What can we hope about output tracking of bilinear quantum systems?

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March 16, 2009

Abstract

We consider a non-resonant bilinear Schrödinger equation with discrete spectrum driven by a scalar control. We prove that this system can approximately track any given trajectory, up to the phase of the coordinates, with arbitrary small controls. The result is valid both for bounded and unbounded Schrödinger operators, and cn be extended to the simulatneous control of several Schrdinger equations. The method used relies on finite-dimensional control techniques applied to Lie groups. We provide also an example showing that no approximate tracking of both modulus and phase is possible, even when controls are not assumed to be essentially bounded.

1 Introduction

1.1 Physical Context

The Schrödinger equation describes the evolution of the probability distribution of the position of a particle in the space. The evolution of the Schrödinger equation can be modified by acting on the electric field, e.g., through the action of a laser.

We will be interested in this paper in non-relativistic and non-stochastic Schrödinger equations on a domain (i.e., an open connected subset) Ω of $\mathbf{R}^d$ that is either bounded or equal to the whole $\mathbf{R}^d$ ($d \in \mathbf{N}$). To each equation, we associate a *Schrödinger operator* defined as

$$(x \mapsto \psi(x)) \mapsto (x \mapsto -\Delta\psi(x) + V(x)\psi(x)), \quad x \in \Omega,$$

where ψ denotes the wave function and the real-valued function V is called the *potential* of the Schrödinger operator. The wave function verifies $\int_\Omega \psi^2 = 1$. We assume moreover that V is extended as $+\infty$ on $\mathbf{R}^d \setminus \Omega$, so that, in the case Ω bounded, ψ satisfies the boundary condition $\psi|_{\partial\Omega} = 0$. The controlled Schrödinger equation with one scalar control is the evolution equation

$$i\frac{d\psi}{dt} = -\Delta\psi(x,t) + V(x)\psi(x,t) + u(t)W(x)\psi(x,t), \tag{1}$$

where the real-valued function W is the *controlled potential*.

The control function $u : [0,T] \to \mathbf{R}$ is chosen in order to steer the quantum particle from its initial state to a prescribed target. A classical result asserts that exact controllability in L^2 is hopeless (see [4] and [9]).

The approximate controllability of one particular system of the type (1) has already be proved by Beauchard using Coron's return method (see [5] and references therein). Approximate controllability results for general systems under generic hypotheses were proved later with completely different methods in [6].

*This work was supported by BQR Région Lorraine-Nancy Université

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1.2 Mathematical framework

We give below the abstract mathematical framework which will be used to formulate the controllability results later applied to the Schrödinger equation. The fact that the Schrödinger equation fits the abstract framework has already been discussed in [6, Section 3].

Let U be a subset of $\mathbf{R}$. Let H be an Hilbert space, n be an integer, $A : D(A) \subset H \to H$ be a densely defined (not necessarily bounded) essentially skew-adjoint operator and $B : D(B) \subset H \to H$ be a densely defined (not necessarily bounded) linear operator.

We assume that (A,B,U) satisfies the following three conditions: (H1) A and B are skew-adjoint, (H2) there exists an orthonormal basis $(\phi_k)_{k=1}^{\infty}$ of H made of eigenvectors of A, and all these eigenvectors are associated to simple eigenvalues (H3) $\phi_k \in D(B)$ for every $k \in \mathbf{N}$. A crucial consequence of these hypotheses is that for every $u \in U$, $A+uB$ has a skew adjoint extension on a dense subdomain of H and generates a group of unitary transformations $e^{t(A+uB)} : H \to H$. In particular, $e^{t(A+uB)}(\mathbf{S}) = \mathbf{S}$ for every $u \in U$ and every $t \geq 0$, where $\mathbf{S}$ is the unit sphere of H.

We consider the conservative diagonal single input control systems

$$\frac{d\psi}{dt}(t) \quad = \quad A(\psi(t)) + u(t)B(\psi(t)) \tag{2}$$

with initial conditions to be specified later.

A point ψ^0 of H and a piecewise constant function $u : [0,T] \to U$, $u = \sum_{l=1}^{L} \chi_{[t_l,t_{l+1})} u_l$ being given, we say that the solution of (2) with initial condition $\psi^0 \in H$ and corresponding to the control function $u : [0,T] \to U$ is the curve $\psi : [0,T] \to H$ defined by $\psi(t) = e^{(t-\sum_{l=1}^{l_t-1} t_l)(A+u_l B)} \circ \cdots \circ e^{t_1(A+u_1 B)}(\psi^0)$ where $\sum_{l=1}^{l_t-1} t_l \leq t < \sum_{l=1}^{l_t} t_l$ and $u(\tau) = u_j$ if $\sum_{l=1}^{j-1} t_l \leq \tau < \sum_{l=1}^{j} t_l$. Notice that such a $\psi(\cdot)$ satisfies, for every $n \in \mathbf{N}$ and almost every $t \in [0,T]$ the differential equation

$$\frac{d}{dt}\langle \psi(t), \phi^n \rangle = \langle \psi(t), (A+u(t)B)\phi^n \rangle. \tag{3}$$

A piecewise constant function $u : [0,T] \to \mathbf{R}$ being given, the propagator of the control system (2) will be denoted by Φ. By definition, $\Phi(t,\psi^0) = \psi(t) = e^{(t-\sum_{l=1}^{k-1} t_l)(A+u_k B)} \circ e^{t_{k-1}(A+u_{k-1}B)} \circ \cdots \circ e^{t_1(A+u_1 B)}(\psi^0)$ for any t in $[0,T]$ and any ψ^0 in H.

For $(k,l) \in \mathbf{N}^2$, we define also the numbers $a(k,l) = \langle A\phi_k, \phi_l \rangle$ and $b(k,l) = \langle B\phi_k, \phi_l \rangle$. A finite sequence $(k_1, \ldots, k_l)$ of $\mathbf{N}$ is said to *connect* the two levels k and k' for the diagonal conservative diagonal single-input control system (A,B) if $k_1 = k$, $k_l = k'$ and $\prod_{q=1}^{l-1} b(k_q, k_{q+1}) \neq 0$. A subset S of $\mathbf{N}^2$ is called a *connectedness chain* of (A,B) if for arbitrary large N, for (k,k') in $\mathbf{N}^2$, $1 \leq k,k' \leq N$, there exists a finite sequence $(q_1, q_2, \ldots, q_l)$ in $[0,N]$ that connects k and k' for (A,B) and such that (q_r, q_{r+1}) belongs to S for every $1 \leq r \leq l-1$.

1.3 Main result

Theorem 1.1. *Assume that the system (2) satisfies (i) the spectrum of A is a $\mathbf{Q}$-linearly independent family (ii) there exists a connectedness chain S of (A,B) and (iii) U contains a neigborhood of zero. Let $c : [0,T] \to L(H,H)$ be a continuous curve taking value in the set of the unitary operators of H and such that $c(0) = Id_H$. Let N be an integer. Then, for every $\varepsilon > 0$, there exist $T_u > T$, a continuous non-decreasing bijection $s : [0,T] \to [0,T_u]$ and a piecewise constant control $u : [0,T_u] \to U$ such that the corresponding propagator $\Phi : [0,T_u] \times H \to H$ of system (2) satisfies i) for every t in $[0,T]$, $\big| |\langle \Phi(s(t),\phi_l),\phi_k \rangle| - |\langle c(t)(\phi_l),\phi_k \rangle| \big| < \varepsilon$ for every k in $\mathbf{N}$, $1 \leq l \leq N$, and ii) $\|\Phi(T_u,\phi_l) - c(T)(\phi_l)\| < \varepsilon$ for every $1 \leq l \leq N$.*

1.4 Content of the paper

In Section 2, we explain how to choose a Galerkyn approximation of the original infinite dimensional control problem (2) in some space $SU(m)$. In Section 3, we use the Lie group structure of $SU(m)$ to compute the dimensions of some Lie subalgebras of $\mathfrak{su}(m)$ and to prove that the Galerkyn approximations obtained in Section 2 have some good tracking properties. A sketch of the proof of Theorem 1.1 and an estimation of the L^1 norm of the control are given in

Section 4, where we prove that the original system (2) share the tracking properties of the Galerkyn approximations established in Section 3. A partial counterpart of Theorem 1.1 (impossibility of approximate tracking of both the modulus and the phase) is given in Section 5.

2 Choice of Galerkyn approximations

2.1 Control and time-reparametrization

We may assume without loss of generality that U has the special form $U = (0, \delta]$. Remark that, if $u \neq 0$, $e^{t(A+uB)} = e^{tu((1/u)A+B)}$. Associate with any piecewise constant $u = \sum_{l=1}^{k-1} \chi_{[t_l,t_{l+1}[} u_l \in PC([0,T_u],U)$ the function $v = \sum_{l=1}^{k-1} \chi_{[\tau_l,\tau_{l+1}[}$ $1/u_l \in PC([0,T_v],1/U)$, with $T_v = \sum_l u_l(t_{l+1}-t_l)$ and τ_l defined by $\tau_1 = t_1$ and $\tau_{l+1} = \tau_l + u_l(t_{l+1}-t_l)$. Up to the time and control reparametrization given above, it is enough to prove Theorem 1.1 for the system $\left(B,A,(\frac{1}{\delta},+\infty)\right)$:

$$\frac{d\psi}{dt}(t) \quad = \quad v(t)A(\psi(t)) + B(\psi(t)) \tag{4}$$

where the set of admissible controls is the set $PC\left(\mathbf{R}^+,(\frac{1}{\delta},+\infty)\right)$.

Remark 2.1. *A feature of this reparametrization from* $u : [0,T_u] \to U$ *to* $v : [0,T_v] \to 1/U$ *is that* $\|u\|_{L^1} = T_v$.

2.2 Galerkyn approximation

For a fixed piecewise constant control $v : \mathbf{R}^+ \to 1/U$ and a fixed ψ^0 in H, we consider the solution ψ of the system (4) of conservative diagonal single-input control systems with initial condition ψ^0.

For $k \in \mathbf{N}$, we define the function $x^k = \langle \psi, \phi_k \rangle : \mathbf{R} \to \mathbf{C}$. With our definition of solution, x^k is absolutely continuous and for almost all t in $\mathbf{R}^+$

$$\frac{d}{dt}x^k = v(t)a(k,k) + \sum_{l\in\mathbf{N}} b(k,l)x^l.$$

Proposition 2.1. *For every continuous curve* $s : [0,T_s] \to H$ *taking value in the unit sphere of* H *(that is,* $\|s(t)\| = 1$ *for all* t *in* $[0,T_s]$*), define the family* $f_l = |\langle s, \phi_l \rangle|^2$, $l \in \mathbf{N}$. *Then, for any strictly positive* ε*, there exists an integer* $N(\varepsilon)$ *such that for all* t *in* $[0,T_s]$, $\sum_{l=1}^{N(\varepsilon)} f_l(t) > 1-\varepsilon$.

We define $\pi^m : H \to \mathbf{C}^m$ by $\pi^m(v) = \sum_{k=1}^m \langle v, \phi^k \rangle e_k^m$ for every v in H, where e_k is the k^{th} element of the canonical basis of $\mathbf{C}^m$.

Proposition 2.2. *Fix a reference curve* $c : [0,T] \to L(H,H)$ *as in the hypotheses of Theorem 1.1,* $\varepsilon > 0$ *and* N *a positive integer. Then there exists a continuous curve* $M : [0,T] \to SU(m)$ *such that* $\|\pi^m(c(t)\phi^k) - M(t)\pi^m\phi^k\| < \varepsilon$ *for every* t *in* $[0,T]$ *and every* k *in* $\{1..N\}$.

For $r \geq 1$, define the $r \times r$ matrices $A^r = [a(k,l)]_{1\leq k,l\leq r}$ and $B^r = [b(k,l)]_{1\leq k,l\leq r}$. The Galerkyn approximation of order m of the system (4) is

$$\frac{dx}{dt}(t) \quad = \quad v(t)A^m(x(t)) + B^m(x(t)) \tag{5}$$

The system (5) defines a control system on the differentiable manifold $\mathbf{C}^m$. Since the system (5) is linear, it is possible to lift it to the group of matrices of the resolvent.

We now proceed to a technical change of variable (variation of the constant) and define for every piecewise constant control function v in $PC(\mathbf{R},1/U)$ and every positive t, $y(t) = e^{-A^m\int_0^t v(s)ds}x(t)$.

Recalling that for all $m \times m$ matrices a,b, $e^{-a}be^a = e^{ad(a)}b$, one checks that y verifies

$$\frac{dy}{dt}(t) \quad = \quad e^{ad(\int_0^t v(s)dsA^m)}B^m y(t) \tag{6}$$

The system (6) defines a control system on the differentiable manifold $SU(m)$, and for every positive t and every $1 \leq k,l \leq m$, $|\langle x(t)\phi_k, \phi_l \rangle| = |\langle y(t)\phi_k, \phi_l \rangle|$.

3 Tracking properties of the Galerkyn approximations

First, we have to recall some classical definitions and results for invariant Lie groups control systems, see [8] and [3].

Let G be a semi-simple compact Lie groups, with Lie algebra $\mathfrak{g} = T_{Id}G$. Consider a smooth right invariant control system on G of the form

$$\begin{cases} \frac{d}{dt}g(t) &= dR_{g(t)}f(u(t)) \\ g(0) &= g_0 \end{cases} \qquad (\Sigma)$$

where U is a subset of $\mathbf{R}$, $u : \mathbf{R} \to U$ is a L^∞ control function, $f : U \to \mathfrak{g}$ is a smooth application, g_0 is a given initial condition and $dR_a b$ denotes the value of the differential of the right translation by a taken at point b. We define the set $\mathscr{V} = \overline{\mathrm{conv}\{f(u), u \in U\}}$ as the topological closure of the convex hull of all admissible velocities at point Id. The topological closure of the convex hull of all admissible velocities at point g is $dR_g(\mathscr{V})$.

Proposition 3.1. *Let P be a Lie-subgroup of G with Lie algebra $\mathfrak{p}$. If $\mathscr{V}$ contains some bounded symmetric set S such that $\mathfrak{p} \subset Lie(S)$, then for any continuous curve $c : [0,T] \to P$, for any $\varepsilon > 0$, there exist $T_u > 0$, a L^∞ control function $u : [0,T_u] \to U$, and an increasing continuous bijection $\phi : [0,T_u] \to [0,T]$ such that the trajectory $g : [0,T_u] \to G$ of (Σ) with control u and initial condition $c(0)$ satisfies (i) $d_G(c(\phi(t)), g(t)) < \varepsilon$ for every t in $[0,T_u]$ (ii) $\phi(T_u) = T$ and (iii) $c(T) = g(T_u)$.*

To obtain trackabillity properties for the system (6), it is enough to check that the finite dimensional systems (6) satisfies the conditions on S given in Proposition 3.1 for a suitable $\mathfrak{p}$. Applying the results of [3, Appendix A] to the set $\mathscr{V}$ defined above, one gets that all the matrices $\sum_{i=1}^{p}\sum_{j=1}^{n} b(k,l)^i E_{k,l} + b(l,k)E_{l,k}$ belong to $\mathscr{V}$. We define S as the set of matrices $S = \{\pm b(k,l)E_{k,l} + b(l,k)E_{l,k}, 1 \le k,l \le m\}$. S is a symmetric and bounded subset of $\mathscr{V}$. The fact that $Lie(S) = \mathfrak{p}$ follows by the hypothesis of connectedness (see [6, Proposition 4.1] for a detailed computation).

4 Infinite dimensional tracking

4.1 Tracking in the phase variables

For the proof of Theorem 1.1, we follow the method introduced in [6]. From the application $c : \mathbf{R} \to L(H,H)$ and the tolerance ε given in the hypotheses of Theorem 1.1, we use the results presented in Section II.B to find an integer m, the finite dimensional control system (6) and the trajectory $t \mapsto \prod M(t)$ to be tracked in $SU\left(m\sum_{i=1}^{p} n\right)$. Proposition 3.1 gives the existence of some time $T_v > 0$ and some control function v in $PC([0,T_v], 1/U)$ such that the corresponding trajectory $(y_{1,1}, .., y_{p,n_p})$ of (6) tracks the trajectory $t \mapsto \prod M(t)$ with an error less than ε on each coordinate.

Since for every $1 \le k \le m$, the sequence $(b(k,l))_{l\ge 1}$ is in ℓ^2, there exists some N_1 in $\mathbf{N}$ such that $\sum_{l=N_1+1}^{\infty} |b(k,l)|^2 < \frac{\varepsilon}{NT_v}$ for every $1 \le k \le m$. The next result asserts that any trajectory of the system (6) can actually be tracked (up to ε), with the N_1-Galerkyn approximation of system (4).

Proposition 4.1. *There exists a sequence $(v_k)_k$ in $PC\left(\mathbf{R}^+, (\frac{1}{\delta}, +\infty)\right)$ such that the sequence of matrix valued curves $t \mapsto e^{ad \int vA^{(m)}} B^{(m)}$ converges in the integral sense to $t \mapsto \left(\begin{array}{c|c} M(t) & 0_{m,N_1-m} \\ \hline 0_{N_1-m,m} & G(t) \end{array}\right)$, where $t \mapsto G(t)$ is some continuous curve in $U(N_1 - m)$.*

Proof. The proof is a direct application of [6, Claim 4.3], dealing with the convergence of the sequence $e^{ad \int v_k \prod A^{(N_1)}}$. □

Proposition 4.2. *For k large enough, the control function $v = v_k$ given by Proposition 4.1 satisfies the conclusion (i) of Theorem 1.1.*

Proof. This is a direct application of [6, Claim 4.4]. □

4.2 Final phase adjustment

After time reparametrization, we get a control function $u \in PC([0,T_u],U)$ from v. Up to prolongation with the constant zero function, the control function $u : [0,T_u] \to U$ obtained in Proposition 4.2 can always be assumed to satisfy $T_u > T$ (the prolongation obviously still satisfies conclusion (i) of Theorem 1.1).

To achieve the proof of Theorem 1.1, one has to change u in such a way that it satisfies the conclusion (ii) of Theorem 1.1. One gets the result with a straightforward application of [6, Proposition 4.5].

4.3 Estimates of the L^1-norm of the control

Combining the Remark 2.1 and the estimates of [3, Prop 2.7-2.8], one gets an easily computable estimation of the L^1-norm of the control u. We denote with $\mu(t) = \sqrt{\langle M^{-1}(t)M'(t), M^{-1}(t)M'(t)\rangle}$ the velocity at time t of the trajectory to be tracked in $SU(m)$.

Proposition 4.3. *In Theorem 1.1, one can choose the control u in such a way that*

$$\|u\|_{L^1} \leq \frac{m^2 \sum \|\mu\|_{L^1}}{\min_{0\leq k,l\leq m} |b(k,l)|}.$$

This estimate is valid for every b, yet is sometimes trivial or too conservative when some $b(j,k)$ is close to zero. For these anisotropic situations, when some directions are much easier to follow than others, one can obtain sharper estimates using [3, Theorem 2.13], the expressions being slightly more intricate.

5 To track both the phase and the modulus is impossible

In this Section, we give a partial counterpart to Theorem 1.1. Indeed, we exhibit an example for which it is not possible to track both the phase and the modulus. The proof can easily be extended to a wide range of systems.

Consider one single control system in an Hilbert space H

$$\left\{ \begin{array}{lcl} \dot{x} & = & Ax + uBx \\ x(0) & = & \phi_1 \end{array} \right. \tag{7}$$

where $A : H \to H$ is a diagonal operator in the Hilbert base $(\phi_l)_{l\in\mathbf{N}}$ of H, with purely imaginary eigenvalues $(i\lambda_l)_{l\in\mathbf{N}}$ and B is a skew adjoint operator whose domain contains ϕ_l for every l in $\mathbf{N}$, satisfying $b_{i,j} = \langle B\phi_i, \phi_j\rangle \in \mathbf{R}$ for every i, j in $\mathbf{N}$. Define as admissible control functions all piecewise constant functions $u : \mathbf{R} \to \mathbf{R}^+$. For $l \in \mathbf{N}$, we note $x_l = \langle x, \phi_l\rangle$ the component of the solution of system (7) and we define $a_l = \Re(x_l)$, $b_l = \Im(x_l)$.

Proposition 5.1. *Assume $\lambda_1, b_{2,1} > 0$. Then, for $\varepsilon < \frac{b_{2,1}}{b_{2,1}+\|B\phi_2\|}$, for every piecewise constant control function $u : \mathbf{R} \to \mathbf{R}^+$, there exists $\tau > 0$, there exists i in $\mathbf{N}$, $i > 1$ such that $|x(\tau)| > \varepsilon$. In other words, it is not possible to track with an arbitrary precision the constant trajectory $x_1 \equiv 1$.*

Proof. We proceed by contradiction and assume that there exists some admissible control function $u : \mathbf{R} \to \mathbf{R}^+$ such that the corresponding trajectory of (7) remains ε-close to ϕ_1 for every time. From system (7), we see that

$$\frac{d}{dt}x_1 = i\lambda_1 x_1 + u\left(\sum_{j=2}^{+\infty} \langle B\phi_1, \phi_j\rangle x_j\right),$$

that is

$$\dot{a}_1 = -\lambda_1 b_1 + u\Re\left(\sum_{j=2}^{+\infty}\langle B\phi_1, \phi_j\rangle x_j\right) = -\lambda_1 b_1 + u\sum_{j=2}^{+\infty} b_{1,j}a_j, \tag{8}$$

$$\dot{b}_1 = \lambda_1 a_1 + u\Im\left(\sum_{j=2}^{+\infty}\langle B\phi_1, \phi_j\rangle x_j\right) = \lambda_1 a_1 + u\sum_{j=2}^{+\infty} b_{1,j}b_j. \tag{9}$$

For any positive t, the integration of (9) on $[0,t]$ yields $b_1(t) = \lambda_1 \int_0^t a_1(s)ds + \int_0^t u(s)\sum_{j=2}^{\infty} b_{1,j}b_j(s)ds$, that is $-\varepsilon\|B\phi_1\|\int_0^t u(s)ds < \int_0^t u(s)\sum_{i=2}^{\infty} b_{1,i}b(s)ds = b_1(t) - \lambda_1 \int_0^t a_1 < \varepsilon - \lambda_1(1-\varepsilon)t$ and

$$\int_0^t u(s)ds > \frac{\lambda_1(1-\varepsilon)t}{\varepsilon\|B\phi_1\|} - \frac{1}{\|B\phi_1\|}. \tag{10}$$

Integrating now $\dot{a}_2(s) = -\lambda_2 b_2(s) + u(s)\sum_{i\neq 2} b_{2,i}a_i(s)$ on $[0,t]$ for any $t > 0$, one finds $a_2(t) \geq -\lambda_2\varepsilon - \int_0^t u(s)ds\|B\phi_2\|\varepsilon$ $+b_{2,1}\int_0^t u(s)a_1(s)ds \geq (b_{2,1}(1-\varepsilon) - \varepsilon\|B\phi_2\|)\int_0^t u(s)ds - \lambda_2\varepsilon$. For ε small enough, $K = (b_{2,1}(1-\varepsilon) - \varepsilon\|B\phi_2\|) > 0$, and from (10), we get $a_2(t) \geq Kt\lambda_1/\|B\phi_1\|$ for t large enough. Hence, $a_2(t)$ tends to infinity as t tends to infinity, what is impossible since $|a_2| \leq \|x\|$ which is constant equal to 1. This gives the desired contradiction. □

Remark 5.1. *In the case where B is bounded, it is possible to define solutions of (7) for u in* $L^1(\mathbf{R},\mathbf{R}^+)$. *The result and the proof are easily extended to integrable controls that are not necessary piecewise constant (in particular, to controls that may be not essentially bounded).*

6 Conclusion and perspectives

In this paper, we prove that approximate tracking of any given trajectory is possible, in modulus, for a large class of Schrödinger equations driven by one scalar control. It appears on an example that complete tracking (phase and modulus) is hopeless in general. Approximate complete tracking with two scalar controls is possible for the Galerkyn approximations. Extensions to complete approximate tracking of the Schrödinger equation driven by two controls are still under investigation.

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