DESIGN OF AMPLITUDE DEATH IN TIME-DELAY OSCILLATORS COUPLED BY A DELAYED CONNECTION

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Abstract

The present paper considers amplitude death in a pair of identical time-delay oscillators coupled by a delayed connection. A stability analysis allows us to derive a systematic procedure to design the connection (i.e., coupling strength and connection delay) for induction of amplitude death. The main advantage of this procedure is that the designed connection guarantees the induction of amplitude death even if the oscillators have long time delays. Our analytical results are verified by numerical examples.

Key words

amplitude death, time-delay nonlinear oscillators, stability analysis.

1 Introduction

Coupled oscillators have been investigated both from a viewpoint of academic interest (Endo and Mori, 1976; Pikovsky et al., 2001) and from a standpoint of engineering applications (Holmes et al., 2006; Ishiguro et al., 2006; Hong and Scaglione, 2005). For over two decades, amplitude death, which is a diffusive-coupling induced stabilization of unstable fixed points, has been investigated by many researchers (Yamaguchi and Shimizu, 1984; Aronson et al., 1990). It is known that this phenomenon never occurs in identical coupled oscillators (Aronson et al., 1990; Konishi, 2005). However, it was reported that a time-delay connection can induce death (Reddy et al., 1998). This time-delay induced death was experimentally observed in electronic circuits (Reddy et al., 2000) and thermo-optical oscillators (Herrero et al., 2000). Furthermore, it has been studied from various viewpoints (Reddy et al., 1999; Atay, 2003a; Atay, 2003b; Atay, 2006; Atay and Karabacak, 2006; Cheng, 2009; Yang, 2007; Konishi, 2003; Konishi, 2004; Konishi, 2005; Konishi et al., 2010a; Konishi et al., 2010b; Prasad, 2005; Dodla et al., 2004; Mehta and Sen, 2006; Song et al., 2007; Zou and Zhan, 2009).

It is well known that time delays in engineering nonlinear systems, such as a metal cutting process (Moon, 1998; Radons and Neugebauer, 2004) and contact rotating systems (Sowa et al., 2006), have a potential to induce harmful oscillations. Therefore, it is significant to achieve the stabilization of the time-delay induced oscillations. Since amplitude death can be regarded as a stabilization of unstable behavior in coupled oscillators, death must have a great deal of potential in engineering. Unfortunately, most of the previous studies on amplitude death focused on the oscillators which do not include delay times. In recent years, we analytically and experimentally investigated amplitude death in a pair of scalar time-delay nonlinear oscillators coupled by a delayed connection (Konishi et al., 2008). For practical situations where one wants to obtain death, a systematic procedure for designing the connection parameters (i.e., coupling strength and connection delay) is useful, since the procedure does not require a trial-and-error testing. Our previous study (Konishi et al., 2008), however, focused only on the stability analysis and did not provide such procedure.

The present paper deals with amplitude death in a pair of scalar time-delay nonlinear oscillators coupled by the delayed connection. A simple systematic procedure for designing the coupling strength and the connection delay is provided. Furthermore, these analytical results are verified by numerical simulations.

2 Time-delay nonlinear oscillators

Two identical scalar time-delay oscillators as illustrated in Fig. 1,

$$\dot{x}_{1,2} = -\alpha x_{1,2} + f(x_{1\tau,2\tau}) + u_{1,2}, \tag{1}$$

are considered. $x_{1,2} \in \boldsymbol{R}$ are the scalar variables, $f : \boldsymbol{R} \to \boldsymbol{R}$ is a nonlinear scalar function, $x_{1\tau,2\tau} := x_{1,2}(t-\tau)$ are the delayed variables, $u_{1,2} \in \boldsymbol{R}$ are coupling signals, and $\alpha > 0$ is a parameter. The individual nonlinear oscillator without coupling (i.e., $u_{1,2} \equiv 0$)

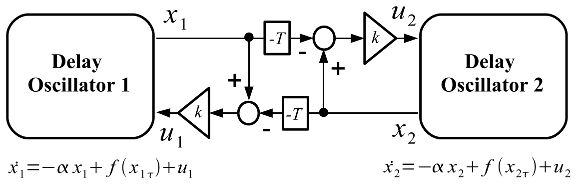

Figure 1. Block diagram of delay nonlinear oscillators coupled by a delayed connection.

has the fixed points,

$$x^* : 0 = -\alpha x^* + f(x^*). \tag{2}$$

Throughout this paper, all the fixed points x^* are assumed to be unstable.

Oscillators (1) are coupled by the delayed connection,

$$u_{1,2} = k(x_{1,2} - x_{2T,1T}), \tag{3}$$

where $x_{2T,1T} := x_{2,1}(t - T)$ are the delayed variables and $T \geq 0$ is the connection delay. The coupling strength is denoted by $k \in \boldsymbol{R}$. This connection cannot move the location of x^*, but can change its stability.

The characteristic equation of oscillators (1) with connection (3) at the fixed point x^* is described by

$$g(\lambda) := g_1(\lambda) g_2(\lambda) = 0, \tag{4}$$

where

$$g_1(\lambda) := \lambda + \alpha - k(1 - e^{-\lambda T}) - \beta(x^*) e^{-\lambda\tau}, \tag{5}$$

$$g_2(\lambda) := \lambda + \alpha - k(1 + e^{-\lambda T}) - \beta(x^*) e^{-\lambda\tau}. \tag{6}$$

Here $\beta(x^*) := \{\mathrm{d}f(x)/\mathrm{d}x\}_{x=x^*}$ is the derivative of $f(x)$ at $x = x^*$.

3 Design of delayed connection

According to our previous study (Konishi et al., 2008), the amplitude death at fixed point (2) never occurs if $\alpha < \beta(x^*)$ (i.e., odd number property for our system). In addition, if $|\beta(x^*)| < \alpha$ holds, fixed point (2) of oscillators (1) without coupling (i.e., $k = 0$) is stable for any $\tau \geq 0$ (Hale and Lunel, 1993). This condition contradicts our assumption: oscillators (1) without coupling have unstable fixed point (2). From these facts, the present paper assumes that the condition,

$$\beta(x^*) < -\alpha < 0, \tag{7}$$

always holds.

Let us show our main result: systematic procedure to design the coupling strength k and the connection delay T such that fixed point (2) is stable.

Theorem 1. *Assume that the oscillator parameters, α, $\beta(x^*)$, and τ, are given and condition (7) holds. Fixed point (2) of oscillators (1) coupled by delayed connection (3) is stable if the connection delay T is set to*

$$T = \frac{1}{2}\tau \tag{8}$$

and the coupling strength $k < 0$ is chosen from

$$k \in \left(4\beta - 2\sqrt{2\beta(\beta - \alpha)}, 4\beta + 2\sqrt{2\beta(\beta - \alpha)}\right). \tag{9}$$

Proof. Let us divide this proof into the following two cases: (i) $T = \tau = 0$; (ii) $T = \tau/2 \geq 0$. We shall prove that all the roots of $g(\lambda) = 0$ are in the open left-half complex plane for case (i) and the roots never cross the imaginary axis for case (ii). For case (i), substituting $T = \tau = 0$ into Eqs. (5) and (6), we have

$$g_1(\lambda) = \lambda + \alpha - \beta(x^*),\ g_2(\lambda) = \lambda + \alpha - 2k - \beta(x^*).$$

From condition (7), it is clear that, for $k < 0$, all the roots λ of $g_1(\lambda) = 0$ and $g_2(\lambda) = 0$ are in open left-half complex plane. For case (ii), substituting $\lambda = j\omega$ into Eqs. (5) and (6) yields $g_1(j\omega) = \mathrm{Re}[g_1(j\omega)] + j\mathrm{Im}[g_1(j\omega)]$ and $g_2(j\omega) = \mathrm{Re}[g_2(j\omega)] + j\mathrm{Im}[g_2(j\omega)]$. It is obvious that if at least one of $\mathrm{Re}[g_1(j\omega)] = 0$ and $\mathrm{Im}[g_1(j\omega)] = 0$ does not hold, $g_1(j\omega) = 0$ is not satisfied for any $\omega \in \boldsymbol{R}$ (i.e., all the roots of $g_1(\lambda) = 0$ never cross the the imaginary axis). The same holds true for $g_2(j\omega) = 0$. Now we show that $\mathrm{Re}[g_1(j\omega)] = 0$ under condition (8) does not hold for any $\omega \in \boldsymbol{R}$. Substituting condition (8) into $\mathrm{Re}[g_1(j\omega)]$, we have

$$\mathrm{Re}[g_1(j\omega)] = \alpha - k + \beta + h(\omega\tau),$$

where

$$h(\omega\tau) := k\cos\frac{\omega\tau}{2} - 2\beta\cos^2\frac{\omega\tau}{2}.$$

A simple algebraic computation allows us to obtain

$$h(\omega\tau) \geq \frac{k^2}{8\beta},\ \forall\omega \in \boldsymbol{R}.$$

Thus, if the condition,

$$\alpha - k + \beta + \frac{k^2}{8\beta} > 0 \Leftrightarrow k^2 - 8\beta k + 8\beta(\alpha + \beta) < 0, \tag{10}$$

is satisfied, then $\mathrm{Re}[g_1(j\omega)] = 0$ under condition (8) does not hold for any $\omega \in \boldsymbol{R}$. For $\mathrm{Re}[g_2(j\omega)] = 0$, we obtain the same condition (10). We see that k satisfying condition (10) is described by condition (9).

□

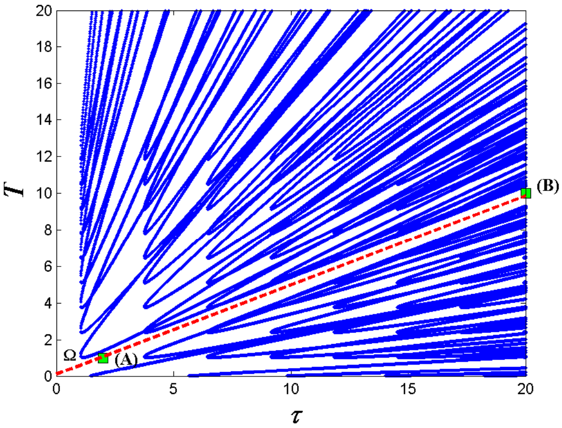

Figure 2. Stability region and stability boundary curves ($\alpha = 1.0, \beta = -1.8, k = -2$).

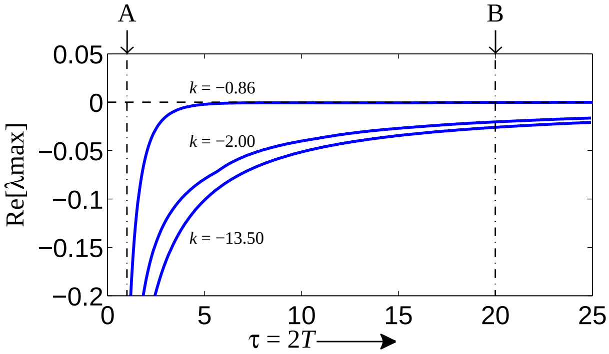

Figure 3. Largest real part of of equation (4) for $k = -0.86, -2.0, -13.50$.

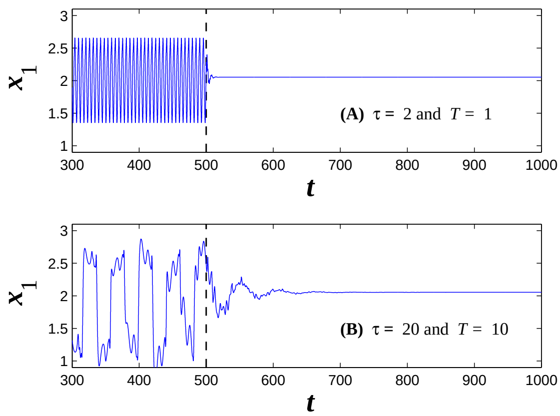

Figure 4. Time series data of $x_1(t)$ for two parameter set (A) and (B) just before and just after coupling ($\alpha = 1.0, \beta = -1.8, k = -2$).

Remark that condition (9) and Eq. (8) are independent of each other. Therefore, the designed k is valid for any $\tau > 0$.

4 Numerical examples

In this section, the analytical results derived in the preceding section is confirmed by numerical examples: time-delay oscillators with

$$\alpha = 1, \ \beta = -1.8. \tag{11}$$

The following nonlinear function is considered:

$$f(x) = \begin{cases} 0.95x + 1.4 & \text{if } x \leq 1.7 \\ -1.8x + 5.75 & \text{if } 1.7 < x \leq 4.3 \\ -1.82 & \text{if } x > 4.3 \end{cases}.$$

The individual oscillator has one fixed point $x^* = 2.21$. The time delay $\tau > 0$ is assumed to be arbitrary chosen.

Let us design the coupling strength k and the connection delay T according to Theorem 1. First, we confirm that parameters (11) satisfy assumption (7). This fact guarantees that there exist k and T for Theorem 1. Second, the connection delay is set as condition (8). Third, the coupling strength is chosen from condition (9): $k \in (-13.5498, -0.8502)$. This example chooses $k = -2$.

Figure 2 illustrates the boundary curves where the roots of Eq. (4) cross the imaginary axis. Since it is obvious that Eq. (4) does not have unstable roots at the origin (i.e., $T = \tau = 0$), we notice that there do not exist unstable roots in region Ω. It must be emphasized that Ω has a long strip including the dashed line given by condition (8). Theorem 1 guarantees that this strip exists for any $\tau > 0$ if k is chosen from condition (9). Figure 3 shows the largest real part of the roots of Eq. (4) for several coupling strengths satisfying condition (9), where the connection delay is set to condition (8). It can be seen that the largest real part never exceeds zero for any k and any $\tau > 0$. Figure 4 shows the time series data of the oscillators for two parameter sets: (A) ($\tau = 2, T = 1$) and (B) ($\tau = 20, T = 10$) in Fig. 2. For parameter set (A), the individual oscillator without coupling behaves periodically until $t = 500$. At $t = 500$ the two oscillators are coupled, and then their trajectories converge on x^*. Similarly, for parameter set (B), the individual oscillator before coupling behaves chaotically, and then x^* is stabilized after coupling.

5 Conclusion

This paper investigated amplitude death phenomenon in two time-delay oscillators coupled by a delayed connection. We proposed a procedure for designing the connection parameters (i.e., coupling strength and connection delay) based on the observation of the root locus movement of the characteristic equation. The numerical examples successfully verified our analysis results.

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