A MODIFIED FREQUENCY-DOMAIN CRITERION FOR GRADIENT-LIKE BEHAVIOR OF DISCRETE PHASE SYSTEMS

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Abstract: Asymptotic behavior of a class of multidimensional discrete control systems with periodic nonlinearities and denumerable set of equilibria is investigated. By means of discrete version of Yakubovich-Kalman theorem and certain modification of Lur'e-Postnikov function a frequency-domain criterion which guarantees that every solution of a system tends to an equilibrium is obtained.

Keywords: Discrete phase systems, frequency-domain criteria, gradient-like behavior, Yakubovich-Kalman theorem, second Lyapunov method

1. INTRODUCTION

In this paper we go on with a number of publications devoted to asymptotic behavior of discrete phase control systems (Koryakin and Leonov, 1976; Leonov and Smirnova, 2000; Smirnova *et al.*, 2006; Leonov, 2006). All these published works have common object of investigation – discrete phase control systems, common goal of investigation – the asymptotic behavior of such systems and common methods of investigation. The latter are the second Lyapunov method and Yakubovich-Kalman frequency theorem in discrete case (Yakubovich, 1973). So all the results are formulated uniformly, in the form of frequency-domain criteria with a number of varying parameters.

Phase systems are the systems of specific character. They include periodic nonlinear functions of phase coordinates and have a denumerable set of equilibria (which may be both stable and unstable). That is why the Lyapunov function of Lur'e-Postnikov type which is traditional for control systems is of no use here.

In all the works (Koryakin and Leonov, 1976; Leonov and Smirnova, 2000; Smirnova *et al.*, 2006; Leonov, 2006) another Lyapunov function is used. It also has the form of "quadratic form plus integral of nonlinearity" just as the Lur'e-Postnikov function has. But in contrast to the Lur'e-Postnikov function the nonlinearity under integral sign does not coincide with the nonlinear function included into the phase system.

The nonlinearity, which has been used in published works (Koryakin and Leonov, 1976; Leonov and Smirnova, 2000; Smirnova *et al.*, 2006; Leonov, 2006), is constructed on the base of the periodic nonlinear function included into the system. It also is periodic but it has a zero mean value on the period. As a result the Lyapunov function has a periodic summand.

In publications (Koryakin and Leonov, 1976; Leonov and Smirnova, 2000; Smirnova *et al.*, 2006; Leonov, 2006) the nonlinear periodic function with a zero mean value on the period is constructed by means of a special procedure, borrowed from the paper (Bakaev and Guzh, 1965). Being united with Yakubovich-Kalman theorem this procedure leads to certain restrictions on the varying parameters of the frequency-domain inequality.

In this paper the periodic nonlinearity with zero mean value is borrowed from papers (Brockett, 1982; Leonov *et al.*, 1992). The restrictions on the varying parameters which appear on this track differ from those which appear in connection with Bakaev-Guzh procedure (Bakaev and Guzh, 1965).

2. DESCRIPTION OF THE PROBLEM. MAIN RESULT

Consider a discrete control system of the type

$$\begin{aligned} x(n+1) &= Ax(n) + b\xi(n), \\ \sigma(n+1) &= \sigma(n) + c^* x(n) - \rho\xi(n), \\ \xi(n) &= \varphi(\sigma(n)), \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{1}$$

Here A is a real $(m \times m)$- matrix, b and c are real m-vectors, ρ is a number. Values x and σ are respectively m-dimensional and scalar components of the state vector of the system, $\varphi(\sigma)$ is a scalar continuously differentiable Δ-periodic function. Symbol * means an Hermite conjugation.

Suppose that all eigenvalues of A are situated inside the unit circle, the pair (A, b) is controllable and the pair (A, c) is observable. Suppose also that function $\varphi(\sigma)$ has two simple zeros: $0 \leq \sigma_1 < \sigma_2 < \Delta$, and $\varphi\prime(\sigma_1) > 0, \varphi\prime(\sigma_2) < 0$. Suppose further that

$$\int_0^\Delta \varphi(\sigma)\, d\sigma < 0. \tag{2}$$

System (1) has a denumerable set of equilibria. It consists of $(m+1)$-vectors $\left\| \begin{matrix} 0 \\ \sigma_1 + \Delta k \end{matrix} \right\|, \left\| \begin{matrix} 0 \\ \sigma_2 + \Delta k \end{matrix} \right\|$ $(k = 0, \pm 1, \pm 2 \ldots)$. The goal of this paper is to establish the conditions which guarantee that every solution of (1) tends to a certain equilibrium as n tends to the infinity.

All the results of this paper are formulated in terms of transfer function of the linear part of system (1)

$$\chi(p) = c^*(A - pE_m)^{-1} b + \rho, \ (p \in C).$$

Let now α_1, α_2 be such numbers that

$$\alpha_1 \leqslant \frac{d\varphi}{d\sigma} \leqslant \alpha_2. \tag{3}$$

Notice that $\alpha_1 \alpha_2 < 0$.

Further in order to construct a Δ - periodic nonlinearity inside the Lyapunov function we shall need the constant

$$\nu = \frac{\int_0^\Delta \varphi(\sigma) d\sigma}{\int_0^\Delta |\varphi(\sigma)| \sqrt{(\alpha_2 - \varphi\prime(\sigma))(\varphi\prime(\sigma) - \alpha_1)} d\sigma}.$$

We shall also need the constants $k_1 = 2\alpha_1 - \alpha_2$ and $k_2 = 2\alpha_2 - \alpha_1$.

Theorem 1. Suppose that all eigenvalues of matrix A are situated inside the open unit circle, the pair (A, b) is controllable and the pair (A, c) is observable. Suppose that there exist such numbers $\varepsilon > 0, \eta > 0, æ \neq 0, \tau > 0$ that for all $p \in C, |p| = 1$ the inequality

$$\Re e \ \{æ\chi(p) - \varepsilon|\chi(p)|^2 - \eta +$$

$$\tau(k_1\chi(p) + (p-1))^*((p-1) + k_2\chi(p))\} \geq 0 \tag{4}$$

is valid and the following inequalities are true:

$$4\varepsilon > æ\alpha_0(2 + |\nu|\sqrt{2(\alpha_2 - \alpha_1)}), \tag{5}$$

where $\alpha_0 = \alpha_2$ if $æ \geq 0$ and $\alpha_0 = \alpha_1$ if $æ < 0$, and

$$4\tau\eta > æ^2\nu^2. \tag{6}$$

Then

$$1) \lim_{n \to \infty} \varphi(\sigma(n)) = 0, \tag{7}$$

$$2) \lim_{n \to \infty} (\sigma(n+1) - \sigma(n)) = 0, \tag{8}$$

$$3) \lim_{n \to \infty} \sigma(n) = \sigma_0, \tag{9}$$

where $\varphi(\sigma_0) = 0$.

3. PRELIMINARY CONSIDERATION

First of all let us extend the state space of (1) as it has been done in (Leonov and Smirnova, 2000). For the purpose we introduce the notations

$$y = \left\| \begin{matrix} x \\ \varphi(\sigma) \end{matrix} \right\|, \qquad P = \left\| \begin{matrix} A & b \\ 0 & 1 \end{matrix} \right\|,$$

$$L = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \qquad c_1^* = \left\| \begin{matrix} c^* & -\rho \end{matrix} \right\|,$$

$\xi_1(n) = \varphi(\sigma(n+1)) - \varphi(\sigma(n))$.

Then system (1) can be represented as

$$\begin{aligned} y(n+1) &= Py(n) + L\xi_1(n), \\ \sigma(n+1) &= \sigma(n) + c_1^* y(n), \qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{10}$$

Remark 1. (Leonov and Smirnova, 2000). If the pair (A, b) is controllable then the pair (P, L) is controllable too.

Remark 2. (Leonov and Smirnova, 2000). If $p \neq 1$ the following equalities are true:

$$c_1^*(P - pE)^{-1}L = \frac{1}{p-1}\chi(p),$$

$$L^*(P-pE)^{-1}L = -\frac{1}{p-1}.$$

Let us define quadratic forms of $(m+1)$ - vector y and a scalar ξ_1:

$$M(y,\xi_1) = (Py+L\xi_1)^*H(Py+L\xi_1) - y^*Hy + F_1(y,\xi_1),$$

$$F_1(y,\xi_1) = æy^*Lc_1^*y + \varepsilon y^*c_1c_1^*y + \eta y^*LL^*y$$
$$+\tau(k_1c_1^*y - \xi_1)^*(\xi_1 - k_2c_1^*y),$$

where $H = H^*$ is a $(m+1)\times(m+1)$ - matrix and $æ, \varepsilon, \eta, \tau$, are scalar parameters.

The following assertion is true.

Lemma (Smirnova *et al.*, 2006). Suppose the pair (A,b) is controllable and the pair (A,c) is observable and all eigenvalues of matrix A are located inside the unit circle. If there exist such numbers $\varepsilon > 0, \eta > 0, æ \neq 0, \tau > 0$ that the inequality (4) is true for all $p \in C, |p| = 1$ then there exists matrix $H = H^*$ such that $M(y,\xi_1) \leq 0$ for all $y \in R^{m+1}, \xi_1 \in R$.

Corollary. Suppose the hypotheses of Lemma are fulfilled. Consider then the sequence

$$W(n) = y^*(n)Hy(n) \ (n = 0, 1.2....)$$

where $y(n)$ satisfies system (10). The sequence $W(n)$ is bounded.

The assertion of the corollary follows immediately from the property of eigenvalues of matrix A and the boundedness of $\varphi(\sigma)$.

4. PROOF OF THE THEOREM 1

Let us introduce the functions

$$\varphi_1(\sigma) = \sqrt{(\alpha_2 - \varphi\prime(\sigma))(\varphi\prime(\sigma) - \alpha_1)},$$

$$\Phi(\sigma) = \varphi(\sigma) - \nu|\varphi(\sigma)|\varphi_1(\sigma).$$

Notice that $\Phi(\sigma)$ is a Δ - periodic function with zero mean value on $[0,\Delta)$, i.e.

$$\int_0^\Delta \Phi(\sigma)\,d\sigma = 0. \tag{11}$$

Let us define the sequence

$$V(n) = W(n) + æ\int_{\sigma(0)}^{\sigma(n)} \Phi(\sigma)d\sigma$$

and consider the difference

$$V(n+1) - V(n) = W(n+1) - W(n) + æ\int_{\sigma(n)}^{\sigma(n+1)} \Phi(\sigma)d\sigma.$$

First of all we shall carry out the estimation of the integral

$$\int_{\sigma(n)}^{\sigma(n+1)} \Phi(\sigma)d\sigma.$$

For the purpose we shall represent it in the form

$$\int_{\sigma(n)}^{\sigma(n+1)} \Phi(\sigma)d\sigma = \int_{\sigma(n)}^{\sigma(n+1)} \varphi(\sigma)d\sigma +$$
$$|\nu|\int_{\sigma(n)}^{\sigma(n+1)} |\varphi(\sigma)|\varphi_1(\sigma)d\sigma. \tag{12}$$

and apply to the second summand in the right part of (12) a mean value theorem. Then

$$\int_{\sigma(n)}^{\sigma(n+1)} |\varphi(\sigma)|\varphi_1(\sigma)d\sigma = \varphi_1(\sigma_n')\int_{\sigma(n)}^{\sigma(n+1)} |\varphi(\sigma)|d\sigma,$$

where $\sigma(n) \lesseqgtr \sigma_n' \lesseqgtr \sigma(n+1)$ and

$$\int_{\sigma(n)}^{\sigma(n+1)} \Phi(\sigma)d\sigma = \int_{\sigma(n)}^{\sigma(n+1)} (\varphi(\sigma) + |\nu|\varphi_1(\sigma_n')|\varphi(\sigma|))d\sigma.$$

Now we can use the estimates (5.4.11) (Leonov and Smirnova, 2000) destined just for the integral

$$\int_a^b (\varphi(\sigma) + \Theta|\varphi(\sigma)|)\,d\sigma (\Theta > 0).$$

In virtue of these estimates we have

$$(\varphi(\sigma(n)) + \Theta|\varphi(\sigma(n))|)(\sigma(n+1) - \sigma(n)) +$$
$$\frac{\alpha_1}{2}(1+\Theta)(\sigma(n+1) - \sigma(n))^2 \leq \int_{\sigma(n)}^{\sigma(n+1)} \Phi(\sigma)d\sigma \leq$$
$$(\varphi(\sigma(n)) + \Theta|\varphi(\sigma(n))|)(\sigma(n+1) - \sigma(n)) +$$
$$\frac{\alpha_2}{2}(1+\Theta)(\sigma(n+1) - \sigma(n))^2, \tag{13}$$

where $\Theta = |\nu|\varphi_1(\sigma_n')$. Consequently,

$$V(n+1) - V(n) \leq W(n+1) - W(n) + æ(\varphi(\sigma(n)) +$$
$$\Theta|\varphi(\sigma(n))|)[\sigma(n+1) - \sigma(n)] +$$
$$\frac{æ\alpha_0}{2}(1+\Theta)(\sigma(n+1) - \sigma(n))^2. \tag{14}$$

Let us define the function

$$Z(n) = W(n+1) - W(n) + æ\varphi(\sigma(n)[\sigma(n+1) - \sigma(n)]$$
$$+\varepsilon(\sigma(n+1) - \sigma(n))^2 + \eta\varphi^2(\sigma(n))$$
$$+\tau[k_1(\sigma(n+1) - \sigma(n)) - (\varphi(\sigma(n+1) - \varphi(\sigma(n))]^* \cdot$$
$$[\varphi(\sigma(n+1))) - \varphi(\sigma(n)) - k_2(\sigma(n+1) - \sigma(n)].$$

Then (14) can be written as follows

$$V(n+1) - V(n) \leq Z(n) + æ\Theta|\varphi(\sigma(n)|(\sigma(n+1) -$$
$$\sigma(n)) + (\frac{æ\alpha_0}{2}(1+\Theta) - \varepsilon)(\sigma(n+1) - \sigma(n))^2 - \eta\varphi^2$$
$$-\tau(k_1c_1^*y(n) - \xi_1(n))^*(\xi_1(n) - k_2c_1^*y(n)). \tag{15}$$

It follows from (5) and the form of $\varphi_1(\sigma(n)$ that

$$\frac{æ\alpha_0}{2}(1+\Theta) - \varepsilon < 0. \tag{16}$$

As soon as $y(n)$ is a solution of system (10) we have

$$W(n+1)-W(n) = (y^*(n)P+L^*\xi_1(n))H(P(y(n)) +L\xi_1(n)) - y^*(n)H(y(n)); \varphi(\sigma(n) = y^*L;$$
$$\sigma(n+1) - \sigma(n) = c_1^* y(n).$$

Hence

$$Z(n) = M(y(n), \xi_1(n)).$$

It follows from the hypotheses of the theorem and the lemma that

$$Z(n) \leq 0. \tag{17}$$

It is easy to see that

$$[k_1(\sigma(n+1) - \sigma(n)) - (\varphi(\sigma(n+1)) - \varphi(\sigma(n)))]\cdot$$
$$[(\varphi(\sigma(n+1)) - \varphi(\sigma(n))) - k_2(\sigma(n+1) - \sigma(n))] =$$
$$(\varphi'(\sigma_n^{'''})-k_2)(k_1-\varphi'(\sigma_n^{''}))(\sigma(n+1)-\sigma(n))^2, \tag{18}$$

where $\sigma(n) \lesseqgtr \sigma_n^{''} \lesseqgtr \sigma(n+1)$ and $\sigma(n) \lesseqgtr \sigma_n^{'''} \lesseqgtr \sigma(n+1)$.

Let us consider the right part of (18):

$$(\varphi'(\sigma_n^{''}) - k_1)(k_2 - \varphi'(\sigma_n^{'''})) = [(\varphi'(\sigma_n^{''}) - \alpha_1)+$$
$$(\varphi'(\sigma_n^{'}) - \alpha_1) + (\alpha_2 - \varphi'(\sigma_n^{'}))][(\alpha_2 - \varphi'(\sigma_n^{'''}))+$$
$$(\alpha_2 - \varphi'(\sigma_n^{'})) + (\varphi'(\sigma_n^{'}) - \alpha_1)] \geq$$
$$(\alpha_2 - \varphi'(\sigma_n^{'}))(\varphi'(\sigma_n^{'}) - \alpha_1) = \varphi_1^2(\sigma_n^{'}). \tag{19}$$

It follows from (15)-(19) that

$$V(n+1) - V(n) \leq -\tau\varphi_1^2(\sigma_n^{'})(\sigma(n+1) - \sigma(n))^2-$$
$$\eta\varphi^2(\sigma(n)) + æ|\nu|\varphi_1(\sigma_n^{'})(\sigma(n+1) - \sigma(n)). \tag{20}$$

The right part of (20) is a quadratic form with respect to the values $\varphi_1(\sigma_n^{'})(\sigma(n+1) - \sigma(n))$ and $\varphi(\sigma(n)$. In virtue of (6) we have that

$$V(n+1) - V(n) \leq -\delta|\varphi(\sigma(n)|^2(\delta > 0). \tag{21}$$

Since the sequence W(n) (n= 0,1,2, ...) is bounded and the function $\Phi(\sigma)$ has a zero mean value on the period the sequence V(n) (n= 0,1,2, ...) is bounded as well. Then it follows from (21) that the series

$$\sum_{n=1}^{\infty} |\varphi(n)|^2$$

converges. Hence

$$\lim_{n\to\infty} \varphi(\sigma(n)) = 0. \tag{22}$$

As all the eigenvalues of matrix A are situated inside the unit circle, the relation (22) implies

$$\lim_{n\to\infty} x(n)) = 0. \tag{23}$$

Then

$$\sigma(n+1) - \sigma(n)) \to 0 \ as \ n \to +\infty \tag{24}$$

and consequently (Leonov and Smirnova, 2000)

$$\sigma(n) \to \sigma_0 \ as \ n \to +\infty,$$

with $\varphi(\sigma_0) = 0$. Theorem is proved.

5. EXTENSION OF THE MAIN RESULT

The frequency-domain criterion of gradient-like behavior proved for system (1) can be extended easily for the system

$$\begin{aligned} x(n+1) &= Ax(n) + Bf(\sigma(n)), \\ \sigma(n+1) &= \sigma(n) + C^*x(n) - Rf(\sigma(n)), \\ n = 0, 1, 2, \ldots, & \end{aligned} \tag{25}$$

where B,C and R are real matrices of order $m \times l$, $m \times l$ and $l \times l$ respectively and $f(\sigma)$ is a vector-value function having the property $f(\sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2), ..., \varphi_l(\sigma_l))$ for $\sigma = (\sigma_l, ...\sigma_l)$. We assume that every component $\varphi_j(\sigma_j)$ is Δ_j - periodic, belongs to $\mathcal{C}^1$ and has two zeros on $[0, \Delta_j)$. Assume also that

$$\alpha_{1j} \leqslant \frac{d\varphi_j}{d\sigma} \leqslant \alpha_{2j}. \tag{26}$$

for all $\sigma \in \mathcal{R}$ where $\alpha_{1j} < 0 < \alpha_{2j} (j = 1, ...l)$ are certain numbers.

Let us determine for each $j = 1, ..., l$ the value

$$\nu_j = \frac{\int_0^\Delta \varphi_j(\sigma) d\sigma}{\int_0^\Delta |\varphi_j(\sigma)| \sqrt{(\alpha_{2j} - \varphi_j\prime(\sigma))(\varphi_j\prime(\sigma) - \alpha_{1j})} d\sigma}.$$

and define the $l \times l$ - matrix

$N = \text{diag}\{|\nu_1|, ..., |\nu_l|\}$.

Let $k_{1j} = 2\alpha_{1j} - \alpha_{2j}, k_{2j} = 2\alpha_{2j} - \alpha_{1j}$ and

$K_1 = \text{diag}\{k_{11}, ..., k_{1l}\}$, $K_2 = \text{diag}\{k_{21}, ..., k_{2l}\}$.

The transfer matrix of the linear part of (25) is as follows

$$K(p) = C^*(A - pE_m)^{-1}B + R, \ (p \in C).$$

Theorem 2. Suppose that all eigenvalues of matrix A lie inside the open unit circle, the pair (A, B) is controllable and the pair (A, C) is observable. Suppose that there exist diagonal $l \times l$ - matrices

$E = \text{diag}\{\varepsilon_1, ..., \varepsilon_l\} > 0$, $D = \text{diag}\{\eta_1, ..., \eta_l\} > 0$,

$T = \text{diag}\{\tau_1, ..., \tau_l\} > 0$, $J = \text{diag}\{æ_1, ..., æ_l\}$

such that the following properties hold:

1.
$$J\Re eK(p) - K(p)^*EK(p) - D+$$
$$\Re e[K_1K(p)+(p-1)E_l]^*T[(p-1)E_l+K_2K(p)] \geq 0 \tag{27}$$

for all complex p with $|p| = 1$;

2.
$$4E > JM_0(2E + NP),$$

where $M_0 = \text{diag}\{\alpha_{01}, ..., \alpha_{0l}\}$ with $\alpha_{0j} = \alpha_{1j}$ if $æ_j < 0$ and $\alpha_{0j} = \alpha_{2j}$ if $æ_j \geq 0$, and

$P = \mathrm{diag}\{\sqrt{\alpha_{21} - \alpha_{11}}, ..., \sqrt{\alpha_{2l} - \alpha_{1l}}\}$;

3.

$$4TD > (JN)^2.$$

Then

$$\lim_{n\to\infty} f(\sigma(n)) = 0,$$
$$\lim_{n\to\infty} x(n) = 0,$$
$$\lim_{n\to\infty} \sigma(n) = const.$$

6. CONCLUSIONS

The problem of asymptotic behavior of discrete phase systems with multiple equilibria is considered. By means of Lyapunov function with a periodic summand and the discrete version of Yakubovich-Kalman theorem a frequency-domain criterion for gradient-like behavior of discrete phase systems is obtained.

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