ON APPLICATION OF FIRST INTEGRALS IN ANALYSIS OF CONSERVATIVE SYSTEMS

Valtntin D. Irtegov

Abstract— **The paper proposes a new approach to obtaining and qualitative investigation of invariant manifolds for the systems of differential equations, which have smooth first integrals. Examples of application of the technique proposed to some problems of rigid body dynamics are given.**

I. INTRODUCTION

The approach proposed is based on the following theorem:

Theorem 1: If a system of differential equations

$$\dot{x}_i = X_i(x,t,), \quad i = 1,...,n \tag{1.1}$$

has a family of smooth first integrals $V(x,t,\lambda) = c$ (where λ is the family's parameter), and if partial derivatives of this family with respect to the variables $x_i, \quad i = 1,...,n$ may be represented in the form

$$\frac{\partial V}{\partial x_i} = \sum_{l=1}^{k} a_{i,l}(x,t,\lambda)\varphi_l(x,t,\lambda)$$

$$+\sum_{l=1}^{k}\sum_{p=1}^{k} a_{i,l,p}(x,t,\lambda)\varphi_l(x,t,\lambda)\varphi_p(x,t,\lambda) + \dots, \tag{1.2}$$

and the rank of the matrix $A = ||a_{i,l}(x,t,\lambda)||$ is "k" on the family of manifolds

$$\varphi_l(x,t,\lambda) = 0, \quad l = 1,\dots,k, \tag{1.3}$$

then the equations $\varphi_l(x,t,\lambda) = 0, \quad l = 1,...,k$ define a family of system's (1.1) invariant manifolds, whose elements attribute stationary values to the corresponding elements of the family of first integrals $V(x,t,\lambda) = c$. Such families of invariant manifolds (1.3) will be called the invariant manifolds of steady motions (IMSMs).

Proof. Let us compute the derivative of integral $V(x,t,\lambda)$ due to initial differential equations (1.1) :

$$\frac{dV}{dt} = \sum_{j=1}^{n}\frac{\partial V}{\partial x_j}X_j + \frac{\partial V}{\partial t} = 0. \tag{1.4}$$

After that, compute the partial derivatives of the left-hand side expression (1.4) with respect to $x_i \quad i = 1,\dots,n$. As a result, after obvious transformations, we have:

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(\frac{\partial V}{\partial x_i})X_j + \frac{\partial}{\partial t}(\frac{\partial V}{\partial x_i}) = -\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\frac{\partial X_j}{\partial x_i},$$

$$i = 1,\dots,n.$$

The work has been supported by the INTAS-SB RAS grant No. 06-1000013-9019

V. D. Irtegov, Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, 134 Lermontov st., Irkutsk, 664033, Russia; irteg@icc.ru

Having substituted expressions for the partial derivatives $\partial V/\partial x_i$ (1.2) into the latter equations, from the theorem's conditions, we have:

$$\sum_{l=1}^{k} a_{il}(\sum_{j=1}^{n}\frac{\partial\varphi_l}{\partial x_j}X_j + \frac{\partial\varphi_l}{\partial t}) = F_i(\varphi_1,...,\varphi_k,x,\lambda),$$

$$i = 1,\dots,n,$$

where $F_i(0,\dots,0,x,\lambda) = 0$ on the manifold $\varphi_l = 0, \quad l = 1,\dots,k$, for all the values of $i = 1,...,n$. Since due to the theorem we may assume the matrix rank $||a_{il}(x,t,\lambda)||$ equal to k, the last equations on the manifold (1.3) will be reduced to the system

$$\sum_{l=1}^{k}\frac{\partial\varphi_l}{\partial x_j}X_j + \frac{\partial\varphi_l}{\partial t} = 0, \quad l = 1,...,k,$$

what proves the invariance of the family of manifolds $\varphi_l(x,t,\lambda) = 0, l = 1,...,k$, for the initial system of differential equations (1.1) [3].

Stationarity of this family of invariant manifolds follows from the fact that expressions for partial derivatives of the family of first integrals $V(x,t,\lambda)$ with respect to the phase variables turn zero on the manifold. The latter follows from the form of these partial derivatives (1.2).

Remark 1: If for some relations of the form $\psi_q(x,t,\lambda) = 0, q = 1,...,m$ the rank of some submatrices of the rectangular matrix A decreases, then equations $\varphi_l(x,t,\lambda) = 0, \quad \psi_q(x,t,\lambda) = 0, l = 1,...,k; q = 1,...,m$, may define the families of invariant manifolds, which lie on the IMSM $\varphi_l(x,t,\lambda) = 0, \ l = 1,...,k$. The latter may fail to be stationary, i.e. may fail to attribute a stationary value to the first integral $V(x,t,\lambda)$.

Remark 2: In many cases, when the first integrals are polynomial, application of the above theorem for the purpose of obtaining and investigation of invariant manifolds appears to be efficient. Furthermore, it is often possible to apply one of several possible techniques in order to represent the corresponding families of first integrals of the differential equations in the form of a polynomial or a series of some set of functions.

Remark 3: It is obvious from the form of expression (1.2) that the family of first integrals, which corresponds to above representation, shall have the form:

$$V = \sum_{l,p=1}^{k} b_{lp}(x,t,\lambda)\varphi_l\varphi_p + \sum_{l,p,q=1}^{k} b_{lpq}(x,t,\lambda)\varphi_l\varphi_p\varphi_q + ...$$

and if the expression (which begins from the quadratic form with respect to $\varphi_p, \quad p = 1, ..., k$) is a sign-definite function with respect to φ_p, then, due to V.I.Zubov's theorem [2], it is possible to conclude on stability of the elements of our family of IMSMs.

Consider some examples related to application of the approach to some problems of mechanics.

II. ON INVARIANT MANIFOLDS OF KIRCHHOFF'S EQUATIONS

Consider the following equations of motion for a rigid body in fluid in Sokolov's case [1]:

$$\begin{aligned}
\dot{s}_1 &= -\alpha^2 r_2 r_3 - \alpha r_1 s_2 - (\beta r_3 - s_2)(\beta r_2 - s_3),\\
\dot{s}_2 &= (\alpha^2+\beta^2) r_1 r_3 - (\alpha r_1 + \beta r_2) s_1 + (\alpha r_3 - s_1) s_3,\\
\dot{s}_3 &= (\beta r_1 - \alpha r_2) s_3,\\
\dot{r}_1 &= r_2(\alpha r_1 + \beta r_2 + 2 s_3) - r_3 s_2,\\
\dot{r}_2 &= r_3 s_1 - r_1(\alpha r_1 + \beta r_2 + 2 s_3),\\
\dot{r}_3 &= r_1 s_2 - r_2 s_1.
\end{aligned} \tag{2.5}$$

These equations possess 2 first integrals:

$$\begin{aligned}
2H &= s_1^2 + s_2^2 + 2s_3^2 + 2(\alpha r_1 + \beta r_2) s_3 -\\
&\quad (\alpha^2+\beta^2) r_3^2 = 2h,\\
2V &= (\alpha^2+\beta^2)(r_1 s_1 + r_2 s_2)^2 +\\
&\quad 2(r_1 s_1 + r_2 s_2) s_3 \times (\alpha s_1 + \beta s_2) +\\
&\quad s_3^2 (s_1^2 + s_2^2 + (\alpha r_1 + \beta r_2 + s_3)^2) = c.
\end{aligned}$$

The variables $s_i, \ r_i, \quad i = 1,2,3$ are linear forms of the projections of the body's mass center velocity vector and of the vector of its angular rate. It can be readily seen from the structure of integral V that the use of the functions

$$\varphi_1 = r_1 s_1 + r_2 s_2, \quad \varphi_2 = s_3,$$

allows one to represent the first integral in the following form:

$$\begin{aligned}
2\tilde{V} &= (\alpha^2+\beta^2)\varphi_1^2 + 2(\alpha s_1 + \beta s_2)\varphi_1\varphi_2 + (s_1^2 + s_2^2 +\\
&\quad (\alpha r_1 + \beta r_2)^2)\varphi_2^2 + 2(\alpha s_1 + \beta s_2)\varphi_2^3 + \varphi_2^4.
\end{aligned}$$

Partial derivatives of the first integral $\tilde{V}$ with respect to the phase variables with the precision up to the first-order terms with respect to φ_1, φ_2 may be written as:

$$\begin{aligned}
\frac{\partial \tilde{V}}{\partial s_1} &= (\alpha^2+\beta^2) r_1 \varphi_1 + \ldots,\\
\frac{\partial \tilde{V}}{\partial s_2} &= (\alpha^2+\beta^2) r_2 \varphi_1 + \ldots,\\
\frac{\partial \tilde{V}}{\partial s_3} &= (\alpha s_1 + \beta s_2)\varphi_1 + (s_1^2 + s_2^2\\
&\quad + (\alpha r_1 + \beta r_2)^2)\varphi_2 + \ldots,\\
\frac{\partial \tilde{V}}{\partial r_1} &= (\alpha^2+\beta^2) s_1 \varphi_1 + \ldots,\\
\frac{\partial \tilde{V}}{\partial r_2} &= (\alpha^2+\beta^2) s_2 \varphi_1 + \ldots,\\
\frac{\partial \tilde{V}}{\partial r_3} &= 0.
\end{aligned} \tag{2.6}$$

Obviously, there exists the second order minor for the matrix of the linear part of the system (2.6) with respect to variables φ_1, φ_2, which is not identically zero on the manifold $\varphi_1 = \varphi_2 = 0$. This minor writes:

$$\begin{aligned}
\Delta_2 = s_2((\alpha^2+\beta^2)(s_1^2 + s_2^2 + r_2^2 s_1^{-2}(\beta s_1 - \alpha s_2)^2)\\
-(\alpha s_1 + \beta s_2)^2).
\end{aligned}$$

Then, according to Theorem 1, the manifold $\varphi_l = \varphi_2 = 0$ is the IMSM of the system of differential equations (2.5). The vector field on the invariant manifold $\varphi_2 = s_3 = 0$ has the form:

$$\begin{aligned}
\dot{s}_1 &= -(\alpha^2+\beta^2) r_2 r_3 + (\alpha r_1 + \beta r_2) s_2,\\
\dot{r}_1 &= r_2(\alpha r_1 + \beta r_2) - r_3 s_2,\\
\dot{s}_2 &= (\alpha^2+\beta^2) r_1 r_3 - (\alpha r_1 + \beta r_2) s_1,\\
\dot{r}_2 &= r_3 s_1 - r_1(\alpha r_1 + \beta r_2),\\
\dot{r}_3 &= r_1 s_2 - r_2 s_1.
\end{aligned} \tag{2.7}$$

The latter equations have the first integral:

$$W = r_1 s_1 + r_2 s_2 = const$$

Consequently, the IMSM obtained above may be interpreted geometrically as an intersection of the invariant hyperplane $s_3 = 0$ and the hypersurface of the first integral

$$\tilde{W} = r_1 s_1 + r_2 s_2 + r_3 s_3 = 0.$$

Investigation of a set, on which the determinant Δ_2 turns zero, allows one to obtain degenerate invariant manifolds (submanifolds), which belong to the invariant manifold $\varphi_1 = r_1 s_1 + r_2 s_2 = 0, \quad \varphi_2 = s_3 = 0$.

Since the expression obtained above for Δ_2 may be written as

$$\Delta_2 = s_2 s_1^{-2}(\beta s_1 - \alpha s_2)^2((\alpha^2+\beta^2) r_2^2 + s_1^2),$$

then one of the degenerate invariant manifolds, which lies on the IMSM $\varphi_l = \varphi_2 = 0$, is defined by the following system of equations:

$$r_1 s_1 + r_2 s_2 = 0, \quad s_3 = 0, \quad \beta s_1 - \alpha s_2 = 0,$$

which may be complemented with the following equation:

$$(\beta r_1 - \alpha r_2) = 0.$$

The latter represents a condition of existence of a nontrivial solution for s_1 and s_2 of the first two equations of our manifold. The vector field on this invariant manifold has the form:

$$\dot{r_1} = r_2(\alpha r_1 + \beta r_2), \quad \dot{r_2} = -r_1(\alpha r_1 + \beta r_2).$$

It can easily be verified that this system of differential equations has the first integral:

$$W_1 = r_1^2 + r_2^2 = const.$$

Due to the known Lyapunov's theorem, the integral allows one to state that the trivial solution $r_1 = r_2 = 0$ of the latter system (2.7) is stable. Hence, there is a stable trivial solution lying on the invariant manifold

$$r_1 s_1 + r_2 s_2 = 0, \ \ \beta s_1 - \alpha s_2 = 0, \ \ s_3 = 0. \ \ (\beta r_1 - \alpha r_2) = 0$$

Consider now the following family of the problem's first integrals:

$$2K = 2\lambda_0 H - \lambda V = 2\lambda_0 (s_1^2 + s_2^2 + 2s_3^2$$
$$+ 2(\alpha r_1 + \beta r_2)s_3 - (\alpha^2 + \beta^2) r_3^2) -$$
$$\lambda[(\alpha^2 + \beta^2)(r_1 s_1 + r_2 s_2)^2 + 2(r_1 s_1 + r_2 s_2) s_3 (\alpha s_1 + \beta s_2)$$
$$+ s_3^2 (s_1^2 + s_2^2 + (\alpha r_1 + \beta r_2 + s_3)^2)]$$

and write down stationary conditions for the latter expression:

$$\frac{\partial K}{\partial s_1} = \lambda_0 s_1 - \lambda\{s_1[(\alpha^2 + \beta^2) r_1^2 + 2\alpha r_1 s_3 + s_3^2]$$
$$+ s_2[(\alpha^2 + \beta^2) r_1 r_2 + s_3(\beta r_1 + \alpha r_2)]\} = 0,$$
$$\frac{\partial K}{\partial s_2} = \lambda_0 s_2 - \lambda\{s_1[(\alpha^2 + \beta^2) r_1 r_2 + s_3(\beta r_1 + \alpha r_2)]$$
$$+ s_2[(\alpha^2 + \beta^2) r_2^2 + 2\beta r_2 s_3 + s_3^2]\} = 0,$$
$$\frac{\partial K}{\partial s_3} = \lambda_0 (2s_3 + \alpha r_1 + \beta r_2) - \lambda\{s_1[r_1(\alpha s_1 + \beta s_2) + s_1 s_3]$$
$$+ s_2[(r_2(\alpha s_1 + \beta s_2) + s_2 s_3] +$$
$$(2s_3 + \alpha r_1 + \beta r_2)[\lambda_0 - \lambda(s_3 + \alpha r_1 + \beta r_2)s_3]\} = 0,$$
$$\frac{\partial K}{\partial r_1} = -\lambda s_1^2[(\alpha^2 + \beta^2) r_1 + \alpha s_3] - \lambda s_2 s_1[(\alpha^2 + \beta^2) r_2 + \beta s_3] +$$
$$\alpha s_3[\lambda_0 - \lambda(s_3 + \alpha r_1 + \beta r_2)s_3] = 0,$$
$$\frac{\partial K}{\partial r_2} = -\lambda s_1 s_2[(\alpha^2 + \beta^2) r_1 + \alpha s_3]$$
$$- \lambda s_2^2[(\alpha^2 + \beta^2) r_2 + \beta s_3] +$$
$$\beta s_3[\lambda_0 - \lambda(s_3 + \alpha r_1 + \beta r_2)s_3] = 0,$$
$$\frac{\partial K}{\partial r_3} = -\lambda_0(\alpha^2 + \beta^2) r_3 = 0.$$

If the denotations $\varphi_1 = s_1, \ \varphi_2 = s_2, \ \varphi_3 = r_3, \ \varphi_4 = \lambda_0 - \lambda(s_3 + \alpha r_1 + \beta r_2)s_3$ are introduced, then the latter equations assume the form of equations (1.2). The transposed matrix of coefficients for the linear part of the system on the manifold $\varphi_i = 0, \quad i = 1, ..., 4$, has the form:

$$\begin{pmatrix} A & B & 0 & 0 & 0 & 0 \\ B & C & 0 & 0 & 0 & 0 \\ 0 & 0 & D & \alpha s_3 & \beta s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\alpha^2 + \beta^2) \end{pmatrix}$$

where:

$$A = \lambda_0 - \lambda[(\alpha^2 + \beta^2) r_1^2 + 2\alpha r_1 s_3 + s_3^2],$$
$$B = -\lambda[(\alpha^2 + \beta^2) r_1 r_2 + s_3(\beta r_1 + \alpha r_2)],$$
$$C = \lambda_0 - \lambda[(\alpha^2 + \beta^2) r_2^2 + 2\beta r_2 s_3 + s_3^2],$$
$$D = (2s_3 + \alpha r_1 + \beta r_2).$$

Since the rank of our matrix is 4 on the manifold under scrutiny, the equations

$$s_1 = s_2 = r_3 = 0, \ \lambda_0 - \lambda(s_3 + \alpha r_1 + \beta r_2)s_3 = 0 \quad (2.8)$$

due to Theorem 1 define a family of IMSMs for differential equations (2.5) of the body's motion.

Likewise in the example considered above, it can readily be verified that the equations

$$s_1 = s_2 = r_3 = 0$$

define the invariant manifold (IM) of system (2.5). The vector field on the given IM writes

$$\dot{s}_3 = (\beta r_1 - \alpha r_2)s_3, \quad \dot{r}_1 = r_2(\alpha r_1 + \beta r_2 + 2s_3),$$
$$\dot{r}_2 = -r_1(\alpha r_1 + \beta r_2 + 2s_3).$$

The latter differential equations have the first integral:

$$U = (s_3 + \alpha r_1 + \beta r_2)s_3 = const.$$

So, the family of IMSMs obtained above is the intersection of the hyperplane $s_1 = s_2 = r_3 = 0$ and the family of hypersurfaces corresponding to the first integral of energy H for arbitrary value of constant h.

It is obvious from the form of the matrix of coefficients that the rank of some part of submatrices of the matrix can decrease under some additional constraints imposed on the problem's variables. This takes place, for example, when $s_3 = 0$. It can easily be verified that under the above constraint and for $\lambda_0 = 0$ we obtain the following invariant manifold for the system of differential equations (2.5):

$$s_1 = s_2 = r_3 = s_3 = 0. \quad (2.9)$$

Obviously, IM (2.9) is a submanifold of the family of IMSMs (10). The vector field on the given IM writes:

$$\dot{r}_1 = r_2(\alpha r_1 + \beta r_2), \quad \dot{r}_2 = -r_1(\alpha r_1 + \beta r_2).$$

There is a stable equilibrium position $r_1 = r_2 = 0$ on the IM (2.9). In this case, this result may be interpreted as conditional stability of the body's equilibrium position with respect to some part of the variables $r_1, \ r_2$.

III. ON INVARIANT MANIFOLDS OF THE KOVALEVSKAYA TOP

As shown above, in some cases, the procedure of finding IMSMs proposed allows one to obtain sufficient conditions of stability for the IMSMs obtained by Lyapunov's second method. For example, in the case of the Kovalevskaya top, the equations of motion

$$2\dot{p} = qr, \quad 2\dot{q} = -pr + x_0\gamma_3, \quad \dot{r} = -x_0\gamma_2,$$
$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1,$$
$$\dot{\gamma}_3 = q\gamma_1 - p\gamma_2, \quad (3.10)$$

have the Kovalevskaya first integral:

$$2V = (p^2 - q^2 - x_0\gamma_1)^2 + (2pq - x_0\gamma_2)^2.$$

Assuming

$$\varphi_1 = p^2 - q^2 - x_0\gamma_1, \quad \varphi_2 = 2pq - x_0\gamma_2,$$

we can obtain the following representation for the integral:

$$2V = \varphi_1^2 + \varphi_2^2. \tag{3.11}$$

The conditions of stationarity for this integral with respect to the variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ write:

$$\frac{\partial V}{\partial p} = 2\varphi_1 p + 2\varphi_2 q = 0, \frac{\partial V}{\partial q} = -2\varphi_1 q + 2\varphi_2 p = 0,$$
$$\frac{\partial V}{\partial \gamma_1} = -\varphi_1 x_0 = 0, \frac{\partial V}{\partial \gamma_2} = -\varphi_1 x_0 = 0.$$

It can readily be seen, the rank of the matrix of coefficients for $\varphi_1, \quad \varphi_2$ in the expressions of the partial derivatives of V with respect to phase variables is 2. So, according to Theorem 1, the manifold $\varphi_1 = 0, \quad \varphi_2 = 0$ is the IMSM of the differential equations (3.10) of motion for a rigid body (these are well-known Delone invariant manifolds). Since integral (3.11) is a sign-definite function of $\varphi_1, \ \varphi_2$, this IMSM is stable.

References

1. Sokolov V.V. A new integrable case of Kirchhoff's equations // Theoretical and Mathematical Physics, 2001, Vol. 129, No.1, pp.1335-1340.

2. Zubov V.I. Oscillations and Waves.– Leningrad: Leningrad University Publishing, 1986.– 415 p.

3.Levi-Civita T., Amaldi U. Lezioni di Meccanica Rasionale.– Bologna, 1926.– 436 p.