Seaching heteroclinic orbits in time-delayed fully-connected oscillators networks

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Abstract

Connecting orbits in oscillator networks have been well studied for the case of ideal transport delay between nodes and for weakly connected networks. This kind of behavior can emerge either as natural consequence of the network symmetry, appearing as symmetry-breaking bifurcations from equilibria; or in asymmetric networks as a global dynamics. The influence of the transport delay between oscillators in this kind of dynamics has been also studied for the case of scalar oscillators. In *Symmetric bifurcation analysis of synchronous states of time-delayed coupled Phase-Locked Loop oscillators*. Communications in Nonlinear Science and Numerical Simulation, Elsevier BV, 2014, (on-line version), we presented an analysis of bifurcations in time-delay fully-connected second-order PLL[1] networks focusing in bifurcations from the stable equilibria; in this work we continue the analysis, this time looking for heteroclinic orbits at the linearization for unstable equilibria. We begin to explore the emergency of conectiong orbits steady-states steady-states using numerical simulation.

1 Introduction

We briefly explore some connecting orbits emerging from unstable-to-stable equilibria in a fully-connected second-order PLL network. In a previous work [6] we discussed how the time-delay between nodes can lead to degenerate Hopf bifurcations; we use these previous results as a starting point to explore how periodic solutions and steady-states can be connected.

This work is divided as follows: In section 2, the full-phase model used in our analysis is presented and conditions for degenerate Hopf bifurcations are given in terms of free parameters, in section 3 the existence of connecting orbits between equilibria is shown based in numerical evidence; we comment on the influence of the time-delay in these orbits; finally, some comments and questions on the ongoing research are given in section 4.

[1]PLL: Phase Locked Loop.

2 The Full-phase model

In [6] the Full-phase model was used to find Hopf bifurcations in the (μ, τ)-parameter space, the model for a N-node fully connected network is:

$$
\begin{aligned}
&\ddot{\phi}_i(t) + \mu\dot{\phi}_i(t) - \mu\omega_M = \\
&\frac{K\mu}{N-1}\sum_{\substack{j\neq i\\ N\\ j=1}}^{N} [\sin(\phi_j(t-\tau) - \phi_i(t)) + \sin(\phi_j(t-\tau) + \phi_i(t))] \\
&+\epsilon I_n(t) + \eta w(t),
\end{aligned} \tag{1}
$$

here, following [1], we introduce an *impulsive input* $I_n(t)$ with unit magnitude, and $w_n(t)$ the uncorrelated white noise, to test the robustness of the eventual connecting orbits, ϵ and η represent the impulsive input amplitude and the noise strength respectively.

Equilibria in equation (1) are:

$$
\left.\begin{aligned}
\phi^+(n) &= \frac{1}{2}\left(\arcsin\left(-\frac{\omega_M}{K}\right) + 2n\pi\right) \\
\phi^-(n) &= \frac{1}{2}\left(\pi - \arcsin\left(-\frac{\omega_M}{K}\right) + 2n\pi\right)
\end{aligned}\right. , n \in \mathbb{Z}, \tag{2}
$$

For $N = 3$, the unperturbed system $(\epsilon, \eta = 0)$ has $\mathbf{S}_3 = \mathbf{D}_3$ symmetry, the *dihedral* group. Symmetry-breaking Hopf bifurcations can emerge in three families of periodic orbits (modulo symmetry), the spatio-temporal symmetry group $\mathcal{H} = \mathbb{Z}_3$, the spatial symmetry group $\mathcal{K} = \mathbb{Z}_2(\pi_{1,2})$, and the spatio-temporal symmetry group $\mathcal{H} = \mathbb{Z}_2(\pi_{1,2})$, for details see [6, 7]. For $K = 1.05$ and $\omega_M = 1$, equlibrium $\phi^+(n)$ is unstable and $\phi^-(n)$ starts stable at $\tau = 0$, for details see [2]. Curves for symmetry-breaking and symmetry-preserving bifurcations computed in the parameter-space (μ, τ), for the equilibrium $\phi^-(0)$, are shown in figure 1.

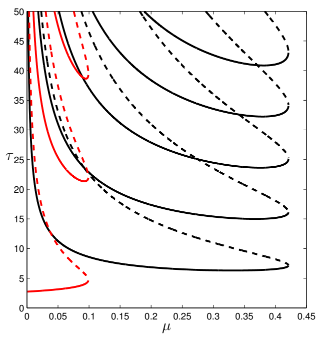
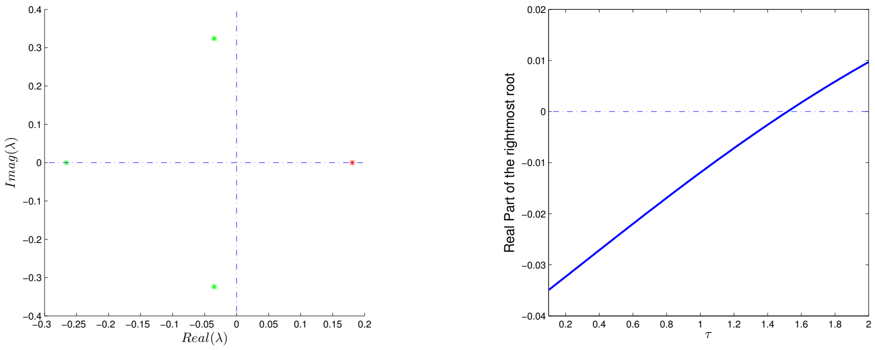


Figure 1: For the equilibrium $\phi^-(0)$, $K = 1.05$, $\omega_M = 1$. Curves of symmetry-breaking bifurcations are shown in red, and curves of symmetry-preserving bifurcations are shown in black; solid lines indicate bifurcations from the left to the right and dashed lines bifurcations from the right to the left. Equilibrium $\phi^+(n)$ is unstable [2].

Using DDE-Biftool [5, 4], we identify an unstable root for the equilibrium $\phi^+(0)$, at $\mu = 0.075$, $K = 1.05$ and $\tau = 0.1$, see figure 2a; by continuating this solution, the branch undergoes to unstable Hopf bifurcation, see figure 2b.



(a) Rightmost roots for $\mu = 0.075$, $K = 1.05$ and $\tau = 0.1$.

(b) Branch of continuation for equilibrium ϕ^+

Figure 2: (a) Unstable root for equilibrium $\phi^+(0)$, we identify the $1-D$ unstable manifold. (b) Branch of solutions going to Hopf Bifurcation.

3 Connecting orbits

A connecting orbit is a collection of solution trajectories connecting sequence of periodic solutions, or equilibria invariant sets via saddle-sink connections [8], for an example of a connecting orbit between $\phi^+(0)$ and $\phi^-(-2)$ see figure 3.

For the system

$$\dot{x}(t) = f(x(t), x(t-\tau), \eta), \tag{3}$$

where $\tau > 0$ is the time-delay, $\eta \in \mathbb{R}^p$ is the vector of parameters and $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, we say that $x(t)$, a solution of (3), is a connecting orbit if:

$$\lim_{t\to-\infty} x(t) = x_1, \qquad \lim_{t\to+\infty} x(t) = x_2. \tag{4}$$

For the same set of parameters used to compute the unstable root and the branch in figures 2a, 2b, and choosing a suitable time-delay, e observe that both branches of the unstable $1D$ manifold in system (1) at the equilibrium $\phi^+(0)$ are connected to the stable manifold for equilibria $\phi^-(n)$; hence, with a suitable perturbation, i.e., $\phi = \phi^+ \pm \delta v$, $(A_0(\eta) + A_\tau(\eta)\mathrm{e}^{-\lambda\tau})v = 0$, with $\delta << 1$, it is possible to reach some $\phi^-(n)$ from $\phi^+(0)$, λ is the only unstable eigenvalue at τ, see figure 2a, and $A_0(\eta) = \dfrac{\partial}{\partial x} f(x^*, x^*, \eta)$, $A_\tau(\eta) = \dfrac{\partial}{\partial x_\tau} f(x^*, x^*, \eta)$.

Simulations were made using sofware DDE-Biftools, and Matlab routines dde23 and ddesd, choosing τ lower than the corresponding to the degenerate Hopf bifurcation point in figure 2b, we observe that for asymptotically stable solutions, $\phi^+(0) \to \phi^-(n)$; in figure 4, three connecting orbits are shown for different values of time-delay ($\tau = \{0.1, 1.17, 1.16, 1.41\}$), the last one is unstable. It is clear that unstable manifold is affected by the time-delay.

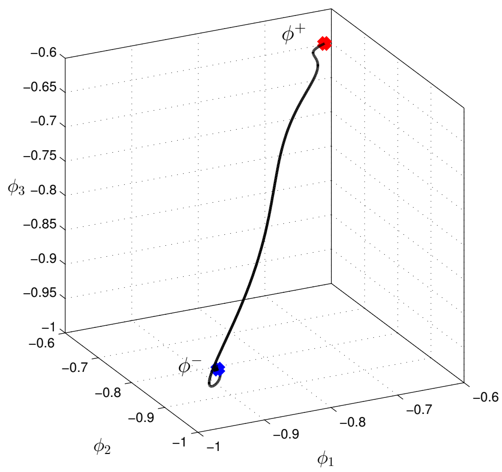
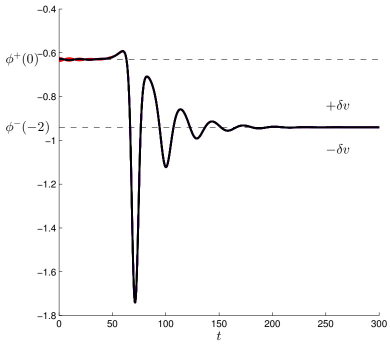


Figure 3: Connecting orbit form equilibrium $\phi^+(0)$ to $\phi^-(-2)$. For $K = 1.05$, $\omega_M = 1$, $\mu = 0.075$ and $\tau = 2.1509$.

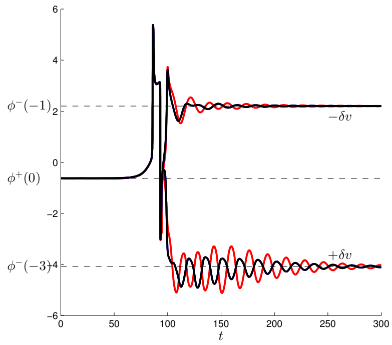
4 Comments and Questions

Finding a connecting orbit between steady state solutions is not a difficult task, provided that an $1D$ unstable manifold can be found. However, even when we know that connecting orbits between periodic solutions, or between periodic solutions and steady-states are also possible due to symmetry, locating such a heteroclinic cycles is a difficult task, a starting point could be the approach introduced in [3] adapted to DDEs. It is important to note that due to symmetry conditions we are dealing with degenerate bifurcations. Bearing this in mind, some questions for further research can be posed:

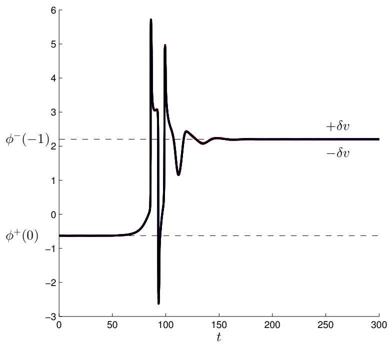
- How the $1D$ unstable manifold is influenced by the time-delay and the other free parameters?.
- Are heteroclinic orbits possible in system (1)?
- If yes. How those orbits can be found?
- If not. What symmetry-breaking condition is needed to make them possible?



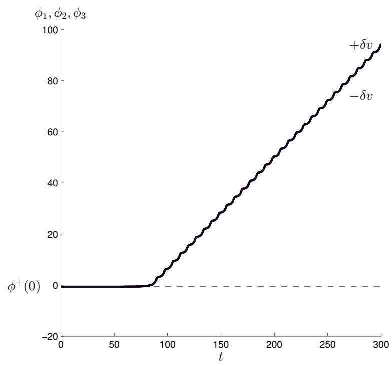
(a) $\tau = 0.1$



(b) $\tau = 1.17$



(c) $\tau = 1.16$



(d) $\tau = 1.41$

Figure 4: Connecting orbits $\phi^+(0) \to \phi^-(n)$, for $\mu = 0.075$ and $K = 1.05$, for different values of τ.

References

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