ON STABILITY OF COPLANAR LIBRATION POINTS IN THE GENERALIZED RESTRICTED CIRCULAR TREE-BODIES PROBLEM

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Abstract

We study relative equilibria of a particle in vicinity of a rigid body, assuming the body motion about mass center is a regular precession. We model the body gravitation as gravitational field of two spheres with centers in the body axis of dynamical symmetry. We deduce the particle motion equations as two-parametrical generalization of Restricted Circular Problem of Three Bodies. We investigate stability of the particle equilibria called the Coplanar Libration Points in a plane crossing the spheres centers and the precession axis if the spheres have equal masses.

Key words

Problem of three bodies, stability, bynary asteroid, libration point

1 Introduction

The Generalized Restricted Circular Problem of Three bodies (GRCP3B) has been formulated in [Beletsky, 2007] as a model of the binary asteroids dynamics. At the difference with other models of binaries dynamics (see, for example, [Kosenko, 1981; Kosenko, 1985; Scheeres and Ostro, 1996; Scheeres, et all, 1998; Scheeres, Williams and Miller, 2000; Scheeres, 2002; Beletsky, 2003; Koon et all, 2004; Vasilkova, 2005; Cendra and Marsden, 2005; Scheeres and Bellerose, 2005; Gabern, Koon and Marsden, 2005; **?**; Fahnestock and Scheeres, 2009; Vasilkova, 2010], etc., etc) in GRCP3B it is assumed that the smaller component of the binary is a particle of infinitesimal mass, that influence of the Sun gravitation can be neglected and that the bigger component is a dynamically symmetric rigid body. From these assumptions it follows that the bigger asteroid motion about mass center is a regular precession. Moreover, in GRCP3B the symmetric rigid body representing the bigger component is replaced with a dumbbell, i.e. with two homogeneous spheres jointed by a weightless rod. It follows that gravitation field of the bigger asteroid can be replaced with the two-centers gravitational field. Evidently, the centers of gravitation coincide with the spheres centers placed in the axis of the bigger asteroid's dynamical symmetry. Note that the particle motion equation can be written in a form generalizing equations of Restricted Circular Problem of Three Bodies (RCP3B)(see [Szebehely, 1967; Markeev, 1978], etc.) by adding two new parameters. It explain the title 'GRCP3B'. New parameters are the angle of nutation and some dimensionless ratio characterising the angular velocity of precession.

There exists a number of the particle equilibria with respect to the axis containing centers of spheres and the axis of nutation. These equilibria can be divided into two types.

Equilibria of the first type belong to the plane crossing the dumbbell mass center perpendicularly to the precession axis. Distances from each equilibrium of this type up to the centers of gravitation are equal. It was a reason to call such equilibrium the Triangular Libration Point (TLP). (By analogy with Lagrangian Libration Points of RCP3B). Existence and stability of TLPs have been studied in [Beletsky, 2007; Beletsky and Rodnikov 2008a]. In particular, it have been proved that the number of TLPs is 0, 1 or 2 and TLPs are stable for the first aproximation if the ratio of gravitating masses is less than $17 - 12\sqrt{2} = 0.02944$. (Note that in the opposite case at the difference with the classical problem both stsbility and instability are possible)

Equilibria of the second type belong to the plane composed by the dumbbell axis and the axis of precession. These equilibria are called Coplanar Libration Points (TLP). TLPs have been partly investigated in [Beletsky and Rodnikov 2008b]. It has been established that if the dumbbell is symmetric, i.e. if the spheres forming the dumbbell have equal masses, then the number of CLPs varies from 3 up to 7.

In this paper we claim that the total number of CLPs is from 3 up to 7 also for a dumbbell composed by spheres of unequal masses. We deduce conditions of CLPs stability in the first approximation. Using these conditions we build a diagram of CLPs stability for the symmet-

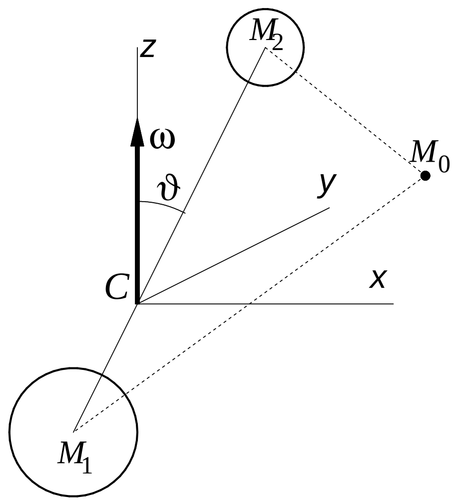

Figure 1.

ric dumbbell in the plane of the problem parameters. In particular, we prove that not more than two of CLPs can be stable and that not more than one stable CLP exists if total number of CLPs is equal to 3.

2 Parameters and designations

Assume the bigger asteroid in the binary can be replaced with a dumbbell consisting of two spheres with centers M_1 and M_2. Suppose the dumbbell motion with respect to mass center C is a regular precession with angular velocity ω and with angle of nutation ϑ. (Without loss of generality, $0 \le \vartheta \le \pi/2$). Let M_0 be a particle that does not influence the dumbbell motion but is influenced by the dumbbell gravitational field (fig. 1). Further, let $Cxyz$ be a Cartesian coordinates system rotating about Cz with angular velocity ω. ($Cz \parallel \omega$, M_1 and M_2 belong to Cxz). Denote by m_1 and m_2 the spheres masses. Let $\mu = m_2/\left(m_1+m_2\right)$. Without loss of generality, $m_2 \le m_1 \Leftrightarrow 0 < \mu \le 1/2$. Moreover, let $\alpha = G\left(m_1+m_2\right)/\omega^2 l^2$, where $l = M_1M_2$, G is Gauss' gravitational constant. Evidently, $\alpha > 0$. Note that $CM_1 = \mu l$ and $CM_2 = (1-\mu)l$. In particular, if $\mu = 1/2$ then $CM_1 = CM_2 = l/2$. It is clear that if $\vartheta = \pi/2$ and $\alpha = 1$ then we have RCP3B.

3 Motion equations and some notes

Let x, y, z be coordinates of M_0 in $Cxyz$. Using dimensionless variables $\xi = x/l$, $\eta = y/l$, $\zeta = z/l$ and dimensionless time τ ($d\tau = \omega dt$) one can present motion equations for the particle M_0 as

$$\begin{cases} \xi'' - 2\eta' - \xi = \partial\Pi/\partial\xi \\ \eta'' + 2\xi' - \eta = \partial\Pi/\partial\eta \\ \zeta'' = \partial\Pi/\partial\zeta, \end{cases} \tag{1}$$

where

$$\Pi = \alpha\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right),$$

$\rho_1 = M_0M_1/l$, $\rho_2 = M_0M_2/l$. (Here by $(\quad)'$ denote derivative w.r.t. τ). From (1) it follows that the particle M_0 can be immovable in $Cxyz$ only if $\zeta = 0$ or $\eta = 0$.

It can easily be checked that $\rho_1 = \rho_2$ for relative equilibria in Cxy ($\zeta = 0$). There exist from 0 up to 2 equilibria called TLPs in this plane (see [Beletsky, 2007] for details).

It have been proved in [Beletsky and Rodnikov 2008b] that the total number of relative equilibria called CLPs in the plane Cxz ($\eta = 0$) is from 3 up to 7 if $\mu = 1/2$.

4 On total number of Coplanar Libration Points

In the particular case of the 'vertical' dumbbell ($\varphi = 0$) there are one isolated libration point in the dumbbell axis and one, two or three circles consisting of libration points. Each of them can be called CLP at a corresponding choice of coordinate axes.

In the particular case of the 'horizontal' dumbbell ($\varphi = \pi/2$) as well as in classical problem there exist 3 CLPs belonging to Oz.

In the general case $0 < \mu < 1/2$, $0 < \theta < \pi/2$ equations for CLPs coordinates can be presented as

$$\xi_1^2 - \frac{2\xi_1}{\Phi_1} + \left(\zeta_1^2 - \frac{2\zeta_1}{\Phi_1}\right)\cot^2\vartheta + \frac{1}{\Phi_1\sin^2\vartheta} = 0 \tag{2}$$

$$\alpha = \frac{(\mu-\xi_1)(1-\zeta_1)(\xi_1^2\sin^2\vartheta + \zeta_1^2\cos^2\vartheta)^{3/2}}{(1-\mu)(\zeta_1-\xi_1)}, \tag{3}$$

where expressions for new variables ξ_1 and ζ_1 read

$$\xi_1 = \xi/\sin\vartheta + \mu, \qquad \zeta_1 = \zeta/\cos\vartheta + \mu \tag{4}$$

and

$$\Phi_1 = 1 - \left(\frac{\mu(1-\zeta_1)}{\zeta_1(1-\mu)}\right)^{\frac{2}{3}} \tag{5}$$

Note that any CLP lays in the strip restricted by the straight lines which pass through M_1 and M_2 in parallel to Cx. Therefore, $0 < \zeta_1 < 1$. Note also that (2) is a quadratic equation of ξ_1. Substituting roots of (2) for ξ_1 in (3), we obtain

$$\alpha = f_{1,2}(\zeta_1; \vartheta, \mu) \tag{6}$$

Analyzing functions f_1 and f_2 behaviour, one can see that the total number of (6) solutions is not less than

3 and not more than 7 for any $\zeta_1 \in (0,1)$ and for any admissible value of α,μ,ϑ. Let us remark that the even number of (6) solutions is a rare situation. Factually, the even number of CLPs (4 or 6) takes place only for some two-dimensional manifold in the three-dimentional space of parameters α, μ, ϑ.

5 On Coplanar Libration Points location for the symmetric dumbbell

In particular, if $\mu = 1/2$ then the even number of CLPs is impossible in general. In this case there exists CLP coinciding with C. This libration point is called 'the central CLP'(CCLP). Other CLPs can be divided into pairs symmetric w.r.t. C. It can be proved that each of quadrants $\xi_1 > 1/2, \zeta_1 > 1/2$ and $\xi_1 < 1/2, \zeta_1 < 1/2$ contains only one CLP called an external CLP (ECLP). ECLPs are analogues of Eulerian libration points L_2 and L_3 in RCP3B. Each of quadrants $\xi_1 > 1/2, \zeta_1 < 1/2$ and $\xi_1 < 1/2, \zeta_1 > 1/2$ contains one or two CLPs that are analogues of L_1. Therefore these CLPs are conditionally called internal CLPs (ICLP). (Here we use traditional designations of Eulerian Libration Points, for instance see [Markeev, 1978])

6 Stability conditions in general

Linearizing (1) in vicinity of any found equilibrium we obtain the system of the first approximation. Equation for eigenvalues λ of this system reads

$$\lambda^6 + 2\lambda^4 + A_2\lambda^2 + A_0 = 0, \qquad (7)$$

where $A_{1,2}$ depends on ξ, η, α, μ, ϑ for TLPs and depends on ζ, η, α, μ, ϑ for CLPs. As the considered dynamical system is conservative, one can see that the chosen equilibrium is stable by the first approximation only if all roots of (7) are various and purely imaginary. (If some roots equal to 0 then an additional studying is required). Thus all roots of

$$x^3 + 2x^2 + A_2x + A_0 = 0$$

should be various and negative real numbers. It follows that stability conditions for any libration points read

$$\begin{aligned} &A_0 > 0, \quad A_2 > 0, \\ &D = \left(\frac{A_0}{2} - \frac{A_2}{3} + \frac{8}{27}\right)^2 + \left(\frac{3A_2 - 4}{9}\right)^3 < 0 \end{aligned} \qquad (8)$$

7 Stability regions

The criterion of TLPs stability has been deduced from (8) in [Beletsky and Rodnikov 2008a]. The brief formulation of this criterion reads: 'If $\mu(1-\mu) < 1/36$ then TLPs are stable by the first approximation otherwise both stability and instability are possible.'

In the particular case $\vartheta = 0$ from (8) it follows that all equilibria are unstable. Reducing stability conditions one can see that the circles of libration points can be stable or unstable.

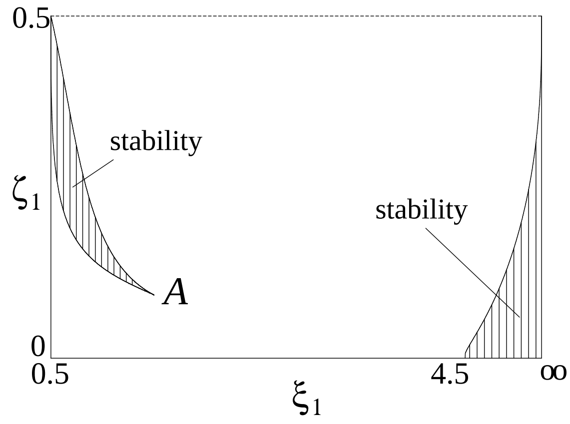

Figure 2.

In the particular case $\vartheta = \pi/2$ from (8) it follows that analogues of L_2 and L_3 are unstable but analogue of L_1 at the difference with the classical problem is stable for $1/9 < \alpha < 1/8$ and can be stable for some values of μ if $\alpha \leq 1/9$.

Consider now CLPs stability for symmetric dumbbell, i.e. for $\mu = 1/2$. Substituting 0 for ξ and for ζ in (8) we obtain stability conditions directly for CCLP. Correspondent regions of stability are the curvilinear triangles BCD and FEG depicted in fig. 3. In this fig. $B(\arccos 1/3; 1/8)$, $C(\pi/2; 1/8)$, $D(\pi/2; 1/9)$, $E(\arccos 1/3; 1/24)$, $F(\arccos\sqrt{5}/3; 1/24)$, $G(\arccos 1/\sqrt{3}; 0)$. Combining the found regions with diagram from [Beletsky and Rodnikov 2008b] one can see that ICLPs exist only if CCLP is unstable. In other words, if CCLP is stable then total number of CLPs equal to 3.

Eliminating α and ϑ from expressions for A_0 and A_1 by (2) and (3) one can see that $A_0 < 0$ in semi-strips $\xi_1 > 1/2, 1/2 < \zeta_1 < 1$ and $\xi_1 < 1/2, 0 < \zeta_1 < 1/2$. Hence ECLPs are always unstable. Nevertheless, (8) are fulfilled in four areas of the strip $0 < \zeta_1 < 1$. Two of these areas are depicted in fig. 2. (In this fig. $A(0.842, 0.093)$). The last two areas are symmetric to depicted ones w.r.t. C.

It can be proved that there exist not more than one pair of stable ICLPs. Thus the total number of stable CLPs is not more 2.

Note that equalities (2,3) are an original mapping from the plane (ξ_1, ζ_1) to the plane (ϑ, α). Using this mapping and combining diagram from [Beletsky and Rodnikov 2008b] with images of stability regions in fig. 2 and with regions of stability for CCLP one can build the final diagram of CLPs stability for $\mu = 1/2$. Regions of this diagram depicted in fig. 3 are marked as $n + m$, where m is a number of stable CLPs and m is a number of unstable CLPs. Coordinates of some points are $A(0.458, 0.046)$, $H(0, 1/8)$, $P(0, 3\sqrt{3}/8)$,

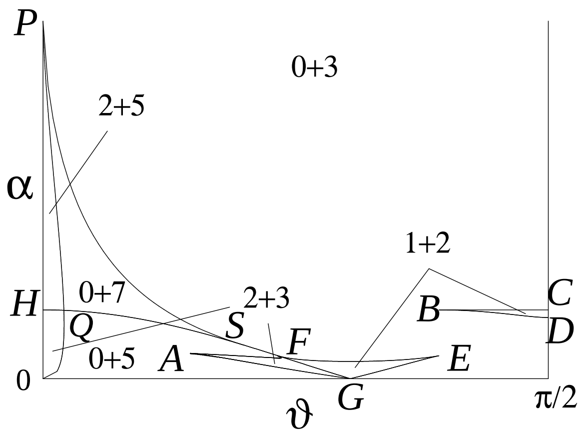

Figure 3.

$Q(0.07, 0.012)$, $S(0.626, 0.0607)$.

8 Conclusion

In this paper stability of a particle equilibria in the plane composed by axis of a rigid body's dynamical symmetry and by axis of the body precession are studied in assumption that the body gravitational field can be replaced with the gravitational field of two spheres of equal masses. Criteria of these equilibria stability by the first approximation are deduced. The number of stable and unstable equilibria is computed. The correspondent diagram in the plane of parameters is built.

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