DIMENSION REDUCTION: A KEY CONCEPT IN DYNAMICS

A. Steindl and H. Troger
Technische Universität Wien
A-1040 Vienna, Austria
Alois.Steindl@tuwien.ac.at
Hans.Troger@tuwien.ac.at

Abstract
The main idea in dimension reduction is the separation of the variables into essential or active modes and into ignorable or passive modes. However, it would be a mistake to assume that the ignorable modes, can be completely ignored in the derivation of the reduced system. Due to nonlinear coupling the elimination of these passive modes is a delicate matter. If these modes are completely ignored in the calculation of the reduced system the so-called standard (linear or flat) Galerkin method, mostly used in engineering, is obtained. However, this approach sometimes gives qualitatively incorrect results and hence better strategies must be looked for. Hence the representation of the passive modes by means of the active modes is the essential step in applying more sophisticated dimension reduction methods. This is done in almost all different approaches by invariant manifolds.

1 Introduction

Usually, for realistically modeled problems in engineering one is confronted in the description of the dynamics of a system with a representation space which is of large, perhaps even of infinite dimension. However, for some classes of systems it is well known, both from numerical simulations and also from experiments, that an accurate description still should be possible by introducing a space of much smaller dimension. This reasoning is also supported by the fact that if, for example, a system possesses limit cycle behaviour, even if this occurs in a high dimensional space (for example, in the description of the motion of a railway car modeled by a system with many degrees of freedom), it should be possible to represent the dynamics of the full high dimensional system by the dynamics in a two dimensional phase space, because all different components of the car oscillate with the same limit cycle frequency but with different amplitudes, as it is known from a Hopf bifurcation analysis via Center Manifold theory. In other words, if the investigated phenomenon is first presented in some arbitrary way, then, if the motion shows limit cycle behaviour, all dependent variables are represented as functions of just two distiguished variables. Such a behaviour can be well studied by the methods of Local Bifurcation theory and can be applied to the analysis of the nonlinear motion, setting in after loss of stability of an equilibrium. Center Manifold theory, which is mathematically well founded, supplies a possible method of dimension reduction.

Locally in the vicinity of a bifurcation point, where a considered state, be it an equilibrium or a periodic solution, looses stability, the Center Manifold attracts all solutions of the system and hence the nonlinear asymptotic dynamics of the full system is represented by the dynamics on the possibly very low dimensional Center Manifold. The main shortcoming of Center Manifold theory is that it is only a local theory, which means that, for example for the problem of loss of stability of an equilibrium at a critical parameter value, only small parameter variations about the critical parameter value are allowed. However there exists a global equivalent to the Center Manifold, namely the Inertial Manifold, which is not restricted to the bifurcation szenario, just described, but which contains the whole bifurcation behaviour.

Moreover it is also well known that in stiff systems often slow motions can be identified, which are dominant and if proper coordinates are selected, can be separated from fast motions, which hardly affect the slow motions. Hence these fast modes of motion can be eliminated resulting in a reduction of the dimension of the original system.

Finally a surprising similarity is found in the concept of *Condensation* which we also explain.

2 How to find a reduced order system?

The purpose of this paper is to show that almost all methods of dimension-reduction are closely related to the concept of invariant manifolds (an exception is the Karhunen-Loeve method). These invariant manifolds play an important role both for the behaviour of dissipative and conservative systems by slaving fast modes to slow modes as it is shown below, for example, in (13).

Hence we consider the following methods and demonstrate that in all cases the essential step is to find an equation similar to (13).

1. Global theory:
   (a) Theoretically important: Inertial manifold theory
   (b) Practically important: Galerkin methods (Approximate inertial manifold theory)
2. Local theory:
   (a) Dynamics: Center manifold theory
   (b) Statics: Liapunov-Schmidt method
3. Slow-fast dynamics in Hamiltonian systems (Nonlinear Normal Modes).
4. Linear Static Problems: Condensation

3 Inertial manifold

In the global process of reducing an infinite dimensional system to a finite dimensional one, four questions must be answered ([1]):

1. Existence and uniqueness of solutions with specified initial conditions. In engineering language this more or less means that the long time behaviour of the solutions must be characterised by a finite dimensional absorbing subset of the phase space. For engineering problems, which include damping, the existence of an absorbing set often can be shown rigorously.
2. Compactness of the universal attractor. If the phase space of the dynamical system is a Hilbert space and solutions are expanded in an orthogonal basis of this Hilbert space, one would like to show that the higher modes in this basis decay strongly.
3. Estimating Hausdorff or fractal dimension of the universal attractor. Quantitative estimates are based upon linearizing the system along its trajectories and computing a Lyapunov spectrum. A rough estimate of part of the Lyapunov spectrum can be obtained from looking at the growth rates of n-dimensional volumes in the linearized flow. If there is some n, for which all n-dimensional volumes decrease along the flow, then n is an upper bound for the dimension of the universal attractor.
4. One can hope that not only the attractors of an infinite dimensional system will be finite, but that there will be a smooth finite dimensional subset that is invariant under the flow and contains the universal attractor. Such a subset is called an *Inertial Manifold*. The existence of inertial manifolds is a more delicate matter than the existence of universal attractors.
   The question that one asks is when a smooth invariant submanifold in a dynamical system will persist under perturbation. The attracting invariant manifold persistent under perturbations must have more extreme Lyapunov exponents in its normal directions than in its tangential directions.
   If the partial differential equations being studied have large gaps in their spectra, then these can be used to look for invariant manifolds that lie close to the linear space spanned by the modes whose eigenvalues lie to the right of a gap in the complex plane. There are many examples for which such spectral gaps exist. For example for a reaction-diffusion equation in one dimension, the eigenvalues of the Laplacian decay in magnitude like $-n^2$ and this leads to the existence of the appropriate gap conditions.

To be more specific, let us consider a dissipative evolutionary equation of the form ([2])

$$\dot{u} = Lu + F(u), \tag{1}$$

where L is a self-adjoint negative operator with compact resolvent, defined on a Hilbert space H and $F(u)$ is the nonlinear part defined on the domain of L. Let $u(t) = S(t)u_0$ denote the solution to (1) at time t satisfying the initial condition $u(t_0) = u_0$. L is usually a (dissipative) spatial differential operator (like the Laplacian, or the biharmonic operator), making (1) a partial differential equation. The operator L has a complete orthogonal set of eigenfunctions $w_1, w_2, \ldots$, with the real parts of the corresponding eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \cdots$. Define P to be the spectral projection onto the span of the first n eigenfunctions and $Q := I - P$ be that onto the remaining ones, i.e.,

$$Pu = p := \sum_{j=1}^{n} \alpha_j(t) w_j := \sum_{j=1}^{n} \frac{(u, w_j)}{(w_j, w_j)} w_j \tag{2}$$

and

$$Qu = q := \sum_{j=n+1}^{\infty} \alpha_j(t) w_j := \sum_{j=n+1}^{\infty} \frac{(u, w_j)}{(w_j, w_j)} w_j \tag{3}$$

where $(\cdot, \cdot)$ denotes the scalar product in H.

Definition: A subset $M \subset H$ is said to be an **Inertial Manifold** for equation (1) if M satisfies the following conditions:

M is a finite-dimensional Lipschitz manifold in H (it can often be shown that M is a C^1 manifold).

M is positively invariant, i.e., if $u_0 \in M$ then $S(t)u_0 \in M$ for all $t > 0$.

M is exponentially attracting, i.e., there is a $\mu > 0$ such that for every $u_0 \in H$ there is a constant $K = K(u_0)$ such that $\text{dist}(S(t)u_0, M) \leq Ke^{-\mu t}$, $t \geq 0$.

The existence of an inertial manifold is very important since it implies that the long term dynamics of the original equation (1) is completely described by a finite dimensional ordinary differential equation, *without error*.

Assuming that the real parts of the eigenvalues of L decay sufficiently fast to satisfy a gap condition, it is possible to obtain an upper bound on n such that M is the graph of a smooth function

$$\Phi : PH \to QH. \tag{4}$$

Under the above assumptions the graph of the function Φ is an n-dimensional manifold in H. For $u \in H$ we set $p = Pu$, $q = Qu$, and using the commutativity relations $PL = LP$ and $QL = LQ$ we can write the partial differential equation (1) equivalently as the following system of ordinary differential equations

$$\dot{p} = Lp + PF(p+q), \tag{5}$$
$$\dot{q} = Lq + QF(p+q). \tag{6}$$

By expressing the q variables in terms of the p variables through the relation

$$q = \Phi(p) \tag{7}$$

we obtain the following reduced system of ordinary differential equations

$$\dot{p} = Lp + PF(p + \Phi(p)) \tag{8}$$

which can now be used to determine the long-term dynamics of the original equation *without error*. System (8) is called an inertial form of (1). Numerical methods ([2]) for solving (1), which in effect compute solutions of (8) with Φ replaced by $\Phi = 0$, are called standard (linear, flat) Galerkin methods. Those, which use nontrivial approximations to the mapping Φ in (8), are referred to as nonlinear Galerkin methods. Further note that the way P and Q are defined in (2) and (3), P is an orthogonal projection on H with finite dimensional range, while Q has infinite dimensional range.

The main problem in the application of Inertial Manifold theory, provided an inertial manifold exists at all, is that usually the estimate of its dimension n is very high [3].

Hence, the dimension obtained in [3] is not useful for practical applications and consequently, various approximative methods called Approximate Inertial Manifold theories or Nonlinear Galerkin methods are used. These basically proceed in the following way, that the infinite dimensional projection Q is approximated by truncating the series (3) by retaining only m modes and therefore equation (3) is replaced by

$$Qu = q := \sum_{j=n+1}^{m} \alpha_j(t) w_j = \sum_{j=n+1}^{m} \frac{(u, w_j)}{(w_j, w_j)} w_j. \tag{9}$$

Because of this truncation, the relation between p and q is replaced by an approximation, say $q = \Phi_a(p)$, which now maps PH into the finite-dimensional space QH, and the corresponding graph M_a is now an approximate inertial manifold of (1), where in addition also n is chosen to be a small number. In [4] these approximate inertial manifold calculations are applied to the Kuramoto-Shivashinski equation.

4 Center manifold

One of the main problems occuring in the approximate inertial manifold calculation, namely the determination of the dimension n, does not occur for the application of Center Manifold theory as we now shortly indicate. We assume that $L = L(\lambda)$ in (1) depends on a parameter λ and consider the loss of stability of the trival solution $u = 0$ of (1) under quasistatic variation of λ. For parameter values below $\lambda = \lambda_c$ the solution $u = 0$ is supposed to be asymptotically stable.

Under certain mild requirements ([5]), where the most important one is that at the critical parameter value $\lambda = \lambda_c$ the eigenvalue with largest real part, crossing the imaginary axis, has finite multiplicity (n), Center Manifold theory is applicable. Then the field variable $u(x,t)$ is decomposed in the form

$$\begin{aligned} u(x,t) &= u_c(x,t) + u_s(x,t) \\ &= \sum_{i=1}^{n} q_i(t) w_i(x) + U(q_i(t), x), \end{aligned} \tag{10}$$

where the $w_i(x)$ are the active spatial modes, obtained from the solution of the eigenvalue problem related to the linear system

$$\dot{u} = L(\lambda_c) u. \tag{11}$$

The $q_i(t)$ are their time dependent amplitudes and $u_s(x,t)$ could be given by an infinite sum. The decomposition is completely analogous to (2) and (3). The key point in Center manifold theory is that the influence of the infinite number of higher modes contained in $u_s(x,t)$ can be expressed in terms of the lower order modes by the function $U(q_i(t), x)$.

We assume that the spectrum of $L(\lambda)$ is discrete and that for $\lambda = \lambda_c$ a finite number (n) of eigenvalues crosses the imaginary axis at the same time. All other eigenvalues have a negative real part. Defining the projections as before, we obtain equations

$$\begin{aligned} \dot{u}_c &= Lu_c + PF(u_c + u_s), \\ \dot{u}_s &= Lu_s + QF(u_c + u_s), \end{aligned} \tag{12}$$

formally completely analogous to (5) and (6). If

$$u_s = \Phi(u_c) \tag{13}$$

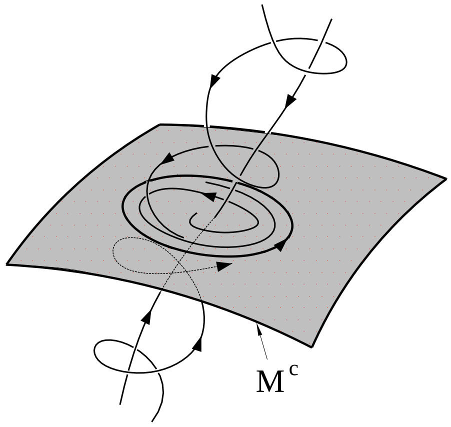

Figure 1. Three-dimensional flow approching the two-dimensional Center Manifold on which a limit cycle represents the asymptotic behaviour after a Hopf bifurcation

is a smooth invariant manifold we call Φ a *center manifold* if $\Phi(0) = \Phi'(0) = 0$. Note that if in (12) $PF = QF = 0$, all solutions tend exponentially fast to solutions of $\dot{u}_c = PLu_c$. That is, the linear n-dimensional equation on the (flat) center manifold determines the asymptotic behaviour of the entire infinite-dimensional linear system, up to exponentially decaying terms. The *center manifold theorem* ([5]) enables us to extend this argument to the nonlinear case, when PF and QF are different from zero and to replace (12), if $|u_c|$ is sufficiently small, by

$$\dot{u}_c = PLu_c + PF(u_c + \Phi(u_c)). \tag{14}$$

The zero solution of (12) has exactly the same stability properties as the zero solution of (14). Further for the determination of $\Phi(u_c)$ the (partial) differential equation

$$\Phi'(u_c)\dot{u}_c = L\Phi(u_c) + QF(u_c + \Phi(u_c)) \tag{15}$$

is obtained, from which a power series approximation of Φ can be calculated.

Basically, the loss of stability is described in terms of the temporal evolution of the amplitudes of certain (active) modes, the determination of which is clear for Center Manifold theory. These modes are those that are either mildly unstable or only slightly damped in linear theory. Their determination requires the solution of the linear eigenvalue problem (11). If the number of these critical modes is finite, a set of ordinary differential equations the *amplitude equations of the critical modes* can be constructed, which govern the long term behaviour of the original system, since the other (possibly infinitely many) modes decay exponentially.

This dimension reduction scenario can be given a geometric interpretation in phase space (Figure 1). The evolution of the flow can be split into two components: The dynamics of the active modes, which dominate the long term behaviour, is governed by the nonlinear system (14), which also takes into account the rapidly decaying terms by the argument $\Phi(u_c)$. If a solution of the full system starts on the invariant manifold, the passive modes are found directly from the equation $u_s = \Phi(u_c)$. For general initial values the solution converges quickly to the invariant manifold.

5 Nonlinear normal modes

The elimination process of the fast (inessential, passive) modes is more generally called slaving of the fast modes to the slow modes. This procedure can also be applied for conservative systems. Here in analogy to the dissipative case the situation is often found that motions in a system are evolving on different time scales. Typically this occurs in stiff systems where one has a slowly evolving time scale describing the salient features of the system and a fast time scale, which is transient, if the system is dissipative and oscillatory if the system is conservative. Simplification of the system dynamics concerning its dimension often can be achieved by elimination of the fast scales.

The equations of such stiff systems can be given a singular perturbation form. For example for the planar motion of a spring pendulum consisting of a mass m and a very stiff spring (constant c, unstrained length l_0) we obtain

$$\dot{T} = A_\mu(R)T + F_\mu(R,T) \tag{16}$$

$$\mu\dot{R} = BR + G_\mu(R,T) \tag{17}$$

where

$$T = (\varphi, \dot{\varphi}), \quad R = (\Delta l, \Delta\dot{l}). \tag{18}$$

Here φ designates the angle and Δl the elongation of the spring. The small parameter

$$\mu = \frac{\omega_p}{\omega_s} = \sqrt{\frac{mg}{l_0 c}} \tag{19}$$

is the ratio of the small pendular frequency to the large extensional frequency. Hence T is the slow variable and R the fast variable.

Now slaving the fast modes by the slow modes similarly as it is done in Center Manifold theory by

$$R = \Phi_\mu(T) \tag{20}$$

results in an equation of the form of (15)

$$B\Phi_\mu + G_\mu(\Phi_\mu, T) = \mu\Phi'_\mu \cdot [A_\mu(\Phi_\mu)T + F_\mu(\Phi_\mu, T)] \tag{21}$$

from which an approximation of the slow manifold can be calculated, as long as the freqencies are not in resonance.

This has been done in [6] for a spring pendulum. On the resulting invariant manifold a two-dimensional motion in the four dimensional space is obtained. If the motion is exactly represented by the motion on the manifold, this motion is called in [8] a Nonlinear Normal Mode. Generalizing this concept we can say that a 2n-dimensional system (n-degrees of freedom), the motion of which takes place on a two-dimensional invariant manifold, possesses a Nonlinear Normal Mode of motion. Hence the Nonlinear Normal Modes of a conservative system of the form (1) are synchronous oscillations of all components ([9]) taking place on a two dimensional invariant manifold.

6 Condensation

An interesting analogy is given by the concept of Condensation presented in [7]. We consider linear problems in structural mechanics, which are often given by an equation of the form ($x \in R^n$)

$$F = Kx$$

which we arrange, after partitioning, in the form

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the forces F_2 are to be zero. From the second equation $B'x_1 + Cx_2 = 0$ follows

$$x_2 = -C^{-1}B'x_1 \tag{22}$$

Hence the variables $x_2 \in R^m$ at which no forces are applied can be eliminated witout any approximation to result in the reduced system

$$F_1 = K_1 x_1 = (A - BC^{-1}B')x_1$$

where $x_1 \in R^{n-m}$ with the reduced stiffness matrix

$$K_1 = A - BC^{-1}B'.$$

Condensation is used to reduce the dimension in FE calculations.

7 Conclusions

In all different cases of reduction of the system dimension the elimination of the passive variables is performed by expressing the passive variables by the active variables in an equation of similar form. These relations are given by (7), (13), (22), (20) and (22).

References

Guckenheimer,J., 1989. Book review: Infinite-dimensional dynamical systems in mechanics and physics, by R. Temam. *Bulletin of the AMS,* vol. 21, 196–198.

Korontinis K.E., Trummer M.R., A finite difference scheme for computing inertial manifolds, *ZAMP* 46 (1995) 419-444.

Bloch, A. M., Titi, E. S., 1990. On the Dynamics of Rotating Elastic Beams. *New Trends in Systems Theory,* Birkhäuser, Boston, pp. 128–135.

Foias, C., Jolly, M. S., Kevrekidis, I. G., Sell, G. R., Titi, E. S., 1988. On the computation of inertial manifolds. *Physics Letters* A 131, 433–437.

Carr, J., 1981. *Applications of Centre Manifold Theory,* Applied Math. Sciences, vol. 35, Springer-Verlag, New York, Heidelberg, Berlin.

Georgiou,I.T., On the Global Geometric Structure of the Dynamics of the Elastic Pendulum, *Nonlinear Dynamics* 18, 51-68 (1999)

R.J.Guyan,1964, *Reduction of Stiffness and Mass Matrices* AIAA Journal, vol 3, No 2, p 380.

Shaw S.W., Pierre C., Normal Modes for Nonlinear Vibratory Systems, *J. Sound and Vibration* (1993) 164, 85-124

Vakakis, A.F., Manevitch L.I., Mikhlin Y.V., Pilipchuk V.N., Zevin A.A., *Normal Modes and Localization in Nonlinear Systems*, Wiley Series in Nonlinear Science, J.Wiley and Sons, New York, 1996