y_1 &&\\
&y(0) = y_0 \in Y_0\,, &&\\
&w(t) = Cy(t)\,, \quad \xi(t) \in \varphi(t, w(t))\,, && (2.2)\\
&\xi(0) = \xi_0 \in \mathcal{E}(y_0)\,, &&\\
&z(t) = Dy(t) + E\,\xi(t)\,. && (2.3)
\end{aligned}$$

In equations (2.1) $-$ (2.3) it is supposed that $C \in \mathcal{L}(Y_{-1}, W), D \in \mathcal{L}(Y_1, Z)$ and $E \in \mathcal{L}(\Xi, Z)$ are linear operators, Ξ, W and Z are real Hilbert spaces, $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Gelfand triple and $A \in \mathcal{L}(Y_0, Y_{-1})$, $B \in \mathcal{L}(\Xi, Y_{-1}), \varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ is a set-valued map, $\psi : Y_1 \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to Y_{-1}$ are given nonlinear maps. The calculation of $\xi(t)$ in (2.2) shows that this value in general also depends on certain "initial state" ξ_0 of φ taken from a set $\mathcal{E}(y_0) \subset \Xi$. This situation is typical for hysteresis nonlinearities [Reitmann, 2005].

In the following we denote by $\|\cdot\|_\Xi$, $\|\cdot\|_W$ and $\|\cdot\|_Z$ the norm in Ξ, W resp. Z.

Let us now introduce the solution space for the problem (2.1), (2.2).

Definition 2.1. *Any pair of functions $\{y\,(\cdot), \xi(\cdot)\}$ with $y \in \mathcal{L}_T$ and $\xi \in L^2_{\mathrm{loc}}(0,\infty;\Xi)$ such that $B\xi \in \mathcal{L}_T$, satisfying (2.1), (2.2) almost everywhere on $(0,T)$, is called* ***solution of the Cauchy problem*** *$y\,(0) = y_0,\ \xi(0) = \xi_0$ defined for (2.1), (2.2).*

In order to have an existence property for (2.1), (2.2) we state the following assumption:

(C1) The Cauchy-problem (2.1), (2.2) has for arbitrary $y_0 \in Y_0$ and $\xi_0 \in \mathcal{E}(y_0) \subset \Xi$ at least one solution $\{y\,(\cdot), \xi(\cdot)\}$.

Assumption **(C1)** is fulfilled, for example, in the following situation [Pankov, 1986].

(C2) a) The nonlinearity $\varphi : \mathbb{R}_+ \times W \to \Xi$ is a function having the property that $\mathcal{A}(t) := -A - B\varphi(t, C\cdot) : Y_1 \to Y_{-1}$ is a family of monotone hemicontinuous operators such that the inequality

$$\|\mathcal{A}(t)y\|_{-1} \leq c_1\|y\|_1 + c_2\,, \quad \forall\, y \in Y_1\,,$$

is satisfied, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants not depending on $t \in [0,T]$. Furthermore for any $y \in Y_1$ and for any bounded set $U \subset Y_1$ the family of functions $\{(\mathcal{A}(t)\eta, y)_{-1,1}\,,\ \eta \in U\}$ is equicontinuous with respect to t on any compact subinterval of $\mathbb{R}_+$.

b) ψ is a proper, convex, and semicontinuous from below function on $\mathcal{D}(\psi) \subset Y_1$.

(C3) $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_{-1})$.

(C4) In the sequel we consider only solutions y of (2.1),(2.2) for which $\dot{y}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}; Y_{-1})$.

Remark 2.1 a) Note that in the special case when $\psi \equiv 0$ in (2.1) the evolutionary variational inequality is equivalent for a.a. $t \in [0,T]$ to the equation

$$\begin{aligned}
&\dot{y} = Ay + B\xi + f(t) \quad \text{in } Y_{-1}\,,\\
&y(0) = Y_0\,,\ w(t) = Cy(t)\,,\ \xi(t) \in \varphi(t, w(t))\,,\\
&\qquad\qquad\qquad\qquad \xi(0) \in \mathcal{E}(y_0)\,,\\
&z(t) = Dy(t) + E\xi(t).
\end{aligned}$$

Under the assumption that φ is a single valued function this class was considered in [Banks, Gilliam and Shubov, 1997; Banks and Ito, 1988; Brusin, 1976]. $\square$

Definition 2.2. *a) Suppose F and G are quadratic forms on $Y_1 \times \Xi$. The* ***class of nonlinearities*** *$\mathcal{N}(F,G)$ defined by F and G consists of all maps $\varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ such that for any $y(\cdot) \in L^2_{\mathrm{loc}}(0,\infty;Y_1)$ with $\dot{y}(\cdot) \in L^2_{\mathrm{loc}}(0,\infty;Y_{-1})$ and any $\xi(\cdot) \in L^2_{\mathrm{loc}}(0,\infty;\Xi)$ with $\xi(t) \in \varphi(t, Cy(t))$ for a.e. $t \geq 0$, it follows that $F(y(t),\xi(t)) \geq 0$ for a.e. $t \geq 0$ and (for any such pair $\{y,\xi\}$) there exists a continuous functional $\Phi : W \to \mathbb{R}$ such that for any times $0 \leq s < t$ we have*

$$\int_s^t G(y(\tau),\xi(\tau))d\tau \geq \Phi(Cy(t)) - \Phi(Cy(s))\,.$$

b) The ***class of functionals*** *$\mathcal{M}(d)$ defined by a constant $d > 0$ consists of all maps $\psi : Y_1 \to \mathbb{R}_+$ such that for any $y \in L^2_{\mathrm{loc}}(0,\infty;Y_0)$ with $\dot{y} \in L^2_{\mathrm{loc}}(0,\infty;Y_1)$ the function $t \mapsto \psi(y(t))$ belongs to $L^1(0,\infty;\mathbb{R})$ satisfying $\int_0^\infty \psi(y(t))dt \leq d$ and for any $\varphi \in \mathcal{N}(F,G)$ and any $\psi \in \mathcal{M}(d)$ the Cauchy-problem (2.1) $-$ (2.3) has a solution $\{y(\cdot),\xi(\cdot)\}$ on any time interval $[0,T]$.*

3 Basic assumptions

(F1) The operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular* [Duvant and Lions, 1976; Likhtarnikov and Yakubovich, 1976], i.e., for any $T > 0, y_0 \in Y_1, \psi_T \in Y_1$ and $f \in L^2(0,T;Y_0)$ the solutions of the direct problem

$$\dot{y} = Ay + f(t)\,,\ y(0) = y_0\,, \quad \text{a.a. } t \in [0,T]$$

and of the dual problem

$$\dot{\psi} = -A^*\psi + f(t)\,,\ \psi(T) = \psi_T\,, \quad \text{a.a. } t \in [0,T]$$

are strongly continuous in t in the norm of Y_1. Here (and in the following) $A^* \in \mathcal{L}(Y_{-1}, Y_0)$ denotes the adjoint to A, i.e., $(Ay,\eta)_{-1,1} = (y, A^*\eta)_{-1,1}\,,\ \forall\, y,\eta \in Y_1$.

Remark 3.1 The assumption **(F1)** is satisfied [Likhtarnikov and Yakubovich, 1976] if the embedding $Y_1 \subset Y_0$ is completely continuous, i.e., transforms bounded sets from Y_1 into compact sets in Y_0. $\square$

(F2) The pair (A, B) is L^2-*controllable*, [Brusin, 1976; Likhtarnikov and Yakubovich, 1976] i.e., for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0, \infty; \Xi)$ such that the problem

$$\dot{y} = Ay + B\xi\,, \quad y(0) = y_0$$

is well-posed on the semiaxis $[0, +\infty)$, i.e., there exists a solution $y(\cdot) \in \mathcal{L}_\infty$ with $y\,(0) = y_0$.

It is easy to see that a pair (A, B) is L^2-controllable if this pair is *exponentially stabilizable*, i.e., if an operator $K \in \mathcal{L}(Y_0, \Xi)$ exists such that the solution $y(\cdot)$ of the Cauchy-problem $\dot{y} = (A + BK)\, y,\ y(0) = y_0$, decreases exponentially as $t \to \infty$, i.e.,

$$\exists\, c > 0 \quad \exists\, \varepsilon > 0 : \, \|y(t)\|_0 \le c\, e^{-\varepsilon t}\, \|y_0\|_0\,,\ \forall\, t \ge 0\,.$$

(F3) Let $F(y, \xi)$ be a Hermitian form on $Y_1 \times \Xi$, i.e.,

$$F(y, \xi) = (F_1 y, y)_{-1,1} + 2\,\mathrm{Re}\,(F_2 y, \xi)_\Xi + (F_3\, \xi, \xi)_\Xi,$$

where

$$F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1})\,,\ F_2 \in \mathcal{L}(Y_0, \Xi)\,,$$
$$F_3 = F_3^* \in \mathcal{L}(\Xi, \Xi)\,.$$

Define the *frequency-domain condition*

$$\alpha := \sup_{\omega, y, \xi} (\|y\|_1^2 + \|\xi\|_\Xi^2)^{-1} F(y, \xi)\,,$$

where the supremum is taken over all triples $(\omega, y, \xi) \in \mathbb{R}_+ \times Y_1 \times \Xi$ such that $i\omega y = Ay + B\xi$.

Remark 3.2 a) Let, in addition to the above assumption, A be the generator of a C_0-*group* on Y_0 and the pair $(A, -B)$ be L^2-controllable. Then the condition $\alpha \le 0$, where α is from **(F3)**, is sufficient for the application of a theorem by [Likhtarnikov and Yakubovich, 1976]. Note that the existence of C_0-groups is given for conservative wave equations, plate problems, and other important PDE classes [Flandoli, Lasiecka and Triggiani, 1988]. $\square$

4 Absolute observation - stability of evolutionary inequalities

We continue the investigation of energy like properties for the observation operators from the inequality problem (2.1), (2.2) with $f \equiv 0$.

The next definition generalizes the concepts which are introduced in [Likhtarnikov and Yakubovich, 1976] for output operators of evolution equations, namely in extending them to the observation operators of a class of evolutionary variational inequalities. In the following we denote for a function $z(\cdot) \in L^2\,(\mathbb{R}_+; Z)$ their norm by

$$\|z(\cdot)\|_{2,Z} \;:=\; \left(\int_0^\infty \|z(t)\|_Z^2\, dt \right)^{1/2} .$$

Definition 4.1. *a) The inequality (2.1), (2.2) is said to be* ***absolutely dichotomic*** *(i.e., in the classes* $\mathcal{N}(F, G), \mathcal{M}(d)$*)* ***with respect to the observation*** z *from (2.3) if for any solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2) with* $y(0) = y_0, \xi(0) = \xi_0 \in \mathcal{E}(y_0)$ *the following is true: Either* $y(\cdot)$ *is unbounded on* $[0, \infty)$ *in the* Y_0*-norm or* $y(\cdot)$ *is bounded in* Y_0 *in this norm and there exist constants* c_1 *and* c_2 *(which depend only on* $A, B, \mathcal{N}(F, G)$ *and* $\mathcal{M}(d)$*) such that*

$$\|Dy(\cdot) + E\xi(\cdot)\|_{2,Z}^2 \le c_1(\|y_0\|_0^2 + c_2)\,. \qquad (4.1)$$

b) The inequality (2.1), (2.2) is said to be ***absolutely stable with respect to the observation*** z *from (2.3) if (4.1) holds for any solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2).*

Definition 4.2. *The inequality (2.1)–(2.3) with* $f \equiv 0$ *is said to be* ***minimally stable*** *if the resulting equation for* $\psi \equiv 0$ *is minimally stable, i.e., there exists a bounded linear operator* $K : Y_1 \to \Xi$ *such that the operator* $A + BK$ *is stable, i.e. for some* $\varepsilon > 0$

$$\sigma(A + BK) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s \le -\varepsilon < 0\}$$
$$\textit{with} \qquad F(y, Ky) \ge 0\,, \qquad \forall\, y \in Y_1\,, \qquad (4.2)$$

$$\textit{and} \quad \int_s^t G(y(\tau)\,,\ Ky(\tau))d\tau \ge 0\,,$$
$$\forall\, s, t : 0 \le s < t\,, \quad \forall\, y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1)\,. \qquad (4.3)$$

With the superscript ^{c} we denote the complexification of spaces and operators and the extension of quadratic forms to Hermitian forms.

Theorem 4.1. *Consider the evolution problem* (2.1) $-$ (2.3) *with* $\varphi \in \mathcal{N}(F, G)$ *and* $\psi \in \mathcal{M}(d)$*. Suppose that for the operators* A^c, B^c *the assumptions* **(F1)** *and* **(F2)** *are satisfied. Suppose also that there exist an* $\alpha > 0$ *such that with the transfer operator*

$$\chi^{(z)}(s) = D^c(sI^c - A^c)^{-1}B^c + E^c \qquad (s \notin \sigma(A^c)) \qquad (4.4)$$

the frequency-domain condition

$$F^c\,((i\omega I^c - A^c)^{-1}B^c\xi, \xi)$$
$$+\, G^c\,((i\omega I^c - A^c)^{-1}B^c\xi, \xi) \le -\alpha \|\chi^{(z)}(i\omega)\xi\|_{Z^c}^2$$
$$\forall\, \omega \in \mathbb{R} : i\omega \notin \sigma(A^c)\,, \quad \forall\, \xi \in \Xi^c$$

is satisfied and the functional

$$J(y(\cdot),\xi(\cdot)) := \int_0^\infty [F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau)) + \alpha\|D^c y(\tau) + E^c\xi(\tau)\|^2_{Z^c}]\, d\tau$$

is bounded from above on any set

$$\mathfrak{M}_{y_0} := \{y(\cdot),\xi(\cdot) : \dot y = Ay + B\xi \text{ on } \mathbb{R}_+, \; y(0) = y_0\,,\, y(\cdot) \in \mathcal{L}_\infty\,,\, \xi(\cdot) \in L^2(0,\infty;\Xi)\}\,.$$

Suppose further that the inequality (2.1−(2.3) with $f \equiv 0$ is minimally stable, i.e., (4.2) and (4.3) are satisfied with some operator $K \in \mathcal{L}(Y_1,\Xi)$ and that the pair $(A+BK, D+EK)$ is observable in the sense of Kalman [Brusin, 1976], i.e., for any solution $y(\cdot)$ of

$$\dot y = (A+BK)y\,, \quad y(0) = y_0\,,$$

with $z(t) = (D+EK)y\,(t) = 0$ for a.a. $t \geq 0$ it follows that $y(0) = y_0 = 0$.
Then inequality (2.1), (2.2) is absolutely stable with respect to the observation z from (2.3).

Proof Under the assumptions of the given theorem there exist by [Likhtarnikov and Yakubovich, 1976] a (real) operator $P = P^* \in \mathcal{L}\,(Y_{-1},Y_0) \cap \mathcal{L}(Y_0,Y_1)$ and a number $\delta > 0$ such that the dissipation inequality is satisfied. Setting in this inequality $\xi = Ky$ from (4.2) with arbitrary $y \in Y_1$ we get with (4.3) the property

$$((A+BK)y\,,\, Py)_{-1,1} \leq -\delta\,\|Dy + EKy\,\|^2_Z\,, \quad \forall\, y \in Y_1\,. \tag{4.5}$$

Using the fact that $A+BK$ is a stable operator and the pair $(A+BK, D+EK)$ is observable, it follows [Brusin, 1976] from (4.5) that $P = P^* \geq 0$. Suppose now that $\{y(\cdot),\xi(\cdot)\}$ is an arbitrary solution of (2.1), (2.2) with $f \equiv 0$. With the Lyapunov-functional $V(y) = (y, Py)_0 \geq 0$ it follows from the dissipation inequality that for arbitrary $t \geq 0$

$$-V(y_0) - \Phi\,(Cy_0) + \int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]d\tau + \delta \int_0^t \|Dy(\tau) + E\,\xi(\tau)\|^2_Z\, d\tau \leq 0\,. \tag{4.6}$$

Since by assumption $\int_0^t [\psi(y(\tau) - \psi(-Py(\tau) + y(\tau))]d\tau \geq -c_2 > -\infty$ we get from (4.6) for arbitrary $t \geq 0$ the inequalities

$$\delta \int_0^t \|Dy(\tau) + E\,\xi(\tau)\|^2_Z\, d\tau \leq V(y_0) + \Phi(Cy_0) + c_2 \leq V(y_0) + c\,\|y_0\|^2_0 + c_2\,. \tag{4.7}$$

The properties (4.7) imply now the estimate (4.1) . ■

5 Application of observation stability to the beam equation

Consider the equation of a beam of length l, with damping and Hookean material, given as

$$\rho\,\mathbf{A}\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}\tilde g\left(\frac{\partial u}{\partial x}\right)\right) = 0\,, \tag{5.1}$$

$$u(0,t) = u(l,t) = 0 \quad \text{for} \quad t > 0\,, \tag{5.2}$$

$$u(x,0) = u_0(x)\,,\, u_t(x,0) = u_1(x) \quad \text{for} \quad x \in (0,l)\,. \tag{5.3}$$

Here u is the deformation in the x direction. Assume that the cross section area $\mathbf{A}$, the viscose damping γ, the mass density ρ and the generalized modulus of elasticity $\mathbf{E}$ are constant. The nonlinear stress-strain law $\tilde g$, is given by

$$\tilde g(w) = 1 + w - (1+w)^{-2}\,, \quad w \in (-1,1)\,. \tag{5.4}$$

Let us break the stress-strain law into the sum of a linear term and a nonlinear term as $\tilde g(w) = g(w) + w$. Then the above model (5.1) can be rewritten as

$$\rho\mathbf{A}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}\frac{\partial u}{\partial x}\right) + \gamma\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}g\left(\frac{\partial u}{\partial x}\right)\right) = 0\,. \tag{5.5}$$

Assume the Gelfand triple $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ with

$$\mathcal{V}_0 := L^2(0,l)\,, \quad \mathcal{V}_1 := H^1_0(0,l) \quad \text{and} \quad \mathcal{V}_{-1} := H^{-1}(0,l)\,. \tag{5.6}$$

Then equation (5.1) − (5.3) can be rewritten in $\mathcal{V}_{-1}$ as

$$\rho\mathbf{A}u_{tt} + \mathcal{A}_1 u + \mathcal{A}_2 u_t + \mathcal{C}^* g(\mathcal{C}u) = 0\,, \tag{5.7}$$

$$u(0) = u_0\,, \quad u_t(0) = u_1\,, \tag{5.8}$$

with $\mathcal{A}_1 \in \mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$, $\mathcal{A}_2 \in \mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$ (strong damping), $\mathcal{C} \in \mathcal{L}(\mathcal{V}_1,\mathcal{V}_0)$ and $g : \mathcal{V}_0 \to \mathcal{V}_0$. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are associated with their bilinear forms $a_i : \mathcal{V}_1 \times \mathcal{V}_1 \to \mathbb{R}$ $(i = 1,2)$ through $(\mathcal{A}_i v, w)_{\mathcal{V}_{-1},\mathcal{V}_1} = a_i(v,w)$, $\forall\, v, w \in \mathcal{V}_0$.

In order to get a variational interpretation of (5.7), (5.8) we make the following assumptions [Banks, Gilliam and Shubov, 1997; Banks and Ito, 1988] :

(A1)

a) The form a_1 is symmetric on $\mathcal{V}_0 \times \mathcal{V}$;

b) a_1 is $\mathcal{V}_1$ continuous, i.e., for some $c_1 > 0$ holds $|a_1(v,w)| \le c_1 \|v\|_{\mathcal{V}_1} \|w\|_{\mathcal{V}_1}$, $\quad \forall\, v, w \in \mathcal{V}_1$;

c) a_1 is strictly $\mathcal{V}_1$-elliptic, i.e., for some $k_1 > 0$ holds $a_1(v,v) \ge k_1 \|v\|^2_{\mathcal{V}_1}$, $\quad \forall\, v \in \mathcal{V}_1$.

(A2)

a) The form a_2 is $\mathcal{V}_1$ continuous, i.e., for some $c_2 > 0$ holds $|a_2(v,w)| \le c_2 \|v\|_{\mathcal{V}_1} \|w\|_{\mathcal{V}_1}$, $\forall\, v, w \in \mathcal{V}_1$.

b) The form a_2 is $\mathcal{V}_1$ coercive and symmetric, i.e., there are $k_2 > 0$ and $\lambda_0 \ge 0$ s.t.
$a_2(v,v) + \lambda_0 \|v\|^2_{\mathcal{V}_0} \ge k_2 \|v\|^2_{\mathcal{V}_1}$ and
$a_2(v,w) = a_2(w,v)$, $\quad \forall\, v, w \in \mathcal{V}_1$.

(A3)

a) The operator $\mathcal{C} \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ satisfies with some $k \ge 0$ the inequality

$$\|\mathcal{C} v\|_{\mathcal{V}_0} \le \sqrt{k} \|v\|_{\mathcal{V}_1} , \quad \forall\, v \in \mathcal{V}_1 .$$

$g : \mathcal{V}_0 \to \mathcal{V}_0$ is continuous and $\|g(v)\|_{\mathcal{V}_0} \le c_1 \|v\|_{\mathcal{V}_0} + c_2$ for $v \in \mathcal{V}_0$, where c_1 and c_2 are nonnegative constants.

b) g is of gradient type, i.e., there exists a coninuous Frechét-differentiable functional $G : \mathcal{V}_0 \to \mathbb{R}$, whose Frechét derivative $G'(v) \in \mathcal{L}(\mathcal{V}_0 , \mathbb{R})$ at any $v \in \mathcal{V}_0$ can be represented in the form

$$G'(v) w = (g(v), w)_{\mathcal{V}_0} , \quad \forall\, w \in \mathcal{V}_0 .$$

c) $g(0) = 0$ and for some positive $\varepsilon < 1$ we have for all $v, w \in \mathcal{V}_0$

$$\begin{aligned} &(g(v) - g(w), v - w)_{\mathcal{V}_0} \\ &\ge -\varepsilon k_1 k^{-1} \|v - w\|^2_{\mathcal{V}_0} . \end{aligned} \tag{5.9}$$

We say that $u \in \mathcal{L}_T$ is a *weak solution* of (5.7), (5.8) if

$$\begin{aligned} &(u_{tt}, \eta)_{\mathcal{V}_{-1},\mathcal{V}_1} + a_1(u,\eta) + a_2(u_t,\eta) \\ &+ (g(\mathcal{C}u), \mathcal{C}u)_0 = 0 \qquad \forall\, \eta \in \mathcal{L}_T ,\ \text{a.a.}\, t \in [0,T] . \end{aligned} \tag{5.10}$$

Let us formulate our problem (5.10) in first order form on the energetic space $Y_0 := \mathcal{V}_1 \times \mathcal{V}_0$ in the coordinates $y = (y_1, y_2) = (u, u_t)$. Define for this $Y_1 := \mathcal{V}_1 \times \mathcal{V}_1$ and $a : Y_1 \times Y_1 \to \mathbb{R}$ by

$$\begin{aligned} & a((v_1,v_2),(w_1,w_2)) \\ =\ & (v_2, w_1)_{\mathcal{V}_1} - a_1(v_1,w_2) - a_2(v_2,w_2) , \\ & \forall\, (v_1,v_2),(w_1,w_2) \in Y_1 \times Y_1 . \end{aligned} \tag{5.11}$$

The norms in the product spaces Y_0 and Y_1 are given in the standard way by

$$\|(y_1,y_2)\|_0^2 := \|y_1\|^2_{\mathcal{V}_1} + \|y_2\|^2_{\mathcal{V}_0} , \quad (y_1,y_2) \in Y_0 ,$$

and

$$\|(y_1,y_2)\|_1^2 := \|y_1\|^2_{\mathcal{V}_1} + \|y_2\|^2_{\mathcal{V}_1} , \quad (y_1,y_2) \in Y_1 .$$

Then (5.10) can be rewritten as

$$\begin{aligned} (\dot{y}, \eta)_{-1,1} - a(y,\eta) &= (B\varphi(Cy), \eta)_{-1,1} , \\ y(0) = (u_0, u_1) , &\quad \forall\, \eta \in Y_1 , \end{aligned} \tag{5.12}$$

where

$$B\varphi(Cy) := \begin{pmatrix} 0 \\ -\mathcal{C}^* g(\mathcal{C} y_1) \end{pmatrix} . \tag{5.13}$$

We can also write (5.12), (5.13) formally in the operator form

$$\dot{y} = Ay + B\varphi(Cy),\ y(0) = y_0 , \tag{5.14}$$

where A is defined by

$$a(v,w) = (Av, w)_{-1,1} , \quad \forall\, v, w \in Y_1 ,$$

i.e., $A = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1 & -\mathcal{A}_2 \end{bmatrix}$.

It is shown in [Banks, Gilliam and Shubov, 1997; Banks and Ito, 1988] that the embedding $Y_1 \subset Y_0$ is completely continuous and the operator A generates an analytic semigroup on Y_1, Y_0 and $Y_{-1} = \mathcal{V}_1 \times \mathcal{V}_{-1}$. Furthermore, its semigroup is exponentially stable on Y_1, Y_0 and Y_{-1} . From this it follows that the pair (A, B) is exponentially stabilizable. Let us consider with parameters $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ a more simplified form of (5.1) − (5.3) written as

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + 2\varepsilon \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} \\ &= -\alpha \left(\frac{\partial}{\partial x} \left(-g \left(\frac{\partial u}{\partial x} \right) \right) \right) =: \alpha \frac{\partial}{\partial x} \xi \end{aligned} \tag{5.15}$$

together with the boundary and initial conditions (5.2), (5.3), where we have $\xi = -g = \varphi$ introduced as new nonlinearity. According to (5.9) in **(A3)**a) we can assume that $\varphi \in \mathcal{N}(F)$ with the quadratic form $F(w,\xi) = \mu w^2 - \xi w$ on $\mathbb{R}\times\mathbb{R}$, where $\mu > 0$ is a certain parameter. Note that it is possible to include a second quadratic form G if we use the information from **(A3)**b).

Suppose that $\lambda_k > 0$ and e_k $(k = 1, 2, \ldots)$ are the eigenvalues resp. eigenfunctions of the operator $-\Delta$ with zero boundary conditions. We write formally the Fourier series of the solution $u(x,t)$ and the perturbation $\xi(x,t)$ to the (linear) equation (5.15) as

$$u(x,t) = \sum_{k=1}^{\infty} u^k(t) e_k \quad \text{and} \quad \xi(x,t) = \sum_{k=1}^{\infty} \xi^k(t) e_k\,. \tag{5.16}$$

If we introduce the Fourier transforms $\tilde{u}$ and $\tilde{\xi}$ of (5.16) with respect to the time variable we get from (5.15) for $k = 1, 2, \ldots$ the equations

$$-\omega^2 \tilde{u}^k(i\omega) + 2i\omega\varepsilon\tilde{u}^k(i\omega) + \lambda_k \tilde{u}^k(i\omega) = -\alpha\sqrt{\lambda_k}\,\tilde{\xi}^k(i\omega)\,. \tag{5.17}$$

It follows from (5.17) that for $k = 1, 2, \ldots$

$$\tilde{u}^k = \chi(i\omega,\lambda_k)\,\tilde{\xi}^k\,, \tag{5.18}$$

where

$$\chi(i\omega,\lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}\,(\alpha\lambda_k)\,,\quad \forall\,\omega\in\mathbb{R} : -\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k \neq 0\,. \tag{5.19}$$

In order to check the sufficient conditions for Theorem 4.1 we consider the functional

$$J(w,\xi) := \mathrm{Re}\int_0^\infty \int_0^l (\mu|w|^2 - w\xi^*)\,dx dt\,. \tag{5.20}$$

Using the Parseval equality for (5.20) with

$$|\tilde{w}|^2 = \sum_{k=1}^{\infty}\lambda_k|\tilde{u}^k|^2 = \sum_{k=1}^{\infty}\lambda_k|\tilde{u}^k|^2 = \sum_{k=1}^{\infty}\lambda_k|\chi(i\omega,\lambda_k)|^2|\tilde{\xi}^k|^2$$

and

$$\tilde{w}\,\tilde{\xi}^* = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\tilde{u}^k\,(\tilde{\xi}^k)^* = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\chi(i\omega,\lambda_k)|\tilde{\xi}^k|^2,$$

we conclude [Arov and Yakubovich, 1982] that the functional (5.20) is bounded from above if and only if the functional

$$\mathrm{Re}\int_{-\infty}^{+\infty}\int_0^l [\mu\left(\sum_{k=1}^{\infty}\lambda_k|\chi(i\omega,\lambda_k)|^2|\tilde{\xi}^k|^2 - \sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\chi(i\omega,\lambda_k)|\tilde{\xi}^k|^2\;\right] dx d\omega \tag{5.21}$$

is bounded on the subspace of Fourier-transforms defined by (5.18), (5.19) or, using again a result of [Arov and Yakubovich, 1982], that the frequency-domain condition

$$\mu\lambda_k|\chi(i\omega,\lambda_k)|^2 - \sqrt{\lambda_k}\,\mathrm{Re}\,\chi(i\omega,\lambda_k) < 0\,, \quad \forall\,\omega\in\mathbb{R} : -\omega^2 + 2\,i\omega\varepsilon + \alpha\lambda_k \neq 0\,,\ k = 1, 2, \ldots, \tag{5.22}$$

is satisfied, where $\chi(i\omega,\lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}(-\alpha\sqrt{\lambda_k})$. Clearly, (5.22) describes a certain domain Q in the space of parameters $\mu > 0, \varepsilon > 0, \alpha \in \mathbb{R}$. Theorem 4.1 shows that (5.14), associated with (5.15),(5.2),(5.3) is absolutely stable with respect to the observation $z = (y_1, y_2)$, if the parameter from Q also guarantee the minimal stability of (5.14).

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