QUANTILE OPTIMIZATION PROBLEM WITH INCOMPLETE INFORMATION

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Abstract
A stochastic optimization problem with incomplete information is considered. Optimal solutions are selected using the minimax quantile criterion. This problem is related to a confidence estimation problem for a random vector with incompletely known distribution. Generalized confidence regions are used as confidence estimates for a statistically uncertain vector. The quantile stochastic optimization problem under incomplete information is solved by means of an optimal choice of a generalized confidence region.

Key words
Stochastic optimization, incomplete information, quantile criterion.

1 Introduction

Stochastic optimization problems with with incomplete information about distributions of random perturbations were studied in [Birge,Wets, 1997; Ermoliev, Gaivoronski and Neveda, 1986; Dupacova, 1986].

In this paper we deal with a quantile optimization problem under uncertainty. Stochastic optimization problems with the quantile criterion were studied in [Precopa, 1995; Kibzun, Kan, 1996]. The estimation problems for statistically uncertain systems were considered in [Kurzanski, Tanaka, 1989; Matasov, 1999; Kats, Timofeeva, 1995].

The quantile optimization problem is related to the confidence estimation problem. There are many confidence regions for a given random vector corresponding to the same probability level, so the confidence estimation problem is usually reformulated as the problem with quantile criterion. On the other hand a quantile optimization problem may be reduced to the generalized minimax problem by an appropriate choice of the confidence set for random perturbations.

In the paper two connected problems are studied:

– to define and to construct generalized confidence regions for a random vector with incompletely known distribution;
– to solve the quantile optimization problem depended on both random and uncertain nonrandom parameters.

2 Problem statement

Let us consider a quantile optimization problem under incomplete information on the random parameter distribution. Let $\tilde{\xi}(\omega, K_\mu)$ be a random vector with incompletely known distribution, i.e. its distribution function $\mu(\cdot)$ is unknown and belongs to the set of possible distributions K_μ.

A function $F(u, \tilde{\xi}(\omega, K_\mu))$ is to be minimized on $u \in U$. Here U is a set of all possible solutions u, the function $F(u, y)$ is measurable with respect to y for any $u \in U$.

Let consider a quantile criterion for the stochastic optimization problem:

$$\tilde{q}_\alpha(u) \to \min, \quad u \in U, \tag{1}$$

where $\tilde{q}_\alpha(u)$ is the worst quantile:

$$\tilde{q}_\alpha(u) = \min\{q : \mathcal{P}\{F(u, \tilde{\xi}(\omega, K_\mu)) \le q\} \ge \alpha\},$$

$$\mathcal{P}\{\tilde{\xi}(\omega, K_\mu) \in B\} \triangleq \inf_{\mu(\cdot) \in K_\mu} P\{\xi(\omega, \mu(\cdot)) \in B\}.$$

We assume further that the measure $\mu(\cdot)$ depends on the uncertain parameter z and the set K_μ has the form: $K_\mu = \{\mu(\cdot, z) \mid z \in Z\}$, where $Z \in \mathbb{R}^m$ is a given compact set of possible values of the parameter z.

In this case the generalized quantile problem (1) may be rewritten as

$$\max_{z \in Z} q_\alpha(u, z) \to \min, \quad u \in U, \tag{2}$$

$$q_\alpha(u,z) = \min\{q : P\{F(u,\xi(\omega,z)) \le q\} \ge \alpha\}.$$

The problem (2) is a quantile optimization problem depended on a random perturbation with a known distribution and on an uncertain parameter.

For example, let us consider a quantile optimization problem (2) for a function $F(u,\tilde{\xi}(\omega,Z))$, where

$$F(u,y) = \max_{j=1,\dots,k} \psi_j(u,y),$$
$$\psi_j(u,y) = a_j^T u - c_j^T y - b_j, \qquad (3)$$
$$\tilde{\xi}(\omega,Z) = \{\eta(\omega) + z \mid z \in Z\},$$

$\eta(\omega)$ is a Gaussian random vector with known statistical moments, $z \in Z$ is an uncertain parameter, $Z \subset \mathbb{R}^n$ is a given convex compact set, $a_j \in \mathbb{R}^l$, $c_j \in \mathbb{R}^n$, $b_j \in \mathbb{R}^1$ are given vectors.

The quantile optimization problem (2)–(3) is rather complicated since the quantile function is not convex or concave [Precopa, 1995]. It is known that a quantile optimization problem cannot be considered as the statistical moment problem.

If we solve the probability optimization problem for arbitrary q:

$$\max_{u\in\mathbb{R}^n} \min_{z\in Z} \alpha_q(u,z),$$
$$\alpha_q(u,z) = P\{F(u,z+\eta(\omega)) \le q\}, \qquad (4)$$

then we could get [Timofeeva, 2007] a solution of the quantile minimization problem. There are special algorithms [Szantai, 1988] to calculate and estimate the probability $P\{\eta(\omega) \in K\}$ for the polyhedron $K \subset \mathbb{R}^n$. In the considered problem (3) this polyhedron is defined by equation

$$K = K(u,z,q) = \{y \in \mathbb{R}^n : F(u,z+y) \le q\}.$$

The probability optimization problem (4) may be solved as the generalized moment problem with uncertainty on the base of methods and algorithms proposed in [Ermoliev, Gaivoronski and Neveda, 1986]. But this approach demands a significant volume of calculations.

The proposed approach allows to find a suboptimal solution of the quantile optimization problem and to estimate the optimal quantile.

3 Properties of the generalized confidence sets

Let us definite a notion of a statistically uncertain random vector and consider its confidence estimate.

Definition 1. [Timofeeva, 2002] *A map $\tilde{\xi}(\omega,Z) : \Omega \times Z \to \mathbb{R}^n$ is called a statistically uncertain random vector if:*

1. *the function $\xi(\omega,z)$ is a random vector for any fixed $z \in Z$, i.e. the set $\{\omega : \xi(\omega,z) \in B\} \in \mathcal{A}$ is measurable for any $B \in \mathcal{B}^{(n)}$, $z \in Z$;*
2. *the probability $P_z(B) = P\{\xi(\omega,z) \in B\}$ is a continuous function with respect to z for any fixed $B \in \mathcal{B}^{(n)}$;*
3. *the set Z is a compact set consisted of more than one point.*

Let $\tilde{\xi}(\omega,Z)$ be a statistically uncertain continuous random vector, $\{X_z^\alpha \mid z \in Z\}$ be a family of confidence sets with the level α, i.e. for any X_z^α the relation

$$P\{\xi(\omega,z) \in X_z^\alpha\} = \alpha$$

holds. Denote by $\hat{X}_\alpha$ the union of the confidence sets

$$\hat{X}_\alpha = \bigcup_{z\in Z} X_z^\alpha. \qquad (5)$$

Usually the union $\hat{X}_\alpha$ of confidence regions corresponding to all permissible distributions is taken as a confidence region in statistically uncertain case. It is shown in [Timofeeva, 2002] that this estimator may be improved in the most cases by means of generalized confidence regions.

Definition 2. *A measurable set $\tilde{X}_\alpha \subset \mathbb{R}^{(n)}$ is called a generalized confidence set with level α for a statistically uncertain random vector $\tilde{\xi}(\omega,Z)$, if*

$$\mathcal{P}\{\tilde{\xi}(\omega,Z) \in \tilde{X}_\alpha\} = \min_{z\in Z} P\{\xi(\omega,z) \in \tilde{X}_\alpha\} = \alpha.$$

Generalized confidence sets (as well as standard confidence sets) are not uniquely defined: there are many generalized confidence regions corresponding to a fixed probability α.

Theorem 1. *If the union $\hat{X}_\alpha$ of the confidence sets is a measurable set then it is a generalized confidence set with a level $\alpha_1 \ge \alpha$ for the statistically uncertain continuous random vector $\tilde{\xi}(\omega,Z)$. The equality $\alpha_1 = \alpha$ holds if and only if there exists a parameter $z^* \in Z$ such that*

$$P\{\xi(\omega,z^*) \in \hat{X}_\alpha\} = \alpha. \qquad (6)$$

Proof. The equality $\hat{X}_\alpha = \bigcup_{z\in Z} X_z^\alpha$ implies the relation

$$\min_{z\in Z} P\{\xi(\omega,z) \in \hat{X}_\alpha\} \ge \min_{z\in Z} P\{\xi(\omega,z) \in X_z^\alpha\} = \alpha.$$

It means that $\hat{X}_\alpha$ is the generalized confidence set with a level $\alpha_1 \ge \alpha$. If we can find z^* such that (6) holds, then

$$\alpha_1 = \mathcal{P}\{\tilde{\xi}(\omega,Z) \in \hat{X}_\alpha\} \le P\{\xi(\omega,z^*) \in \hat{X}_\alpha\} = \alpha,$$

and therefore $\alpha_1 = \alpha$.

Let for all $z \in Z$, the inequality $P\{\xi(\omega, z) \in \hat{X}_\alpha\} < \alpha$ holds. The set Z is closed and the probability $P\{\xi(\omega, z) \in \hat{X}_\alpha\}$ continuously depends on z since $\tilde{\xi}(\omega, Z)$ is statistically uncertain random vector. Thus

$$\min_{z \in Z} P\{\xi(\omega, z) \in \hat{X}_\alpha\} < \alpha,$$

and $\hat{X}_\alpha$ is a confidence set with the level $\alpha_1 < \alpha$. □

For the same statistically uncertain vector a union of confidence sets may be a generalized confidence set with the same probability level or with a greater level. It depends on the forms of the confidence sets.

Theorem 2. *Let $\tilde{X}_\alpha$ be the generalized confidence set with the level α for the statistically uncertain random vector $\tilde{\xi}(\omega, Z)$ then there are confidence sets $X^*_\alpha(z)$ with the level α such that*

1. *$P\{\xi(\omega, z) \in X^*_\alpha(z)\} = \alpha$ for any $z \in Z$;*
2. *$\bigcup_{z \in Z} X^*_\alpha(z) = \tilde{X}_\alpha$;*
3. *there is $z^* \in Z$ such that $X^*_\alpha(z^*) = \tilde{X}_\alpha$.*

Proof. The third condition follows from definition of the generalized confidence set: if $\mathcal{P}\{\tilde{\xi}(\omega, Z) \in \tilde{X}_\alpha\} = \alpha$, then

$$\min_{z \in Z} P\{\xi(\omega, z) \in \tilde{X}_\alpha\} = \alpha.$$

Denote by $z^* \in Z$ the minimizer of the probability, i.e.

$$P\{\xi(\omega, z^*) \in \tilde{X}_\alpha\} = \min_{z \in Z} P\{\xi(\omega, z) \in \tilde{X}_\alpha\} = \alpha.$$

It means that $\tilde{X}_\alpha$ is a confidence set with the level α for $\xi(\omega, z^*)$ and the second condition holds. Let us choose any $z_1 \in Z$ and denote

$$\alpha_1(z_1) = P\{\xi(\omega, z_1) \in \tilde{X}_\alpha\} \geq \alpha.$$

If $\alpha_1(z_1) = \alpha$ then we choose $X^*_\alpha(z_1) = \tilde{X}_\alpha$. If $\alpha_1(z) > \alpha$ then we can find a measurable set $X^*_\alpha(z_1) \subset \tilde{X}_\alpha$ such that $P\{\xi(\omega, z_1) \in X^*_\alpha(z_1)\} = \alpha$ since the random vector $\xi(\omega, z)$ has continuous distribution. Thus we have constructed a family of $X^*_\alpha(z) \subset \tilde{X}_\alpha$ such that

$$\bigcup_{z \in Z} X^*_\alpha(z) = \tilde{X}_\alpha. \ \square$$

The next statement [Timofeeva, 2002] follows from the properties of the probability function.

Theorem 3. *Let $\tilde{\xi}(\omega, z) = \{z + \eta(\omega) \mid z \in Z\}$ be a statistically uncertain vector and the following conditions hold:*

1. *$\eta(\omega)$ is a continuous random vector with a given density function $f_\eta(x)$;*
2. *$f_\eta(x) > 0$ for all $x \in \mathbb{R}^n$;*
3. *$Z \subset \mathbb{R}^n$ is a given convex compact set;*
4. *B_α is a convex compact confidence region with a level $\alpha \in (0.5; 1)$ for $\eta(\omega)$: $P\{\eta(\omega) \in B_\alpha\} = \alpha$;*
5. *$0 \in int(B)$, where $int(B)$ is the set of all interior points of B.*

Then there exists $\varepsilon \in (0, 1)$ such that the set $Z + \varepsilon B_\alpha$ is a generalized confidence region of probability α for $\tilde{\xi}(\omega, Z)$.

The following simple example illustrates the properties of generalized confidence regions.

Example 1. Let

$$\tilde{\xi}(\omega, Z) = \{z + \eta(\omega) \mid z \in Z\},$$

where Z is the the interval $Z = [-a, a]$, $\eta(\omega)$ is a normal distributed random value with given statistical moments: $E\xi = 0, \ E\xi^2 = \sigma^2$.

The set $X_\alpha(0) = [-t_{0.5\alpha}\sigma, t_{0.5\alpha}\sigma]$, is a confidence set for $\eta(\omega)$ with probability α. Here

$$\Phi(t_\alpha) = \alpha, \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dz.$$

The sets

$$X_\alpha(z) = z + X_\alpha(0), \ \ z \in Z$$

are confidence sets with the level α for $\xi(\omega, z) = z + \eta(\omega)$, but

$$\hat{X}_\alpha = \bigcup_{z \in Z} X_\alpha(z) = Z + X_\alpha(0)$$

is not a generalized confidence set with the same level for $\tilde{\xi}(\omega, Z)$ since

$$\min_{z \in [-a,a]} P\{z + \eta(\omega) \in \hat{X}_\alpha\} > \alpha.$$

Let us construct a symmetrical confidence set $X^*_\alpha(z)$ for $\xi(\omega, z)$. The set

$$X^*_\alpha(z) = [-|z| - \Delta_\alpha; |z| + \Delta_\alpha]$$

is a confidence set with the level α if

$$\Delta_\alpha = \sigma \mathrm{gal}(\alpha, \sigma^{-1}|z|)$$

and $g = \mathrm{gal}(\alpha, v)$ is the root of the equation

$$\Phi(g + 2v) + \Phi(g) = \alpha. \tag{7}$$

From the inclusion $X^*_\alpha(z_1) \subset X^*_\alpha(z_2)$ for $|z_2| > |z_1|$, it follows that

$$\bigcup_{z\in Z} X^*_\alpha(z) = X^*_\alpha(\max_{z\in Z}|z|) = X^*_\alpha(a),$$

and the generalized confidence set is

$$\tilde{X}_\alpha = [-\sigma\mathrm{gal}(\alpha,\sigma^{-1}a) - a, a + \sigma\mathrm{gal}(\alpha,\sigma^{-1}a)].$$

The function $\mathrm{gal}(\alpha, v)$ increases on α and decreases on v. Since $\Phi(g) < \Phi(g + 2v) < 0.5$ then from (7) it follows that $t_{\alpha-0.5} < \mathrm{gal}(\alpha, v) < t_{\alpha/2}$ for all $v > 0$. Therefore

$$\tilde{X}_\alpha \subset \hat{X}_\alpha = Z + X_\alpha(0).$$

On the other hand, if we take one-sided confidence regions $Y_\alpha(0) = [-\infty, t_{\alpha-0.5}\sigma]$ for $\eta(\omega)$, then the union

$$\hat{Y}_\alpha = \bigcup_{z\in Z}(z + Y_\alpha(0)) = Z + Y_\alpha(0)$$

is the generalized confidence set with probability α for $\tilde{\xi}(\omega, Z)$:

$$\min_{z\in} P\{\tilde{\xi}(\omega, Z) \in \hat{Y}_\alpha\} = \alpha.$$

The generalized confidence sets for Gaussian n-vector with the incompletely known mean value have the same properties [Timofeeva, 2002].

4 Statistically uncertain quantile optimization problem

Properties of the optimal quantile for stochastic optimization problem with complete information about distributions were studied in [Precopa, 1995; Kibzun, Kan, 1996].

Lemma 1. [Kibzun, Kan, 1996] *Let $\eta(\omega)$ be a continuously distributed random n-vector and $F(y)$ be a measurable function $\mathbb{R}^n \to \mathbb{R}^1$, then the quantile*

$$q^*_\alpha = \min\{q : P\{F(\eta(\omega)) \le q_\alpha\} \ge \alpha$$

satisfies the relation

$$q^*_\alpha = \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} F(y), \tag{8}$$

where $\mathcal{E}_\alpha$ is the family of all confidence sets with level not less than α for $\eta(\omega)$:

$$\mathcal{E}_\alpha = \{E_\alpha \in \mathcal{B}^{(n)} : P\{\eta(\omega) \in E_\alpha\} \ge \alpha\}. \tag{9}$$

Lemma 2. [Kibzun, Kan, 1996] *Let a function $F(v, y)$ be continuous on $V \times \mathbb{R}^n$, $V \subset \mathbb{R}^l$ be a closed set and*

$$P\{|F(v, \eta(\omega)) - q| \le \varepsilon\} > 0 \tag{10}$$

for all $v \in V,\ q \in (q_-(v), q_+(v))$, where

$$q_-(v) = \inf_{y\in\mathbb{R}^n} F(v, y),\ q_+(v) = \sup_{y\in\mathbb{R}^n} F(v, y)$$

Then quantile $q_\alpha(v)$ is continuous with respect to v for all $\alpha \in (0; 1)$.

Let us return to the stochastic optimization problem (2) with the quantile criterion and incomplete information.

Let $\xi(\omega, z) = \varphi(\eta(\omega), z)$ for all $z \in Z$, where the function $\varphi(y, z)$ is measurable on y and continuous on z, $\eta(\omega)$ is a random vector with a given continuous distribution, then $F(u, \varphi(y, z)) = F_1(u, z, y)$ and the quantile optimization problem (2) has a form:

$$\max_{z\in Z} q_\alpha(u, z) \to \min, u \in U, \tag{11}$$

$$q_\alpha(u, z) = \min\{q : P\{F_1(u, z, \eta(\omega)) \le q\} \ge \alpha\}.$$

Theorem 4. *Let $\eta(\omega)$ be a continuous random vector, $Z \subset \mathbb{R}^m$ be a compact set and the conditions of Lemma 2 are carry out for the function $F_1(u, z, y)$ and $v = \{u, z\} \in U \times Z = V$. Then the optimal quantile in problem (2)*

$$\tilde{q}^*_\alpha = \inf_{u\in U} \max_{z\in Z} q_\alpha(u, z) \tag{12}$$

is equal to

$$\tilde{q}^*_\alpha = \inf_{u\in U} \max_{z\in Z} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} F(u, y), \tag{13}$$

where $\mathcal{E}_\alpha$ is the family of confidence sets (9).

Proof. From equality (8) it follows that

$$q_\alpha(u, z) = \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} F_1(u, z, y).$$

Since the conditions of Lemma 2 hold then the quantile function $q_\alpha(u, z)$ is continuous on the compact set Z and there exist an optimal vector $z^*(u)$ such that $q(u, z^*(u)) = \min_{z\in Z} q(u, z)$.

Thus stochastic problem (2) is reduced to a generalized minimax deterministic problem as it had been made for quantile optimization problem with complete information in [Kibzun, Kan, 1996]. □

The obtained problem (13) seems more difficult than the initial quantile optimization problem (2). But one can take an appropriate family of confidence sets $\{E^1_\alpha(z) \mid z \in Z\}$, then solve the problem

$$q^1_\alpha = \inf_{u\in U} \max_{z\in Z} \max_{y\in E^1_\alpha(z)} F_1(u,z,y), \qquad (14)$$

and consider its solution u^1 as a suboptimal solution of the quantile optimization problem (2), and obtain an estimate of the optimal quantile

$$q^1_\alpha \ge \tilde{q}^*_\alpha.$$

The problem is how to choose the family of the confidence sets.

If we take the same confidence set for all $z \in Z$ then $q^1_\alpha \ge \hat{q}_\alpha$, where

$$\hat{q}_\alpha = \inf_{u\in U} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{z\in Z} \max_{y\in E_\alpha} F_1(u,z,y). \qquad (15)$$

Criterion (15) was considered [Kibzun, Kan, 1996] for the statistically uncertain quantile optimization problem.

Obviously an inequality

$$\tilde{q}^*_\alpha \le \hat{q}_\alpha. \qquad (16)$$

is carried out.

Let us note that inequality (16) is as a rule a strict inequality and formulate sufficient conditions for the equality.

Theorem 5. *Let the conditions of Theorem 4 hold and for any $u \in U$ there exists $z^* = z^*(u) \in Z$ such that*

$$\max_{z\in Z} F_1(u,z,y) = F_1(u,z^*(u),y)$$

for all $y \in \mathbb{R}^n$, then the minima of the criteria coincide: $\tilde{q}^*_\alpha = \hat{q}_\alpha$.

Proof. It follows from condition of the Theorem that

$$\hat{q}_\alpha = \inf_{u\in U} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} \max_{z\in Z} F_1(u,z,y) =$$

$$= \inf_{u\in U} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} F_1(u,z^*(u),y) \le$$

$$\le \inf_{u\in U} \max_{z\in Z} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} F_1(u,z,y) = \tilde{q}^*_\alpha.$$

Since the inequality $\tilde{q}^*_\alpha \le \hat{q}_\alpha$ is carried out in any case, we get $\tilde{q}^*_\alpha = \hat{q}_\alpha$. □

Example 2. Let us consider the problem of minimization of the function

$$F_1(u,z,\eta(\omega)) = |z + u + \eta(\omega)|,$$

where $z \in Z = [z_0 - a, z_0 + a]$ is the an incompletely known parameter, $u \in \mathbb{R}^1$, $\eta(\omega)$ is a random perturbation. Let $\eta(\omega)$ have the normal distribution with the known parameters $E\eta = 0, E\eta^2 = \sigma^2$. We choose a control according to the minimax quantile criterion (12):

$$\tilde{q}^*_\alpha = \min_{u\in U} \max_{z\in Z} q_\alpha(u,z)$$

$$q_\alpha(u,z) = \{q : P\{|z + u + \eta(\omega)| \le q\} \ge \alpha.\}$$

According to Theorem 3 we have :

$$\tilde{q}^*_\alpha = \min_{u} \max_{z\in Z} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{y\in E_\alpha} |z + u + y|.$$

From properties of the probability function [Kibzun, Kan, 1996] it follows that the optimal value

$$q_\alpha(u,z) = \min_{E\in\mathcal{E}_\alpha} \max_{y\in E} |z + u + y|$$

is reached on a symmetrical confidence set for any fixed u, z. According to Example 1 we get

$$q_\alpha(u,z) = \sigma \mathrm{gal}(\alpha, |u + z| \cdot \sigma^{-1}) + |u + z|$$

and

$$\tilde{q}^*_\alpha = \min_{u} \max_{z\in Z} (|u + z| + \sigma \mathrm{gal}(\alpha, \sigma^{-1}|u + z|)).$$

Since the quantile $q_\alpha(u,z) = Q_\alpha(|u+z|)$ is monotone with respect to $v = |u + z|$ then the minimax value is reached at the saddle point of the problem

$$\min_{u} \max_{z\in Z} |u + z|,$$

i.e. at the point (u^*, z^*), where $u^* = -z_0$, $z^* = z_0 \pm a$. The optimal quantile is equal to

$$\tilde{q}^*_\alpha = a + \sigma \mathrm{gal}(\alpha, \sigma^{-1}a).$$

If we consider the random and uncertain perturbations together (see (15)), then

$$\hat{q}_\alpha = \min_{u} \min_{E_\alpha\in\mathcal{E}_\alpha} \max_{z\in Z} \max_{y\in E_\alpha} |u + z + y| = a + t_{0.5\alpha} \cdot \sigma.$$

In the considered problem

$$\hat{q}_\alpha < \tilde{q}^*_\alpha \quad \text{for all} \quad a > 0, \quad \alpha \in (0.5, 1).$$

The similar effect is observed in the most stochastic quantile optimization problems with uncertainty.

The optimization over confidence sets can be substituted by the optimization over generalized confidence sets.

Theorem 6. *Let $\tilde{\xi}(\omega, Z)$ be a statistically uncertain continuous random vector, a function $F(u, y)$ be continuous then*

$$\tilde{q}^*_\alpha = \inf_{u \in U} \min_{\tilde{E}_\alpha \in \tilde{\mathcal{E}}_\alpha} \max_{y \in \tilde{E}_\alpha} F(u, y), \tag{17}$$

where $\tilde{\mathcal{E}}_\alpha \subset \mathcal{B}^{(n)}$ is the family of the generalized confidence sets $\tilde{E}_\alpha$ with level not less than α:

$$\tilde{\mathcal{E}}_\alpha = \{\tilde{E}_\alpha \in \mathcal{B}^{(n)} : \mathcal{P}\{\tilde{\xi}(\omega, Z) \in \tilde{E}_\alpha\} \geq \alpha\}.$$

Proof. Any generalized confidence set $\tilde{E}_\alpha$ can be presented as a union of confidence sets $E^*_\alpha(z)$ with the same level. There is $z^* \in Z$ such that $E^*_\alpha(z^*) = \tilde{E}_\alpha$ (see Theorem 2). Therefore for any fixed $u \in U$ the relation

$$\min_{\tilde{E}_\alpha \in \tilde{\mathcal{E}}_\alpha} \max_{y \in \tilde{E}_\alpha} F(u, y) = \max_{z \in Z} \min_{E_\alpha \in \mathcal{E}_\alpha} \max_{y \in E_\alpha} F(u, y)$$

holds. □

The Theorem 6 allows us to find an optimal family of the confidence sets $E_\alpha(z)$ for the estimate (14) of the optimal quantile.

The exact minimax solution of the minimax problem (17) requires a significant calculations. But we can find a suboptimal solution and estimate the optimal quantile.

For example, let us consider again the quantile optimization problem for function (3) and take a generalized confidence region Y^1_α of the given level α for statistically uncertain vector $\tilde{\xi}(\omega, Z) = \{z + \eta(\omega) \mid z \in Z\}$, e.g.:

$$Y^1_\alpha = \{y \in \mathbb{R}^n : \|y\| \leq r(\alpha, Z)\}$$

or $Y^1_\alpha = \{y \in \mathbb{R}^n : c_j^T y \leq \gamma_j(\alpha, Z) \mid j \in J\}$, where $J \subset \{1, \ldots, k\}$.

The optimal quantile is estimated by

$$\tilde{q}^*_\alpha(u) \leq q^1_\alpha(u_1) = \min_{u \in U} \max_{y \in Y^1_\alpha} F(u, y). \tag{18}$$

Here u_1 is the solution of a standard minimax problem:

$$q^1(u_1) = \min_{u \in U} \max_{y \in Y^1_\alpha} F(u, y) = \min_{u \in U} \max_j \max_{y \in Y^1_\alpha} \psi_j(u, y),$$

where the functions $\psi_j(u, y)$ are linear with respect to y and u. Of course, the estimate (18) depends on the generalized confidence set Y^1_α and can be improved by an appropriate choice of the set Y^1_α.

5 Conclusion

The quantile optimization for problem with incomplete information about random parameters distributions is considered. The problem is reduced to the problem of the optimal choice of generalized confidence region for statistically uncertain vector.

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