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CONTROLLABILITY AND MOTION PLANNING OF VIBRATORY SYSTEMS: A FLATNESS APPROACH

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Abstract
We study the control of vibratory systems by means of the flatness approach. For flat control systems, laws can be produced either for stabilization or for optimization under certain regimes. Also, that systems provides an acceptable easy solution for the motion planning problem.

Key words
Vibratory systems, flatness based control, differential algebra, motion planning.

1 Introduction

Vibratory systems consist of dynamical systems that perform oscillations around equilibria, the control of vibrations is important in the modeling and implementation of mechanical devices, uncontrollable vibrations might cause serious problems because of energy loose, fatigue and fracture of mechanisms. The first elementary example of a vibratory system is given by the so-called mass-spring-damper model, which consist of an harmonic oscillator to which a viscous damping mechanism has been attached, this example is the basic prototype for the analysis of vibratory systems and allows the introduction of the main concepts including those of resonance and frequency response analysis.

If a harmonic force $F(\omega)$ is added to the oscillator, it turns out that the mass oscillates at the same frequency of the applied force but with a certain phase shift. The amplitude of the vibration $X(\omega)$ can be written in terms of $F(\omega)$ and the constant of the spring. Furthermore, the vibration problem can be stated as an open-loop system where the force is the input and the vibration, the output. By representing the force and vibration in the frequency domain (magnitude and phase), the following equation can be written: $X(\omega) = H(\omega) \cdot F(\omega)$, where $H(\omega)$ is the so-called frequency response function, also referred to as the transfer function

$$\text{FORCE} \longrightarrow \boxed{\text{FREQUENCY RESPONSE}} \longrightarrow \text{VIBRATION}$$

There is extensive literature on the topic of vibratory systems in both theoretic and applied physics, for details we refer the reader to B. Tongue's book [Tongue, 2002]. In this paper we follow the approach of non-linear control of closed loop systems and, more specifically, we use the so-called flatness techniques. We describe a vibratory system inspired in a robotic mechanism of a prismatic pair coupled with a revolute and containing an oscillatory element in the end-effector.

The concept of flat differential systems finds its mathematical foundations in D. Hilbert's 22th problem about the uniformization of analytic relations by means of meromorphic functions [Hilbert, 1902] and the equivalence method for differential systems of E. Cartan [Cartan, 1914]. More recently flat differential systems have been extensively studied within the non-linear control literature, see for instance M. Fliess et al. [Fliess *et al.*, 1992] and P. Rouchon treatment of control of oscillators [Rouchon, 2005].

Apart from this introduction, the paper contains six sections. In Section 2 the general framework for flat differential systems is presented. Section 3 is devoted to the description of equivalence of differential systems including some general controllability results. In Section 4 a case study for a vibratory system is we present, the corresponding Euler-Lagrange equations are written as a non-linear control system. Section 5 analyzes the flatness of the system and describe some control laws. Section 6 provides some simulations and experimental results, and finally in Section 9 some conclusions and perspectives for future work are presented.

2 Flat Differential Systems

In this section we present the main definitions concerning flatness, we restrict ourselves to the basic statements leaving aside formal demonstrations, for that, we refer the reader to the H. Sira-Ramírez et al. book [Sira-Ramírez and Agrawal, 2004].

A differential system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \le n \tag{1}$$

is said to be differentially flat if there is a vector $y \in \mathbb{R}^m$ such that

1. $y, \dot{y}, \ddot{y}, \ldots$ are linearly independent
2. y is a function of x and a finite number of derivatives of u
3. There are two functions ϕ and ψ such that

$$\left.\begin{array}{l} x = \phi(y, \dot{y}, \ldots, y^{(\alpha)}) \\ u = \psi(y, \dot{y}, \ldots, y^{(\alpha+1)}) \end{array}\right\} \tag{2}$$

for certain multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ and

$$y^{(\alpha)} = \left(\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \ldots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}} \right) \tag{3}$$

2.1 Differential Fields

It is a commutative ring $\mathcal{R}$ with a derivation $\frac{d}{dt} : \mathcal{R} \to \mathcal{R}, \quad a \mapsto \frac{d}{dt}(a) =: \dot{a}$

$$\left.\begin{array}{l} \frac{d}{dt}(a+b) = \dot{a} + \dot{b} \\ \frac{d}{dt}(ab) = \dot{a}\, b + a\, \dot{b} \end{array}\right\} \tag{4}$$

An element $c \in \mathcal{R}$ is a constant if $\dot{c} = 0$.

L/K for two given fields $K \subset L$, in such a way that the derivation of L in K coincides with the derivation of K.

An element $\xi \in L$ is differentially K-algebraic, if there exists a $p \in K[x_1, \ldots, x_n]$ such that

$$p(\xi, \dot{\xi}, \ldots, \xi^{(n)}) = 0 \tag{5}$$

The extension L/K is said to be algebraic if all the elements in L are K-algebraic.

$\xi \in L$ is K-transcendent if and only if is not K-algebraic. The extension L/K is said to be transcendent if there exist at least an element L that is transcendent.

A set $\{\xi_i\}_{i \in I}$ is differenciably K- algebraic independent if $\{\xi_i^{(\nu)} \mid \nu \in \mathbb{N}\}_{i \in I}$ is K-algebraic independent.

Maximal independent sets with respect to the inclusion. The cardinality of a basis is the transcendence differential degree of the extension. Let K be a differential field then

$$K\left[\frac{d}{ds}\right] = \left\{ \sum_{finita} a_\nu \frac{d^\nu}{ds^\nu} \right\} \tag{6}$$

is a principal ideals ring. It is commutative if and only if K is a field of constants.

2.2 Field of Differential Operators

Let $\mathcal{C} = \{f : [0, +\infty) \longrightarrow \mathbb{C}\}$ be a ring of functions with respect to sum and convolution

$$(f \star g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau. \tag{7}$$

$\mathcal{C}$ has no zero divisors (*Titchmarsh*). The field of differential operators is the quotient field of $\mathcal{C}$.

1. Identity element: Dirac in $t = 0$
2. The inverse of the Heaviside function: is the derivation operator

$$\mathbf{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases} \tag{8}$$

3 Equivalence

Let M be a differential manifold and let $F \in C^\infty(TM, \mathbb{R}^{n-m})$, an implicit system is written as follows

$$F(x, \dot{x}) = 0, \quad \text{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m \tag{9}$$

Any system $\dot{x} = f(x,u)$ can be taken into this form: rank $\left(\frac{\partial f}{\partial u}\right) = m$ implies $u = \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)$, for then

$$F_i(x, \dot{x}) = \dot{x}_i - f_i(x, \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)) \tag{10}$$

Two systems $(M,F), (N,G)$ with rank $\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ and rank $\left(\frac{\partial G}{\partial \dot{y}}\right) = p - q$ are equivalent in $x_0 \in M$ and $y_0 \in N$ if:

1. There is $\Phi = (\varphi_1, \varphi_2, \ldots) \in C^\infty(N, M)$ such that

$$\Phi(y_0) = x_0, \quad \frac{d\varphi_i}{dt} = \varphi_{i+1} \tag{11}$$

and any solution $t \mapsto y(t)$ of $G(y, \dot{y}) = 0$ satisfies $F(\varphi_1(y(t)), \varphi_2(y(t))) = 0$

2. There is $\Psi = (\psi_1, \psi_2, \ldots) \in C^\infty(M, N)$ such that

$$\Psi(x_0) = y_0, \quad \frac{d\psi_i}{dt} = \psi_{i+1} \qquad (12)$$

and any solution $t \mapsto y(t)$ of $F(x, \dot{x}) = 0$ satisfies

$$G(\psi_1(x(t)), \psi_2(x(t))) = 0 \qquad (13)$$

If two systems are equivalent then they have the same co-ranks $m = q$.

Given a trajectory $t \mapsto x(t)$ of system $F(x, \dot{x}) = 0, x \in M$ and $\xi \in TM$, the implicit system

$$\left(\frac{\partial F}{\partial x}(x, \dot{x})\right) \xi(t) + \left(\frac{\partial F}{\partial \dot{x}}(x, \dot{x})\right) \dot{\xi}(t) = 0 \qquad (14)$$

is called *the linear approximation* around x.

Proposition 3.1. *If two systems are equivalent then the corresponding linear approximations are also equivalent.*

Definition 3.1. (M, F) *is flat in* x_0 *if it is equivalent to* $(\mathbb{R}^m, 0)$*, that is, if trajectories* $t \mapsto x(t)$ *are the image of a trivialization* Φ*, such that,* $\Phi(y_0) = x_0$*. Equivalently, for each curve* $t \mapsto y(t)$

$$x(t) = (x, \dot{x}, \ldots) = \Phi(\varphi_1(y(t)), \varphi_2(y(t)), \ldots) \qquad (15)$$

Proposition 3.2. *If a system is flat then it is equivalent to its linear approximation.*

Proposition 3.3. *If* (M, F) *is flat in* x_0*, then*

1. *Its linear approximation is controllable.*
2. *If* x_0 *is an equilibrium point, the system is locally controllable around* x_0.

4 The Elasto-Robot

We now present a particular case inspired in a robotic mechanism consisting of a prismatic pair coupled with a revolute and a oscillating end-effector, see Figure 1.

The parameters involved are the following

$a =$ Disk radius
$\theta =$ Angular displacement
$r =$ Parallel displacement
$m_2 =$ Prismatic-pair mass
$z =$ Vibration
$m_3 =$ Terminal-effector mass

In order to write the Euler-Lagrange equations T_i and V_i, we consider the kinetic and potential energies for each of the elements, here κ denotes the constant associated to the vibration.

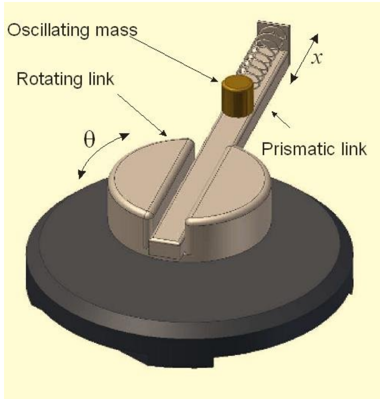

Figure 1. Robot with vibratory end-effector.

Revolute

$$T_1 = \frac{1}{2} I \dot{\theta}^2 \quad \text{y} \quad V_1 = 0 \qquad (16)$$

Prismatic pair

$$T_2 = \frac{1}{2} m_2 (\dot{r}^2 + r^2 \dot{\theta}^2) \quad \text{y} \quad V_2 = 0 \qquad (17)$$

Terminal-effector

$$\left.\begin{array}{l} T_3 = \frac{1}{2} m_3 ((\dot{r}\dot{z})^2 + (r - z)^2) \\ V_3 = \frac{1}{2} (z - r)^2 \kappa \end{array}\right\} \qquad (18)$$

The Lagrangian is the following

$$\left.\begin{array}{l} \mathcal{L} = 2\left[T_1 + T_2 + T_3 - (V_1 + V_2 + V_3)\right] \\ \quad = I\dot{\theta}^2 + (m_2 + m_3)\dot{r}^2 + (m_2 + m_3) r^2 \dot{\theta}^2 \\ \quad + m_3 \dot{z}^2 - r^2 \kappa - z^2 \kappa + 2 r z \kappa \end{array}\right\} \qquad (19)$$

From which we get the Euler-Lagrange equations

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau_1 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \tau_2 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0 \end{array}\right\} \qquad (20)$$

for then

$$\left.\begin{array}{l} \tau_1 = \left(I + m_2 r^2 + m_3 (r - z)^2\right) \ddot{\theta} \\ \quad + 2 m_3 (\dot{r} r - z \dot{r} - \dot{z} r + \dot{z} z) \dot{\theta} + 2 m_2 r \dot{r} \dot{\theta} \end{array}\right\} \qquad (21)$$

$$\left.\begin{array}{l} \tau_2 = (m_2 + m_3) \ddot{r} - m_3 \ddot{z} - m_2 r \dot{\theta}^2 \\ \quad - m_3 r \dot{\theta}^2 + m_3 z \dot{\theta}^2 + \kappa r - \kappa z \end{array}\right\} \qquad (22)$$

$$0 = m_3 \ddot{z} - m_3 \ddot{r} - m_3 z \dot{\theta}^2 + m_3 r \dot{\theta}^2 - \kappa r + \kappa z \qquad (23)$$

The torque forces $(u, v) = (\tau_1, \tau_2)$, are control parameters. We define the state variables

$$\left.\begin{array}{l} x_1 = \theta \\ x_2 = r \\ x_3 = z \\ x_4 = \dot{x}_1 \\ x_5 = \dot{x}_2 \\ x_6 = \dot{x}_3 \end{array}\right\} \quad (24)$$

For then $M(x)\dot{x} + V(x, \dot{x}) + G(x) = \tau$, coordinates $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ in the manifold

$$\mathcal{M} = (0, 2\pi)\times(0, R)\times(0, Q)\times(0, 2\pi)\times(0, R)\times(0, Q) \quad (25)$$

for certain fixed values for R and Q.

$M(x)$ denotes the inertia matrix, $V(x, \dot{x})$ the Coriolis vector and $G(x)$ the vector potential. By writing

$$I + m_2 x_2^2 + m_3(x_2 - x_3)^2 = J,$$

with $J > I$, and assuming $a^2 - x^2 > 0$, $m_2 = 1$ and $J = 1$, with $a = \sqrt{J - I}$, we get

$$x_2 - x_3 = \frac{1}{\sqrt{m_3}}\sqrt{a^2 - x_2^2} \quad (26)$$

In conclusion we obtain the following non-linear control system

$$\dot{x}_1 = x_4 \quad (27)$$

$$\dot{x}_2 = x_5 \quad (28)$$

$$\dot{x}_3 = x_6 \quad (29)$$

$$\left.\begin{array}{l} \dot{x}_4 = -2\sqrt{m_3}x_4(x_5 - x_6)\sqrt{a^2 - x_2^2} \\ \qquad -2x_2x_4x_5 + u \end{array}\right\} \quad (30)$$

$$\dot{x}_5 = x_2 x_4^2 + v \quad (31)$$

$$\dot{x}_6 = \frac{\kappa}{\sqrt{m_3^3}}\sqrt{a^2 - x_2^2} + x_3 x_4^2 + v \quad (32)$$

5 Flatness-Based Control

We now show that the above control system is flat, for that we consider the output

$$y = (L_1, L_2) = (x_1, x_2). \quad (33)$$

Equation (31) yields

$$v = \ddot{L}_2 - L_2\dot{L}_1^2, \quad (34)$$

then, equation (30) implies

$$\ddot{L}_1 = -2L_2\dot{L}_1\dot{L}_2 - 2\sqrt{m_3}\dot{L}_1(\dot{L}_2 - x_6)\sqrt{a^2 - L_2^2} + u \quad (35)$$

Now, using (34) and (26), we get

$$\left.\begin{array}{l} x_3 = L_2 - \frac{1}{\sqrt{m_3}}\sqrt{a^2 - L_2^2} \\ x_6 = \dot{L}_2 + \frac{L_2\dot{L}_2}{\sqrt{m_3}\sqrt{a^2 - L_2^2}} \end{array}\right\} \quad (36)$$

and together with

$$\left.\begin{array}{l} x_1 = L_1 \\ x_2 = L_2 \\ x_4 = \dot{L}_1 \\ x_5 = \dot{L}_2 \end{array}\right\} \quad (37)$$

$$u = \ddot{L}_1 - 4L_2\dot{L}_1\dot{L}_2 \quad (38)$$

completes the description of the system in terms of the flat output and a finite number of derivatives. In conclusion, we can write

$$\left.\begin{array}{l} x = \Theta(y, \dot{y}, \ddot{y}) \\ u = \Phi(y, \dot{y}, \ddot{y}) \\ v = \Psi(y, \dot{y}, \ddot{y}) \end{array}\right\} \quad (39)$$

Once x_1 and x_2 are controlled, so are x_3, x_4, x_5 and x_6 and open-loop controls are given by expressions (34) and (38).

6 Simulation and Numerical Experiments

Given a desired reference trajectory for an angle $\theta = F(t)$ and a displacement $r = G(t)$, represented by $F^*(t)$ and $G^*(t)$, respectively, the desired control law can be obtained by

$$v = \tilde{v} - L_2\dot{L}_1^2, \quad (40)$$

and

$$u = \tilde{u} - 4L_2\dot{L}_1\dot{L}_2 \quad (41)$$

where

$$\left.\begin{array}{l} \tilde{u} = (\ddot{F})^*(t) - \lambda_2(\dot{F}(t) - \dot{F}^*(t)) \\ \quad - \lambda_1(F(t) - F^*(t)) \\ \tilde{v} = (\ddot{G})^*(t) - \gamma_2(\dot{G}(t) - \dot{G}^*(t)) \\ \quad - \gamma_1(G(t) - G^*(t)) \end{array}\right\} \quad (42)$$

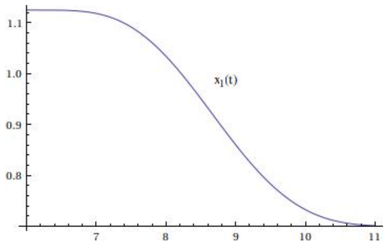

Figure 2. Estimated state variable x_1 of the elasto-robot.

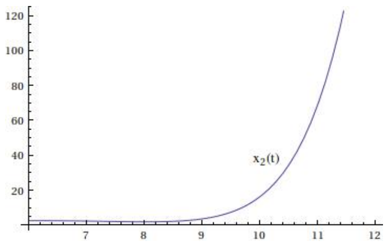

Figure 3. Estimated state variable x_2 of the elasto-robot.

for certain parameters λ_1, λ_2, γ_1 and γ_2. That differential parametrization of the control inputs, u, v in term of the outputs L_1 and L_2, says that its possible controlling the second derivative of L_1 and L_2 by means of u and v, on the proposed flat output trajectory tracking task.

In practice, this linearizing feedback control law is difficult to synthesize, however, through the use of Laplace transform, you can write the system as system string, resulting in a trivial integration. The proposal controller makes the elasto-robot coordinates θ and r track the desired trajectories.

A choice of a nominal planar rotation F^* and a nominal parallel displacement G^*, results in displacement evolution for the end-effector z, with a certain nominal value along the prescribed trajectories. Simulations for nominal trajectories were carried out for the elasto-robot. The following system parameter values were chosen:$m_2 = 1\,[Kg]$, $m_3 = 1\,[Kg]$, $J = 1$, $a = 1$, $k = 1\,[N - m/rad]$. For the required calculations about the reference trajectories $F^*(t)$ and $G^*(t)$, we prescribe a smooth polynomial spline for interpolating the initial and the final desired values of the corresponding states. The nominal displacement variables are specified as

$$F^*(t) = x_1(t_0) + (x_1(T) - x_1(t_0))f(t), \qquad (43)$$

where $x_1(t_0) = 1.1246$, $x_1(T) - x_1(t_0) = 0.4246$, for

$$\left.\begin{array}{l} f(t) = (((t-6)/6)^5) \cdot (252 - 1050 \cdot ((t-6)/6) \\ \quad +1800 \cdot ((t-6)/6)^2 - 1575 \cdot ((t-6)/6)^3 \\ \quad +700 \cdot ((t-6)/6)^4 - 126 \cdot ((t-6)/6)^5) \end{array}\right\} \qquad (44)$$

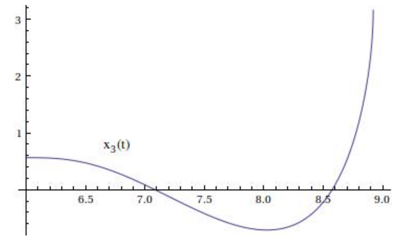

Figure 4. Estimated state variable x_3 of the elasto-robot.

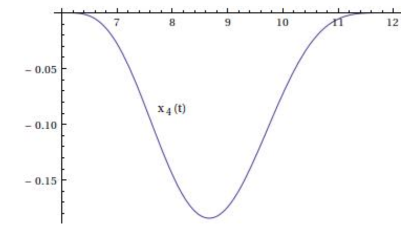

Figure 5. Estimated state variable x_4 of the elasto-robot.

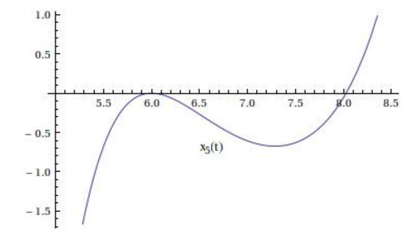

Figure 6. Estimated state variable x_5 of the elasto-robot.

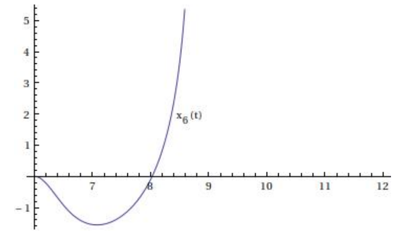

Figure 7. Estimated state variable x_6 of the elasto-robot.

and

$$G^*(t) = x_2(t_0) + (x_2(T) - x_2(t_0))g(t), \qquad (45)$$

where $x_2(t_0) = 2.5$, $x_2(T) - x_2(t_0) = -0.5$, for

$$\left.\begin{array}{l} g(t) = (((t-6)/6)^5) \cdot (1050 - 1800 \cdot ((t-6)/6) \\ \quad +1800 \cdot ((t-6)/6)^2 - 1575 \cdot ((t-6)/6)^3 \\ \quad +700 \cdot ((t-6)/6)^4 - 126 \cdot ((t-6)/6)^5) \end{array}\right\} \qquad (46)$$

Figures 2 and 3 despict computer simulations showing the performance for the previously designed feedback controller for the prescribed trajectory.

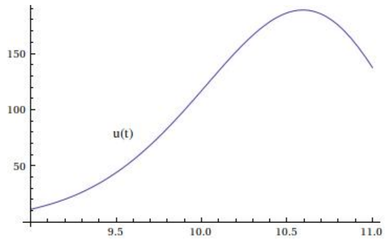

Figure 8. Action of control input u.

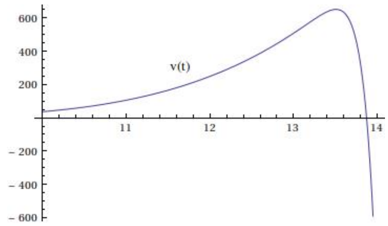

Figure 9. Action of control input v.

7 Motion Planning

In general, a robot is completely described by a kinematic motion (a non-linear system over a manifold) with non-holonomic constraints (given by a distribution over the manifold). Roughly speaking, the motion planning problem consists in finding a collision-free admissible path for the system, for steering the robot from an initial position and velocity, to a goal position and velocity. Moreover, we can request for the trajectory to have an optimal cost. We look the trajectories like $(x(t), u(t))$, where $x(t)$ is a feasible trajectory and $u(t)$ is an open-loop control generating $x(t)$. The solution of the motion planning problem allows the planification of the robot's trajectories to avoid undesired oscillations, vibrations and obstacles.

For the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \leq n \tag{47}$$

two configurations x_I, x_F in the space $\mathbb{R}^n$ and a reference trajectory,

$$x : [t_I, t_F] \to \mathbb{R}^n, \text{ with } x(t_I) = x_I, \ x(t_F) = x_F \tag{48}$$

the motion planning problem consists in finding an admissible trajectory for the system, connecting those configurations, avoiding obstacles and with low cost. The motion planning problem has been extensively studied in the literature under different approaches. In this work we investigate it with the flatness approach.

8 Flatness and Motion Planning

The motion planning algorithms based in the flatness property exploit the existence of a flat output that allows to express all the system variables in function of the flat output and a finite number of its derivatives. Since the initial an final conditions are done over $x(t)$ and $u(t)$, through the surjetivity of the mappings (39) between sufficiently smooth trajectories of the output and feasible trajectories of the system, we can find a trajectory $t \mapsto y(t)$, $2r + 1$ times differentiable that satisfies the corresponding conditions for the flat output. We compute the trajectory by polynomial interpolation for the variables of the flat output, constructing by joining two equilibrium points of the system (1), at rest when starting and to rest at the end (rest-to-rest trajectories). Assume the conditions

$$\left.\begin{array}{l} y(t_I) = y_I, \ \ \dot{y}(t_I) = 0, \ \ \cdots y^{(2r+1)}(t_I) = 0 \\ y(t_F) = y_F, \ \dot{y}(t_F) = 0, \ \ \cdots y^{(2r+1)}(t_F) = 0 \end{array}\right\} \tag{49}$$

To find a trajectory of the flat output satisfying these conditions we construct a polynomial function for a variable of the flat output:

$$\eta(t) = \eta_I - (\eta_I - \eta_F)(\frac{t - t_I}{t_F - t_I})^{r+1} \sum_{j=0}^{r} a_j (\frac{t - t_I}{t_F - t_I})^j \tag{50}$$

where $\eta_I = \eta(t_I)$, $\eta_F = \eta(t_F)$ and the coefficients a_j are independent of t_I, t_F, $\eta(t_I)$, $\eta(t_F)$ and satisfy a linear system of equations

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r+2 & r+3 & & a2r+3 \\ \vdots & \vdots & & \vdots \\ (r+2)! & \frac{(r+3)!}{2} & \cdots & \frac{(2r+3)!}{(r+2)!} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{r+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{51}$$

We refer the reader to the book [Levine, 2009]. For our model, we have the constraints

$$\left.\begin{array}{l} y(t_I) = y_I, \ \ \dot{y}(t_I) = 0, \ \ \ddot{y}(t_I) = 0 \\ y(t_F) = y_F, \ \dot{y}(t_F) = 0, \ \ \ddot{y}(t_F) = 0 \end{array}\right\} \tag{52}$$

and the reference trajectory is given by

$$x_i(t) = x_i^I - (x_i^I - x_i^F)(\frac{t - t_I}{t_F - t_I})^3 \sum_{j=0}^{2} a_j (\frac{t - t_I}{t_F - t_I})^j \tag{53}$$

for the variables x_i, $i = 1, 2$ of the flat output $y = (x_1, x_2)$, where $x_i^I = x_i(t_I)$, $x_i^F = x_i(t_F)$ and the coefficients a_j satisfy the conditions

$$\begin{pmatrix} 1 & 1 & 1 & a_0 \\ 3 & 4 & 5 & a_1 \\ 6 & 12 & 20 & a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{54}$$

so, $a_0 = 10$, $a_1 = -15$ and $a_2 = 6$.

Figure 10 illustrates our reference trajectory solution $x_1(t)$ and the velocities of the components. Figures 12 and 13 show the control outputs.

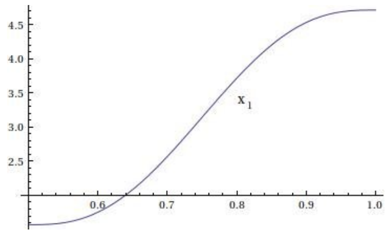

Figure 10. Reference trajectory for angle θ, for the elasto-robot.

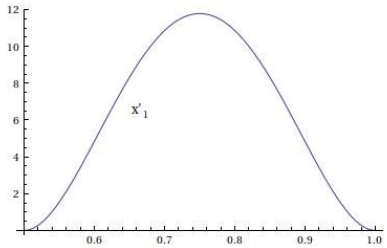

Figure 11. Derivative of the reference trajectory for angle θ.

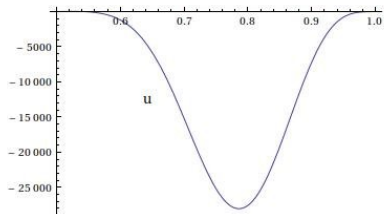

Figure 12. Control inputs.

9 Conclusions

In this paper we have presented the concept of flat control system together with a case of study inspired in a robotic mechanism equipped of with a prismatic pair coupled with a revolute and a oscillating end-effector. We have obtained a control strategy based on differential flatness properties of the elasto-robot. The methods of differential flatness make it possible to control the entire system, through flat output control system. We detail here an example of an open-loop control calculation. For this model, the first experimental results

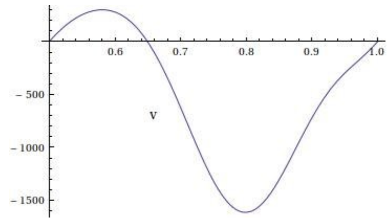

Figure 13. Control inputs.

confirm its effectiveness. Also, we study the motion planning problem related to manipulation of the elasto-robot with the same flatness-based approach. We compute the reference trajectory solution without integration of the model equations, by elementary interpolation.

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