LYAPUNOV TYPE FUNCTIONS FOR NONLINEAR IMPULSIVE CONTROL SYSTEMS: MONOTONICITY CONDITIONS AND APPLICATIONS

Olga Samsonyuk
Institute for System Dynamics and Control Theory of
Siberian Branch of the Russian Academy of Sciences
Russia
samsonuk.olga@rambler.ru

Abstract

This paper is devoted to the study of Lyapunov type functions relative to an impulsive control system with trajectories of bounded variation and impulsive controls (regular vector measures). We focus on the definitions and infinitesimal properties of strongly and weakly monotone Lyapunov type functions. As an application of the Lyapunov type functions, we consider estimates for integral funnels of trajectories of impulsive systems.

Key words

Measure–driven control system, nonsmooth Lyapunov functions.

1 Introduction

Strongly and weakly monotone Lyapunov type functions play an important role in the study of various qualitative properties of control dynamical systems such as stability, stabilization, invariance, attainability, and optimality (see, e.g., [Aubin and Cellina, 1984; Aubin and Frankowska, 1990; Bardi and Capuzzo-Dolcetta, 1997; Bacciotti and Rosier, 2005; Clarke, Ledyaev, Stern, and Wolenski, 1998; Guseinov and Ushakov, 1990; Krotov, 1996; Milyutin and Osmolovskii, 1998; Subbotin, 1994; Vinter, 2000]). Roughly speaking, the property of strong monotonicity is the property of being monotone along all solutions (trajectories) of the control system whereas the property of weak monotonicity means this along at least one trajectory of the system that starts from an arbitrary initial position. Such Lyapunov type functions are smooth or generalized solutions of the corresponding Hamilton–Jacobi inequalities.

As for optimal control theory, a more important role is played by strongly monotone Lyapunov type functions, which allow one to estimate from below the lower bound of the cost functional. The approaches of Caratheodory and Krotov [Krotov, 1996; Clarke, Ledyaev, Stern, and Wolenski, 1998; Vinter, 2000] as well as their modern modifications [Milyutin and Osmolovskii, 1998; Dykhta, 2004] are close to this direction, in which one can also include Bellman's dynamic programming method.

The applications of the weakly monotone functions are much less known in optimal control. Under a certain choice of the boundary conditions, these functions allow one to obtain a quasioptimal control synthesis, upper estimates for the lower bound of the cost functional, methods for improving nonoptimal control, and necessary optimality conditions (see, e.g., [Subbotin, 1994; Clarke, Ledyaev, Stern, and Wolenski, 1998; Bardi and Capuzzo-Dolcetta, 1997]).

The majority of the above-mentioned results deal with ordinary differential control systems with a compact control value set or compact-valued differential inclusions. The extension of these results to systems with unbounded velocity sets faces some serious difficulties, which generally require modifications of the Hamilton–Jacobi differential operators. A typical example of such systems is given by systems linear in control with a convex closed cone as the Pontryagin set. These systems often arise in applications to laser physics, robotics, economics, ecology etc. A natural relaxation extension of the relevant optimal control problems leads to problems of optimizing impulsive processes with trajectories of bounded variation [Dykhta and Samsonyuk, 2003; Miller and Rubinovich, 2003; Motta and Rampazzo, 1995; Sesekin, 1997]. The dynamics of such systems can be formally described by the measure driven equation

$$\left.\begin{array}{l} dx(t) = f\big(t, x(t), u(t)\big)dt + G\big(t, x(t)\big)\mu(dt), \\ u(t) \in U \ \text{ a.e. on } T, \quad \mu(B) \in K \ \ \forall\, B \in \mathcal{B}_T. \end{array}\right\} \quad (1)$$

Here, $T = [a, b]$ is a fixed time interval, U is a compact set in $\mathbb{R}^l$, K is a convex closed cone in $\mathbb{R}^m$, $x(\cdot)$ is a function of bounded variation, $u(\cdot)$ is an $\mathcal{L}$-measurable function, μ is a K-valued bounded Borel measure on T, $\mathcal{B}_T$ is the algebra of Borel subsets of the interval T, and "a.e." signifies "almost everywhere with respect to the Lebesgue measure, $\mathcal{L}$, on $\mathbb{R}$".

This paper deals with the impulsive control system (1). Since we do not assume any commutativity property of the vector fields generated by the columns of G, system (1) may have a nonunique solution even when u, μ, and an initial condition x_0 are fixed. The solution concept we adopt with some modifications was introduced in the papers [Miller, 1982; Miller, 1996], and [Miller and Rubinovich, 2003]. This concept is presented in detail in the next section.

It is well known that, by using a time transformation, the impulsive control system can be converted into an auxiliary conventional control system [Bressan and Rampazzo, 1988; Motta and Rampazzo, 1995; Miller, 1996; Miller and Rubinovich, 2003; Sesekin, 1997; Pereira and Silva, 2000; Silva and Vinter, 1996; Wolenski and Zabic, 2007]. Thus the definitions and monotonicity conditions for Lyapunov type functions can be obtained through the corresponding auxiliary control systems. Let us refer to the paper [Pereira and Silva, 2002], in which this way was realized for autonomous measure-driven differential inclusions; see also [Pereira, Silva, and Oliveira, 2008] and [Code and Silva, 2010], where such way was applied to the invariance and stabilization problems for impulsive control systems, respectively. However, the impulsive dynamics exists in both a slow time, t, and a fast time, denoted as V below, in which jumps of trajectories are realized (namely, the sum $t+V$ is considered as a time for the auxiliary control system). Let us stress that the nonlinear impulsive control systems, except special cases, for example, when $K \subseteq \mathbb{R}^m_+$ or the matrix G satisfies the so-called well-posedness condition of Frobenius type etc, are nonautonomous (in a certain sense) with respect to the fast time. Thus it can be taken into account for the definitions of monotonicity relative to impulsive systems.

In this paper we formulate some definitions of strongly and weakly monotone Lyapunov type functions as well as their proximal characterization and give their applications to estimates of the reachable sets and integral funnels of trajectories of impulsive systems.

This paper is organized as follows. The precise statement of the impulsive control system and the concept of its solutions are given in Section 2. In Section 3, the definitions of strongly and weakly monotone Lyapunov type functions are introduced and infinitesimal conditions in a form of proximal inequalities are obtained. Section 4 is devoted to an application of certain families of Lyapunov type functions to inner and outer estimates of the reachable sets and integral funnels of trajectories.

2 Statement of the impulsive control system

Let K_1 be the set $\{v \in K \mid ||v|| = 1\}$, where $||\cdot||$ is the norm $||v|| = \sum_{j=1}^{m} |v_j|$.

Suppose that the following assumptions hold.

(H1) The functions $f(t,x,u)$, $G(t,x)$ are continuous; for any compact set $Q \subset \mathbb{R}^n$ there exist constants L_1, $L_2 > 0$ such that

$$|f(t,x_1,u) - f(t,x_2,u)| \le L_1|x_1 - x_2|,$$
$$|G(t,x_1) - G(t,x_2)| \le L_2|x_1 - x_2|$$

whenever $(t,x_1,u), (t,x_2,u) \in T \times Q \times U$; moreover, there exist constants $c_1,\ c_2 > 0$ such that

$$|f(t,x,u)| \le c_1(1+|x|), \quad |G(t,x)| \le c_2(1+|x|)$$

whenever $(t,x,u) \in T \times \mathbb{R}^n \times U$. Here, $|\cdot|$ denotes a vector or consistent matrix norm.

(H2) The set $f(t,x,U)$ is a convex set $\forall\ (t,x) \in T \times \mathbb{R}^n$.

In what follows $BV^+\left(T,\mathbb{R}^k\right)$ means the Banach space of $\mathbb{R}^k$-valued functions on T of bounded variation and which are right continuous on $(a,b]$; $\mathcal{U}(T,U)$ is the set of $\mathcal{L}$-measurable functions from T into $\mathbb{R}^l$ with values in U.

Let μ be a K-valued bounded Borel measure on T. Given μ, let $S_d(\mu)$ be the set $\{s \in T \mid \mu(\{s\}) \neq 0\}$ and μ_c be the continuous component in the Lebesgue decomposition of μ. Let $\gamma(\mu)$ denote any family $\{d_s, \omega_s(\cdot)\}_{s\in S}$ whose components satisfy the following conditions:

(a) $S \subset T$ is a set which is at most denumerable and such that $S_d(\mu) \subseteq S$;

(b) for each $s \in S$, the constant $d_s \in \mathbb{R}_+$ and $\mathcal{L}$-measurable function $\omega_s : [0,d_s] \to co\ K_1$ satisfy the conditions

$$d_s \ge ||\mu\big(\{s\}\big)||, \qquad \int_0^{d_s} \omega_s(\tau)d\tau = \mu\big(\{s\}\big);$$

(c) $\sum_{s\in S} d_s < \infty$.

Here, $co\,A$ is the convex hull of a set A. It is clear that there exists a nonunique family $\gamma(\mu)$ corresponding to μ. We denote by $\pi(\mu)$ any pair $\big(\mu,\gamma(\mu)\big)$ in which μ is a K-valued bounded Borel measure on T and $\gamma(\mu)$ is some corresponding family described above. Let $\mathcal{W}(T,K)$ be the set of all such $\pi(\mu)$. Now for each $\pi(\mu) \in \mathcal{W}(T,K)$ we define the function $V(\cdot) = V_{\pi(\mu)}(\cdot) \in BV^+(T,\mathbb{R})$ by the rule: $V(a) = 0$,

$$V(t) = |\mu_c|\big([a,t]\big) + \sum_{s\le t,\ s\in S} d_s, \quad t \in (a,b], \tag{2}$$

where $|\mu_c|$ is a total variation of the measure μ_c.

Consider an impulsive control system of the form

$$\begin{aligned} dx(t) &= f\big(t, x(t), u(t)\big)dt + G\big(t, x(t)\big)\pi(\mu), \\ u(\cdot) &\in \mathcal{U}(T, U), \quad \pi(\mu) \in \mathcal{W}(T, K), \end{aligned} \qquad (\mathcal{D}),$$

where elements of the sets $\mathcal{U}(T,U)$ and $\mathcal{W}(T,K)$ are ordinary and impulsive controls, respectively. The solutions of $(\mathcal{D})$ are understood in the sense of the following definition.

Definition 1. A function $x(\cdot) \in BV^+(T, \mathbb{R}^n)$ is said to be the *solution* of $(\mathcal{D})$ corresponding to a pair $\big(u(\cdot), \pi(\mu)\big) \in \mathcal{U}(T,U) \times \mathcal{W}(T,K)$ if the equality

$$x(t) = x(a) + \sum_{s \le t,\ s \in S} \Big(z_s\big(d_s; x(s-)\big) - x(s-)\Big) + \int_a^t f\big(t, x(t), u(t)\big)dt + \int_a^t G\big(t, x(t)\big)\mu_c(dt) \qquad (3)$$

is fulfilled for all $t \in (a, b]$, where, for each $s \in S$, the function $z_s(\cdot; x)$ is an absolutely continuous function satisfying the system of differential equations

$$\frac{dz_s(\tau)}{d\tau} = G\big(s, z_s(\tau)\big)\omega_s(\tau),\ z_s(0) = x,\ \ \tau \in [0, d_s].$$

Let us briefly comment on system $(\mathcal{D})$. This system is a certain extension of a conventional kind system governed by the standard control dynamics

$$\left.\begin{aligned} &\dot{x}(t) = f\big(t, x(t), u(t)\big) + G\big(t, x(t)\big)v(t), \\ &u(t) \in U, \quad v(t) \in K \ \text{ a.e. } t \in T, \end{aligned}\right\} \qquad (4)$$

where $x(\cdot)$ is an absolutely continuous function, $u(\cdot)$ and $v(\cdot)$ are $\mathcal{L}$-measurable bounded functions. Indeed, the set of $(\mathcal{D})$'s solutions is obtained by closing the set of trajectories of (4) in the weak* topology in the space of functions of bounded variation.

A triple $\big(x(\cdot), u(\cdot), \pi(\mu)\big)$ consisting of ordinary and impulsive controls together with the corresponding trajectory is called an *impulsive process* and denoted by σ. The function $V_{\pi(\mu)}(\cdot)$ corresponding to σ and given by (2) will be also denoted as $V_\sigma(\cdot)$ or even $V(\cdot)$ if σ is easy for the context. We shall use the similar notation for σ's trajectory, that is, $x_\sigma(\cdot)$ or $x(\cdot)$. Since, for given σ, the function $V(\cdot)$ has jumps only at points of the set S and $d_s = [V(s)] := V(s) - V(s-)\ \forall\, s \in S$, we can identify the set S with $S_d(V) := \{s \in T \mid [V(s)] > 0\}$.

Let us note that the variable V, which appears in the impulsive system via (2) and (3), may be also interpreted as an energy variable. Indeed, for given σ there exists a sequence $\{x_k(\cdot), u_k(\cdot), v_k(\cdot)\}$ satisfying (4) such that the sequence $\big\{x_k(\cdot), V_k(\cdot)\big\}$, where $t \to V_k(t) := \displaystyle\int_a^t ||v_k(\tau)||d\tau$, tends to $\big\{x_\sigma(\cdot), V_\sigma(\cdot)\big\}$ as $k \to \infty$ in the sense of weak* convergence in the space of functions of bounded variation (then $\big(x_k(t), V_k(t)\big) \to \big(x_\sigma(t), V_\sigma(t)\big)$ at all the points of continuity $\big(x_\sigma(\cdot), V_\sigma(\cdot)\big)$ and at the points a, b).

Finally we note that any pair $\big(x(\cdot), V(\cdot)\big)$ corresponding to an impulsive process σ is a generalized solution of (4) in the sense of [Miller, 1996; Miller and Rubinovich, 2003], moreover, it is a V-solution of (4) in the sense of [Sesekin, 1997].

For given σ and $\Delta = [t_0, t_1] \subseteq T$, we denote by $\varkappa_{\sigma\Delta}$ the set $\big(x(\cdot), V(\cdot), \{z_s(\cdot)\}_{s \in S \cap \Delta}\big)$, where, for $s \in S \cap \Delta$, $z_s(\cdot) := z_s(\cdot; x(s-))$ corresponds to $\pi(\mu)$, the functions $x(\cdot)$ and $V(\cdot)$ are the restrictions of $x_\sigma(\cdot)$ and $V_\sigma(\cdot)$ to $(t_0, t_1]$, moreover, $x(t_0) = x(t_0-) = x_\sigma(t_0-)$, $V(t_0) = V(t_0-) = V_\sigma(t_0-)$. We shall say that $\varkappa_{\sigma\Delta}$ is an *extended trajectory* of $(\mathcal{D})$ defined on Δ.

The following notation is used:

Σ is the set of all impulsive processes of $(\mathcal{D})$;

$$\mathcal{TE}^+(t_\alpha, x_\alpha, V_\alpha) = \left\{ \varkappa_{\sigma\Delta} \,\middle|\, \begin{array}{l} \sigma \in \Sigma,\ \Delta = [t_\alpha, b], \\ x(t_\alpha-) = x_\alpha, \\ V(t_\alpha-) = V_\alpha \end{array} \right\};$$

$$\mathcal{TE}^-(t_\alpha, x_\alpha, V_\alpha) = \left\{ \varkappa_{\sigma\Delta} \,\middle|\, \begin{array}{l} \sigma \in \Sigma,\ \Delta = [a, t_\alpha], \\ x(t_\alpha) = x_\alpha, \\ V(t_\alpha) = V_\alpha \end{array} \right\}.$$

3 Strongly and weakly monotone Lyapunov type functions

In this section we define some monotonicity properties of a continuous function $\varphi(t, x, V)$ with respect to system $(\mathcal{D})$. Note that the set of conventional variables t and x of Lyapunov type functions is now supplemented with the variable V, which is responsible for the impulsive dynamics of the system and combines the properties of both the time variable and the state variable.

Definition 2. A function φ is *strongly increasing* if, for any $(t_\alpha, x_\alpha, V_\alpha) \in T \times \mathbb{R}^n \times \mathbb{R}_+$, for any $\big(x(\cdot), V(\cdot), \{z_s(\cdot)\}\big)$ from $\mathcal{TE}^+(t_\alpha, x_\alpha, V_\alpha)$, the functions $t \to \varphi\big(t, x(t), V(t)\big)$ and $\tau \to \varphi\big(s, z_s(\tau), V(s-) + \tau\big)$, $s \in S_d(V) \cap [t_\alpha, b]$ do not decrease on the intervals $[t_\alpha, b]$, $\big[0, [V(s)]\big]$, $s \in S_d(V) \cap [t_\alpha, b]$, respectively.

Definition 3. A function φ is *weakly predecreasing* if, for any $(t_\alpha, x_\alpha, V_\alpha) \in T \times \mathbb{R}^n \times \mathbb{R}_+$, there exists an extended trajectory $\big(x(\cdot), V(\cdot), \{z_s(\cdot)\}\big)$ from $\mathcal{TE}^-(t_\alpha, x_\alpha, V_\alpha)$ such that the functions $t \to \varphi\big(t, x(t), V(t)\big)$ and $\tau \to \varphi\big(s, z_s(\tau), V(s-) + \tau\big)$, $s \in S_d(V) \cap [a, t_\alpha]$ do not increase on the intervals $[a, t_\alpha]$, $\big[0, [V(s)]\big]$, $s \in S_d(V) \cap [a, t_\alpha]$, respectively.

Strongly decreasing and weakly preincreasing functions φ are defined in the similar manner.

Now we define some property of φ that, in a certain sense, is wider than the weakly predecreasing property.

Denote by Q_φ the set $\{(t, x, V) \in T \times \mathbb{R}^n \times \mathbb{R}_+ \mid \varphi(t, x, V) \ge 0\}$.

Definition 4. A function φ is *weakly preinvariant* if the set Q_φ is a weakly preinvariant set relative to $(\mathcal{D})$; i.e., for any $(t_\alpha, x_\alpha, V_\alpha) \in Q_\varphi$, there exists an extended trajectory $\big(x(\cdot), V(\cdot), \{z_s(\cdot)\}\big)$ from $\mathcal{TE}^-(t_\alpha, x_\alpha, V_\alpha)$ such that the following inclusions hold

$$\big(t, x(t), V(t)\big) \in Q_\varphi \qquad \forall t \in [a, t_\alpha],$$
$$\big(s, z_s(\tau), V(s-) + \tau\big) \in Q_\varphi$$
$$\forall \tau \in \big[0, [V(s)]\big], \ \forall s \in S_d(V) \cap [a, t_\alpha].$$

We shall say that the above-defined functions are Lyapunov type functions of the impulsive control system $(\mathcal{D})$. We denote by Φ_s, Φ_w^-, and Φ_{wi}^- the sets of all continuous strongly increasing, weakly predecreasing, and weakly preinvariant Lyapunov type functions of $(\mathcal{D})$, respectively.

Now we formulate infinitesimal conditions of strong and weak monotonicity of Lyapunov type functions.

Let us define the functions h_0, h_1, $\mathcal{H}_0$, and $\mathcal{H}_1$ to be

$$h_0(t, x, \psi) = \min_{u \in U} \langle \psi, f(t, x, u) \rangle,$$
$$h_1(t, x, \psi) = \min_{\omega \in K_1} \langle \psi, G(t, x)\omega \rangle,$$
$$\mathcal{H}_0(t, x, \psi) = \max_{u \in U} \langle \psi, f(t, x, u) \rangle,$$
$$\mathcal{H}_1(t, x, \psi) = \max_{\omega \in K_1} \langle \psi, G(t, x)\omega \rangle.$$

Moreover, we define the functions $\overline{h}_0$, $\overline{h}_1$, $\overline{\mathcal{H}}_0$, and $\overline{\mathcal{H}}_1$ as follows:

$$\overline{h}_i(t, x, \psi_1, \psi_2) = \psi_1 + h_i(t, x, \psi_2), \quad i = 0, 1,$$
$$\overline{\mathcal{H}}_i(t, x, \psi_1, \psi_2) = \psi_1 + \mathcal{H}_i(t, x, \psi_2), \quad i = 0, 1.$$

We denote by $\partial_P \varphi(t, x, V)$ and $\partial^P \varphi(t, x, V)$ the proximal subdifferential and superdifferential of the function φ at the point (t, x, V). Let us recall [Clarke, Ledyaev, Stern, and Wolenski, 1998; Vinter, 2000] that a vector p is called a proximal subgradient of $y \to \varphi(y)$ at a point y if there exist a neighborhood Ω of the point y and a constant $c > 0$ such that

$$\varphi(z) \geq \varphi(y) + \langle p, z - y \rangle - c|z - y|^2 \quad \forall z \in \Omega.$$

This inequality implies that locally (in a neighborhood of y) φ has a quadratic lower support function at the point y with gradient p at this point. The proximal subdifferential $\partial_P \varphi(y)$ consists of all such subgradients. It may be an empty set; in this case, the respective proximal inequalities given below are assumed to hold automatically at the point y. Note that $\partial_P \varphi(y) \subset \{\nabla \varphi(y)\}$ for a differentiable φ; moreover, if φ is twice continuously differentiable at the point y, then this inclusion turns into an equality. The proximal superdifferential $\partial^P \varphi$ is introduced in an antisymmetric way and is formally defined by the equality $\partial^P \varphi(y) = -\partial_P\big(-\varphi(y)\big)$.

Denote by $N_Q^P(y)$ the proximal normal cone to Q at y. Recall that a vector $p \in \mathbb{R}^k$ is a *proximal normal vector* to a closed set $Q \subset \mathbb{R}^k$ at $y \in Q$ iff there exists $\alpha > 0$ such that $d_Q(y + \alpha p) = \alpha||p||$, where $d_Q(\cdot)$ is the distance function defined by $d_Q(z) = \inf_{y \in Q} ||z - y||$. The *proximal normal cone* to Q at y is the set of all proximal normal vectors to Q at y.

We shall use the following notation: $\varphi^t(\cdot, \cdot) = \varphi(t, \cdot, \cdot)$, $\varphi^{V_0}(\cdot, \cdot) = \varphi(\cdot, \cdot, 0)$,

$$Q_{(a,b)} = Q \cap \big((a, b) \times \mathbb{R}^n \times (0, +\infty)\big),$$
$$Q_t = \{(x, V) \in \mathbb{R}^n \times (0, +\infty) \mid (t, x, V) \in Q\},$$
$$Q_{V_0} = \{(t, x) \in (a, b) \times \mathbb{R}^n \mid (t, x, 0) \in Q\}$$

whenever Q is a closed set in $T \times \mathbb{R}^n \times \mathbb{R}_+$; moreover, denote by $\overline{Q}_{(a,b)}$, $\overline{Q}_t$, and $\overline{Q}_{V_0}$ the closures of the corresponding sets.

Let us consider the following conditions in the form of systems of proximal Hamilton–Jacobi inequalities with respect to continuous functions $\varphi : T \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$.

Condition (A):

$$\overline{h}_0(t, x, p_t, p_x) \geq 0, \qquad \overline{h}_1(t, x, p_V, p_x) \geq 0$$
$$\forall\, (p_t, p_x, p_V) \in \partial_P \varphi(t, x, V),$$
$$\forall\, (t, x, V) \in (a, b) \times \mathbb{R}^n \times (0, +\infty);$$
$$\overline{h}_1(s, x, p_V, p_x) \geq 0 \quad \forall\, (p_x, p_V) \in \partial_P \varphi^t(x, V),$$
$$\forall\, (t, x, V) \in \{a; b\} \times \mathbb{R}^n \times (0, +\infty).$$

Condition (B):

$$\min_{\substack{\omega_0, \omega_1 \geq 0 \\ \omega_0 + \omega_1 = 1}} \{\overline{h}_0(t, x, p_t, p_x)\omega_0 + \overline{h}_1(t, x, p_V, p_x)\omega_1\} \leq 0$$
$$\forall\, (p_t, p_x, p_V) \in \partial^P \varphi(t, x, V),$$
$$\forall\, (t, x, V) \in (a, b) \times \mathbb{R}^n \times (0, +\infty);$$
$$\overline{h}_0(t, x, p_t, p_x) \leq 0 \quad \forall\, (p_t, p_x) \in \partial^P \varphi^{V_0}(t, x),$$
$$\forall\, (t, x) \in (a, b) \times \mathbb{R}^n;$$
$$\overline{h}_1(t, x, p_V, p_x) \leq 0 \quad \forall\, (p_x, p_V) \in \partial^P \varphi^t(x, V),$$
$$\forall\, (t, x, V) \in \{a; b\} \times \mathbb{R}^n \times (0, +\infty).$$

Condition (C):

$$\max_{\substack{\omega_0, \omega_1 \geq 0 \\ \omega_0 + \omega_1 = 1}} \{\overline{\mathcal{H}}_0(t, x, p_t, p_x)\omega_0 + \overline{\mathcal{H}}_1(t, x, p_V, p_x)\omega_1\} \geq 0$$
$$\forall\, (p_t, p_x, p_V) \in N^P_{\overline{Q}_{(a,b]}}(t, x, V), \ \forall\, (t, x, V) \in Q_{(a,b]};$$
$$\overline{\mathcal{H}}_0(t, x, p_t, p_x) \geq 0$$
$$\forall\, (p_t, p_x) \in N^P_{\overline{Q}_{V_0}}(t, x), \quad \forall\, (t, x) \in Q_{V_0};$$
$$\overline{\mathcal{H}}_1(t, x, p_V, p_x) \geq 0$$
$$\forall\, (p_x, p_V) \in N^P_{\overline{Q}_t}(x, V), \ \forall\, (x, V) \in Q_t, \ \ t = a.$$

Theorem 1.

(a) $\varphi \in \Phi_s$ *iff* (A) *holds.*
(b) *Suppose* (B) *holds. Then* $\varphi \in \Phi_w^-$.
(c) *Suppose* (C) *holds for* $Q = Q_\varphi$. *Then* $\varphi \in \Phi_{wi}^-$.

The proof is based on the description of system $(\mathcal{D})$ via an auxiliary control system obtained by a certain discontinuous time transformation (namely, the so-called space-time systems in the terminology of [Motta and Rampazzo, 1995]). This auxiliary control system has the following form (see [Dykhta and Samsonyuk, 2011])

$$\mathbf{t}'(\tau) = \omega_0(\tau), \quad \mathbf{t}(0) = a, \ \mathbf{t}(\tau_1) = b,$$
$$\mathbf{x}'(\tau) = f\big(\mathbf{t}(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)\big)\omega_0(\tau) + G\big(\mathbf{t}(\tau), \mathbf{x}(\tau)\big)\omega(\tau),$$
$$\mathbf{V}'(\tau) = 1 - \omega_0(\tau), \quad \mathbf{V}(0) = 0,$$
$$\mathbf{u}(\tau) \in U, \quad (\omega_0(\tau), \omega(\tau)) \in co\, \widetilde{K}_1 \quad \text{a.e. } \tau \in [0, \tau_1].$$

Here, $\mathbf{t}(\cdot), \mathbf{x}(\cdot), \mathbf{V}(\cdot)$ are absolutely continuous functions, $\mathbf{u}(\cdot), \omega_0(\cdot), \omega(\cdot)$ are $\mathcal{L}$-measurable functions,

$$\widetilde{K}_1 = \{(\omega_0, \omega) \mid \omega_0 \geq 0, \ \omega \in K, \ \omega_0 + ||\omega|| = 1\},$$

τ_1 is a nonfixed terminal instant of time, and the prime denotes differentiation with respect to τ. By using some results of the papers [Clarke, Ledyaev, Stern, and Wolenski, 1998; Vinter, 2000] the statement of Theorem 1 is obtained.

4 Estimates for the integral funnel of trajectories

In this subsection we dwell on an application of the Lyapunov type functions to inner and outer estimates of the integral funnel of trajectories and reachable set of $(\mathcal{D})$.

Let X be a compact set in $\mathbb{R}^n$. Let $\mathcal{TE}_{[a,t]}(a, X)$ consist of all pairs $\big(x(\cdot), V(\cdot)\big)$ given on the interval $[a, t]$ in which $x(\cdot)$ is a solution of $(\mathcal{D})$ such that $x(a) \in X$ and $V(\cdot)$ is a corresponding function (2). Denote by $\mathcal{RE}(a, X)$ the set

$$\left\{ \big(t, x(t), V(t)\big) \,\middle|\, \begin{array}{l} t \geq a, \\ \big(x(\cdot), V(\cdot)\big) \in \mathcal{TE}_{[a,t]}(a, X) \end{array} \right\}.$$

The set $\mathcal{RE}(a, X)$ is called the *integral funnel of trajectories* of $(\mathcal{D})$ emanating from X. Let us note that $\mathcal{RE}(a, X)$ should be called an extended integral funnel, because there are the both $x(\cdot)$ and any corresponding function $V(\cdot)$; but we prefer the shorter name. Then the cross-section at a fixed time $t > a$ is the reachable set at t for system $(\mathcal{D})$ emanating from (a, X) (or, more precisely, the extended reachable set).

Let us begin with inner estimates of $\mathcal{RE}(a, X)$.

Let Φ_* be a subset of either Φ_{wi}^- or Φ_w^- such that

$$\bigcup_{\varphi \in \Phi_*} \{x \mid \varphi(a, x, 0) \geq 0\} \subseteq X. \qquad (5)$$

We suppose also that the left-hand part of (5) is a nonempty set. Let us introduce the set

$$\mathcal{AE}[\Phi_*](a, X) = \bigcup_{\varphi \in \Phi_*} Q_\varphi.$$

Theorem 2. *The following inclusion holds*

$$\mathcal{AE}[\Phi_*](a, X) \subseteq \mathcal{RE}(a, X).$$

Proof. Let Φ_* be a subset of Φ_{wi}^-. Let us take any $(t_\alpha, x_\alpha, V_\alpha) \in \mathcal{AE}[\Phi_*](a, X)$. Then there exists $\varphi \in \Phi_*$ such that $(t_\alpha, x_\alpha, V_\alpha) \in Q_\varphi$. Since φ is a preinvariant Lyapunov type function, there exists a pair $\big(x(\cdot), V(\cdot)\big) \in \mathcal{TE}_{[a,t_\alpha]}$ for which $x(t_\alpha) = x_\alpha$, $V(t_\alpha) = V_\alpha$, $V(a) = V(a-) = 0$, and $\varphi\big(t, x(t), V(t)\big) \geq 0$ for all $t \in [a, t_\alpha]$. Moreover, from (5), it follows that $x(a) \in X$. Thus, $(t_\alpha, x_\alpha, V_\alpha) \in \mathcal{RE}(a, X)$. The proof for $\Phi_* \subset \Phi_w^-$ is in the same way. The proof is completed.

Now we obtain outer estimates of $\mathcal{RE}(a, X)$. Let Φ^* be an arbitrary subset of Φ_s. We denote by $\mathcal{BE}[\Phi^*](a, X)$ the set

$$\bigcap_{\varphi \in \Phi^*} \left\{ (t, x, V) \in T \times \mathbb{R}^n \times \mathbb{R}_+ \,\middle|\, \begin{array}{l} \exists\, x_a \in X : \\ \varphi(t, x, V) \\ \geq \varphi(a, x_a, 0) \end{array} \right\}.$$

Then the following statement is rather evident.

Theorem 3. *The following inclusion holds*

$$\mathcal{RE}(a, X) \subseteq \mathcal{BE}[\Phi^*](a, X).$$

The proof immediately follows from the strong increase of Lyapunov type functions from Φ^*.

Thus the sets $\mathcal{AE}[\Phi_*](a, X)$ and $\mathcal{BE}[\Phi^*](a, X)$ give inner and outer estimates of $\mathcal{RE}(a, X)$. The inner and outer estimates of the extended reachable set at the time $t > a$ may be obtained as the cross-sections of $\mathcal{AE}[\Phi_*](a, X)$ and $\mathcal{BE}[\Phi^*](a, X)$ at t, respectively.

5 Conclusion

In conclusion, let us note other applications of the Lyapunov type functions for impulsive optimal control problems. In [Dykhta and Samsonyuk, 2011; Dykhta, Samsonyuk, and Sorokin, 2010; Dykhta and Samsonyuk, 2010; Samsonyuk, 2010] some families of the strongly and weakly monotone Lyapunov type functions are applied to necessary and sufficient global optimality conditions corresponding to the approach of the canonical optimality theory. The papers [Fraga and Pereira, 2008; Pereira, Matos, and Silva, 2002; Motta and Rampazzo, 1996] are devoted to the substantiation of the dynamic programming principle for nonlinear impulsive control problems whereas [Daryin and Kurzhanski, 2008], for the linear ones.

Acknowledgements

The research is supported by SB–UrB RAS integration project 85.

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