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ON CONTROL PROBLEM WITH CONSTRAINTS OF ASYMPTOTIC CHARACTER

Alexander Chentsov
N.N. Krasovskii Institute of Mathematics and Mechanics
Russian Federation
Ural Federal University
Russian Federation
chentsov@imm.uran.ru

Artem Baklanov
International Institute for Applied Systems Analysis
Austria
N.N. Krasovskii Institute of Mathematics and Mechanics
Russian Federation
Ural Federal University
Russian Federation
baklanov@iiasa.ac.at

Ilya Savenkov
N.N. Krasovskii Institute of Mathematics and Mechanics
Russian Federation
Ural Federal University
Russian Federation
slaeme@yandex.ru

Abstract

For the double integrator with a discontinuous coefficient at the control, we obtain an attraction set—an asymptotic variant of reachable sets corresponding to constraints of asymptotic character—and study the latter's properties.

These constraints correspond to a control mode in the class of short-time pulses. An additional requirement is to fully consume the available energy resources. To calculate the attraction set numerically, an algorithm was developed and implemented. The results of computational experiments are presented.

Key words

Finitely additive measures, attraction set, constraints of asymptotic character, ultrafilters.

1 Introduction

The paper explores an abstract reachability problem, which encompasses the problem of constructing reachable sets for linear control systems with discontinuous coefficients at the control.

We consider two types of constraints of asymptotic character: the ones generated by a sequential relaxation of standard constraints (e.g., phase constraints) and the ones defined 'naturally' as in the case of short-time control pulses. Specifically, we study a control problem with relaxed phase constraints and the requirement to fully utilize all available energy resources during vanishingly small time. We study attraction sets (ASs), asymptotic versions of reachable sets, which can be considered as a more robust estimate of reachable sets given a potential relaxation of the constraints.

2 General notation

We use the standard set-theoretic notation. We call a "family" a set in which all elements are sets. The pair set of y, z is denoted by $\{y; z\}$; $\{h\}$ is the singleton containing h; an ordered pair $z = (x, y)$ has $\mathrm{pr}_1(z) = x$ as its first element and $\mathrm{pr}_2(z) = y$ as the second one; obviously, $z = \big(\mathrm{pr}_1(z), \mathrm{pr}_2(z)\big)$.

By $\mathcal{P}(X)$ (by $\mathcal{P}'(X)$) we denote the family of all (the family of all nonempty) subsets of a set X. By definition, B^A is the set of all mappings from a set A to a set B. If $g \in B^A$ and $C \in \mathcal{P}(A)$, then $g^1(C) \stackrel{\triangle}{=} \{g(x) : x \in C\} \in \mathcal{P}(B)$ is the image of C under g. Further, $\mathbb{N} \stackrel{\triangle}{=} \{1; 2; \ldots\}$ and $J[k] \stackrel{\triangle}{=} \{l \in \mathbb{N} |\, l \leqslant k\}\ \ \forall k \in \mathbb{N}$. If T is a set and $k \in \mathbb{N}$, then, as per the common convention, we write T^k instead of $T^{J[k]}$. By $(\mathrm{top})[S]$, we denote the family of all topologies on a set S; if $\tau \in (\mathrm{top})[S]$, then (S, τ) is a topological space; if $H \in \mathcal{P}(S)$, then $\mathrm{cl}(H, \tau)$ stands for the closure of H in (S, τ). If (S, τ) is a topological space and $M \in \mathcal{P}(S)$, then $\tau\big|_M \stackrel{\triangle}{=} \{M \cap G : G \in \tau\} \in (\mathrm{top})[M]$, and $(M, \tau\big|_M)$ is a subspace of (S, τ). Let $(\tau - \mathrm{comp})[S]$ stand for the family of all nonempty and compact (in (S, τ)) subsets of S.

If E is a set, then $\beta[E] \stackrel{\triangle}{=} \{\mathcal{E} \in \mathcal{P}'\big(\mathcal{P}(E)\big) |\, \forall \Sigma_1 \in \mathcal{E}\ \forall \Sigma_2 \in \mathcal{E}\ \exists \Sigma_3 \in \mathcal{E} : \Sigma_3 \subset \Sigma_1 \cap \Sigma_2\}$ stands for the family of all nonempty directed subfamilies of $\mathcal{P}(E)$.

Assume that X are Y nonempty sets, $\mathcal{X} \in \mathcal{P}'\big(\mathcal{P}(X)\big)$, $\tau \in (\mathrm{top})[Y]$, and $r \in Y^X$. Then, we define AS $(\mathbf{as})[X; Y; \tau; r; \mathcal{X}]$ as in [Chentsov, 2013a, Section 3]. Sequential AS $(\mathbf{sas})[X; Y; \tau; r; \mathcal{X}]$ is defined by only the sequential limits of points in Y. Note that, if $\mathcal{X} \in \beta[X]$, then

$$(\mathbf{as})[X; Y; \tau; r; \mathcal{X}] = \bigcap_{S \in \mathcal{X}} \mathrm{cl}\big(r^1(S), \tau\big).$$

2.1 Finitely additive measures and ultrafilters

We define $I \triangleq [a,b]$ where $a,b \in \mathbb{R}$ and $a < b$. By $\mathcal{I}$, we denote the family of sets $L \in \mathcal{P}(I)$ such that $\exists c \in I \ \ \exists d \in I : (]c,d[\subset L)\&(L \subset [c,d])$. Let $\mathcal{A}$ be the algebra of subsets of I generated by the semi-algebra $\mathcal{I}$. Let $\chi_L \in \mathbb{R}^I$ be the indicator functions of sets $L \in \mathcal{P}(I)$. Then, $B_0(I,\mathcal{A})$ denotes the linear span of $\{\chi_A : A \in \mathcal{A}\}$. Note that $B_0(I,\mathcal{A})$ is a (linear) manifold in the Banach space $\mathbb{B}(I)$ of all bounded real-valued functions on I endowed with the standard sup-norm [Dunford and Schwartz, 1958, p. 261 of the Russian translation], which we denote by $\|\cdot\|$. Let $B(I,\mathcal{A})$ stand for the closure of $B_0(I,\mathcal{A})$ in $(\mathbb{B}(I),\|\cdot\|)$. Note that $B(I,\mathcal{A})$ with the norm induced by $(\mathbb{B}(I),\|\cdot\|)$ is itself a Banach space, whose topological dual $B^*(I,\mathcal{A})$ is isometrically isomorphic to the space $\mathbb{A}(\mathcal{A})$ of all bounded finitely additive measures on $\mathcal{A}$ endowed with the (strong) norm-variation. Moreover, the isometric isomorphism between $\mathbb{A}(\mathcal{A})$ and $B^*(I,\mathcal{A})$ is defined by the rule

$$\mu \longmapsto \Big(\int_I f\, d\mu\Big)_{f\in B(I,\mathcal{A})} : \ \mathbb{A}(\mathcal{A}) \to B^*(I,\mathcal{A});$$

in the paper, integration is defined through the basic scheme [Chentsov, 2009, Ch. 3]. Assume that $\mathbb{A}(\mathcal{A})$ is endowed with the $*$-weak topology $\tau_*(\mathcal{A})$ corresponding to the duality $\big(B(I,\mathcal{A}),\mathbb{A}(\mathcal{A})\big)$. Thus, $(\mathbb{A}(\mathcal{A}),\tau_*(\mathcal{A}))$ is a locally convex σ-compact space. We will also deal with the topology $\tau_0(\mathcal{A})$ of a subspace of the topological power of $\mathbb{R}$ with the discrete topology with $\mathcal{A}$ as the index set; see the definition of $\tau_0(\mathcal{A})$ in [Chentsov, 1996, (4.2.9)]. Let $(\mathrm{add})_+[\mathcal{A}]$ be the set of all real-valued non-negative finitely additive measures on $\mathcal{A}$; $(\mathrm{add})_+[\mathcal{A}] \subset \mathbb{A}(\mathcal{A})$. Further, $\mathbb{P}(\mathcal{A})$ stands for the set of all finitely additive probabilities; precisely, $\mathbb{P}(\mathcal{A}) \triangleq \{\mu \in (\mathrm{add})_+[\mathcal{A}] \,|\, \mu(I) = 1\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})]$. By definition, put

$$\begin{gathered}\mathbb{T}(\mathcal{A}) \triangleq \{\mu \in \mathbb{P}(\mathcal{A})\,| \\ \forall A \in \mathcal{A} \ \big(\mu(A)=0\big)\vee\big(\mu(A)=1\big)\} \in \\ \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})].\end{gathered}$$

Let $\mathbb{F}_0^*(\mathcal{A})$ be the set of all ultrafilters in the algebra $\mathcal{A}$ (see [Chentsov, 2011a, (3.2)]). For all $\mathcal{L} \in \mathcal{P}(\mathcal{A})$, we define $\mathbb{X}_{\mathcal{L}} \in \mathbb{R}^{\mathcal{A}}$ (the indicator of $\mathcal{L}$) by the rule $\mathbb{X}_{\mathcal{L}}(L) \triangleq 1$ if $L \in \mathcal{L}$ and $\mathbb{X}_{\mathcal{L}}(A) \triangleq 0$ if $A \in \mathcal{A}\setminus\mathcal{L}$. Thus, $\mathbb{X}_{\mathcal{U}} \in \mathbb{T}(\mathcal{A}) \ \ \forall \mathcal{U} \in \mathbb{F}_0^*(\mathcal{A})$. The mapping $\kappa \triangleq (\mathbb{X}_{\mathcal{U}})_{\mathcal{U}\in\mathbb{F}_0^*(\mathcal{A})}$ is a homeomorphism between $\mathbb{F}_0^*(\mathcal{A})$ and $\mathbb{T}(\mathcal{A})$ (see [Chentsov, 2013b, Proposition 4.2]); then, $\mathbb{F}_0^*(\mathcal{A})$ and $\mathbb{T}(\mathcal{A})$ are homeomorphic.

Let us present the structure of $\mathbb{F}_0^*(\mathcal{A})$ (see [Chentsov, 2011b] for the full exposition). First, we define the family $\beta_{\mathcal{A}}^0(I) \triangleq \{\mathcal{B} \in \beta[I] \,|\, (\emptyset \notin \mathcal{B})\&(\mathcal{B} \subset \mathcal{A})\}$ of all bases of filters of I contained in $\mathcal{A}$. Secondly, every $\mathcal{B} \in \beta_{\mathcal{A}}^0(I)$ generates the corresponding filter $(I-\mathbf{fi})[\mathcal{B}|\,\mathcal{A}] \triangleq \{A \in \mathcal{A} \,|\, \exists B \in \mathcal{B} : \ B \subset A\}$ in $\mathcal{A}$. Thirdly, if $t \in]a,b]$, then $\mathcal{J}_t^{(-)} \triangleq \{[c,t[: \ c \in [a,t[\} \in \beta_{\mathcal{A}}^0(I)$ generates the ultrafilter

$$\mathcal{U}_t^{(-)} \triangleq (I-\mathbf{fi})[\mathcal{J}_t^{(-)}|\,\mathcal{A}] \in \mathbb{F}_0^*(\mathcal{A}).$$

Fourthly, if $t \in [a,b[$, then $\mathcal{J}_t^{(+)} \triangleq \{]t,c] : \ c \in]t,b]\} \in \beta_{\mathcal{A}}^0(I)$ generates the ultrafilter

$$\mathcal{U}_t^{(+)} \triangleq (I-\mathbf{fi})[\mathcal{J}_t^{(+)}|\,\mathcal{A}] \in \mathbb{F}_0^*(\mathcal{A}).$$

Note that all ultrafilters mentioned above are free [Engelking, 1977, Section 3.6]. Finally, $\mathbb{F}_0^*(\mathcal{A})$ coincides with the union of the set $\{\mathcal{U}_t^{(-)} : \ t \in]a,b]\} \cup \{\mathcal{U}_t^{(+)} : \ t \in [a,b[\}$ and the set of all trivial ultrafilters in $\mathcal{A}$.

Let η stand for the trace of the Lebesgue measure on the algebra $\mathcal{A}$; $\eta \in (\mathrm{add})_+[\mathcal{A}]$. In what follows, we deal with the compact sets

$$\begin{aligned}\mathbb{P}_\eta(\mathcal{A}) \triangleq \big\{\mu \in \mathbb{P}(\mathcal{A}) \,|\, \forall A \in \mathcal{A} \ \big(\eta(A)=0\big)\Rightarrow \\ \Rightarrow \big(\mu(A)=0\big)\big\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})],\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbb{T}_\eta(\mathcal{A}) \triangleq \big\{\mu \in \mathbb{T}(\mathcal{A}) \,|\, \forall A \in \mathcal{A} \ \big(\eta(A)=0\big)\Rightarrow \\ \big(\mu(A)=0\big)\big\} \in \big(\tau_*(\mathcal{A}) - \mathrm{comp}\big)[\mathbb{A}(\mathcal{A})].\end{aligned}$$

For arbitrary $f \in B(I,\mathcal{A})$, by $f * \eta$ we denote the indefinite η-integral of f. Note that $f*\eta$ is a set function. Let $B_0^+(I,\mathcal{A})$ be the set of all nonnegative functions from $B_0(I,\mathcal{A})$. We define the set of all feasible controls as follows:

$$\mathbf{F} \triangleq \big\{f \in B_0^+(I,\mathcal{A}) \big| \int_I f\, d\eta = 1\big\}.$$

Evidently, $f*\eta \in \mathbb{P}_\eta(\mathcal{A}) \ \ \forall f \in \mathbf{F}$. Let $\mathfrak{I}$ be defined by the rule $f \mapsto f*\eta : \ \mathbf{F} \to \mathbb{P}_\eta(\mathcal{A})$. This mapping allows us to embed $\mathbf{F}$ in the compact set (1) as a dense subset: $\mathbb{P}_\eta(\mathcal{A}) = \mathrm{cl}\big(\mathfrak{I}^1(\mathbf{F}),\tau_*(\mathcal{A})\big) = \mathrm{cl}\big(\mathfrak{I}^1(\mathbf{F}),\tau_0(\mathcal{A})\big)$; see [Chentsov, 1996, Ch. 4]. We stress that (see [Chentsov, 2013c])

$$\begin{aligned}\mathbb{T}_\eta(\mathcal{A}) = \mathbb{T}(\mathcal{A}) \cap \mathrm{cl}\big(\mathfrak{I}^1(\mathbf{F}),\tau_*(\mathcal{A})\big) = \\ = \{\kappa(\mathcal{U}_t^{(-)}) : \ t \in]a,b]\} \cup \{\kappa(\mathcal{U}_t^{(+)}) : \ t \in [a,b[\}.\end{aligned}$$

For every $t \in]a,b[$, we put $\zeta_t^0 \triangleq \inf(\{t-a; b-t\})$ and introduce the set

$$\begin{aligned}\mathbb{P}_\eta^0(\mathcal{A}|\,t) \triangleq \{\mu \in \mathbb{P}_\eta(\mathcal{A})\,| \\ \mu(]t-\varepsilon,t+\varepsilon[) = 1 \ \ \forall \varepsilon \in]0,\zeta_t^0]\} = \\ = \{\alpha\kappa(\mathcal{U}_t^{(-)}) + (1-\alpha)\kappa(\mathcal{U}_t^{(+)}) : \ \alpha \in [0,1]\}.\end{aligned}$$

In this connection, we put by definition

$$\mathbb{P}^0_\eta[\mathcal{A}] \triangleq \Big(\bigcup_{t\in]a,b[} \mathbb{P}^0_\eta(\mathcal{A}|\,t)\Big) \cup \{\kappa(\mathcal{U}^{(+)}_a);\kappa(\mathcal{U}^{(-)}_b)\}.$$

Notice that, for topologies $\tau^*_\eta(\mathcal{A}) \triangleq \tau_*(\mathcal{A})\big|_{\mathbb{P}_\eta(\mathcal{A})}$ and $\tau^0_\eta(\mathcal{A}) \triangleq \tau_0(\mathcal{A})\big|_{\mathbb{P}_\eta(\mathcal{A})}$, the property $\tau^*_\eta(\mathcal{A}) \subset \tau^0_\eta(\mathcal{A})$ holds (see [Chentsov, 1996, Ch. 4]).

3 Constraints of asymptotic character and generalized elements

To arbitrary $f \in \mathbf{F}$, assign the set $\mathrm{supp}(f) \triangleq \{t \in I|\, f(t) \neq 0\} \in \mathcal{P}'(I)$ and two values $\mathbf{t}_0(f) \triangleq \inf\big(\mathrm{supp}(f)\big)\in I$, $\mathbf{t}^0(f) \triangleq \sup\big(\mathrm{supp}(f)\big)\in I$. Given $\varepsilon \in]0,\infty[$, we define $\mathbf{F}_\varepsilon \triangleq \{f \in \mathbf{F}|\, \mathbf{t}^0(f) - \mathbf{t}_0(f) < \varepsilon\}$ and fix $N \in \mathbb{N}$, $(\rho_i)_{i\in J[N]} \in B(I,\mathcal{A})^N$, a nonempty closed set $\mathbf{Y} \in \mathcal{P}'(\mathbb{R}^N)$, and a set $M \in \mathcal{P}(J[N])$ such that $\rho_j \in B_0(I,\mathcal{A})\ \ \forall j \in M$ (the case $M = \emptyset$ is allowed). For every $\varepsilon \in]0,\infty[$, we put by definition

$$\begin{aligned} O(\mathbf{Y},\varepsilon) \triangleq \big\{(z_i)_{i\in J[N]} \in \mathbb{R}^N |\, \exists (y_i)_{i\in J[N]} \in \mathbf{Y}:\\ |y_j - z_j| < \varepsilon\ \ \forall j \in J[N]\big\},\\ \widehat{O}(\mathbf{Y},\varepsilon) \triangleq \big\{(z_i)_{i\in J[N]} \in \mathbb{R}^N |\, \exists (y_i)_{i\in J[N]} \in \mathbf{Y}:\\ (y_j = z_j\ \ \forall j \in M)\,\&\\ \&\,(|y_j - z_j| < \varepsilon\ \ \forall j \in J[N])\big\}. \end{aligned}$$

We introduce the corresponding sets of ε-admissible controls in $\mathbf{F}$:

$$\begin{aligned} \mathbb{Y}_\varepsilon \triangleq \Big\{f \in \mathbf{F}_\varepsilon \big|\big(\int_I \rho_i f\, d\eta\big)_{i\in J[N]} \in O(\mathbf{Y},\varepsilon)\Big\},\\ \widehat{\mathbb{Y}}_\varepsilon \triangleq \Big\{f \in \mathbf{F}_\varepsilon \big|\big(\int_I \rho_i f\, d\eta\big)_{i\in J[N]} \in \widehat{O}(\mathbf{Y},\varepsilon)\Big\} \end{aligned}$$

$\widehat{\mathbb{Y}}_\varepsilon \subset \mathbb{Y}_\varepsilon$. Thus, we derive the following directed families of subsets of $\mathbf{F}$:

$$\begin{aligned} \mathfrak{Y} \triangleq \{\mathbb{Y}_\varepsilon :\, \varepsilon \in]0,\infty[\} \in \beta[\mathbf{F}],\\ \widehat{\mathfrak{Y}} \triangleq \{\widehat{\mathbb{Y}}_\varepsilon :\, \varepsilon \in]0,\infty[\} \in \beta[\mathbf{F}]. \end{aligned}$$

We introduce the set of all admissible generalized elements

$$\widetilde{\mathbb{P}}^0_\eta(\mathcal{A}) \triangleq \Big\{\mu \in \mathbb{P}^0_\eta[\mathcal{A}]\ \big|\ \big(\int_I \rho_i\, d\mu\big)_{i\in J[N]} \in \mathbf{Y}\Big\} \quad (2)$$

playing a major role in our study.

Theorem 1 [Chentsov and Baklanov, 2015, Chentsov et al., 2016]. *A universal AS in the space of generalized elements is defined in (2). Precisely,*

$$\begin{aligned} \widetilde{\mathbb{P}}^0_\eta(\mathcal{A}) = (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^*_\eta(\mathcal{A});\mathfrak{I};\mathfrak{Y}] =\\ = (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^0_\eta(\mathcal{A});\mathfrak{I};\mathfrak{Y}] =\\ = (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^*_\eta(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] =\\ = (\mathbf{as})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^0_\eta(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^0_\eta(\mathcal{A});\mathfrak{I};\mathfrak{Y}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^*_\eta(\mathcal{A});\mathfrak{I};\mathfrak{Y}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^0_\eta(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{P}_\eta(\mathcal{A});\tau^*_\eta(\mathcal{A});\mathfrak{I};\widehat{\mathfrak{Y}}]. \end{aligned}$$

The universality of an AS (in this case, $\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})$) is understood in the sense that the AS coincides for both asymptotic constraints ($\mathfrak{Y}$ and $\widehat{\mathfrak{Y}}$).

4 Attraction sets as asymptotic versions of reachable sets

Fix $n \in \mathbb{N}$ and $(\pi_i)_{i\in J[n]} \in B(I,\mathcal{A})^n$. We assume that $(\pi_i)_{i\in J[n]}$ determines a mapping of elements of $\mathbf{F}$ (controls) to $\mathbb{R}^n$. Namely, Π is defined by the rule

$$f \longmapsto \Big(\int_I \pi_i f\, d\eta\Big)_{i\in J[n]} : \mathbf{F} \longrightarrow \mathbb{R}^n$$

whose values generate the reachable set. We view ASs $(\mathbf{as})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\mathfrak{Y}]$ and $(\mathbf{as})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\widehat{\mathfrak{Y}}]$ as the asymptotic versions of reachable sets. To derive the representation of these ASs, we introduce the generalized operator $\widetilde{\Pi}$ defined by

$$\mu \longmapsto \Big(\int_I \pi_i\, d\mu\Big)_{i\in J[n]} :\ \mathbb{P}_\eta(\mathcal{A}) \longrightarrow \mathbb{R}^n.$$

We stress that $\Pi = \widetilde{\Pi}\circ\mathfrak{I}$ and $\widetilde{\Pi}$ is a continuous mapping w.r.t. $\big(\mathbb{P}_\eta(\mathcal{A}),\tau^*_\eta(\mathcal{A})\big)$ and $(\mathbb{R}^n,\tau^{(n)}_{\mathbb{R}})$; here, $\tau^{(n)}_{\mathbb{R}}$ is the standard topology of coordinatewise convergence in $\mathbb{R}^n$. Combining Theorem 1 and [Chentsov, 1997, Propositions 3.3.1 and 5.2.1], we arrive at the following theorem:

Theorem 2. [Chentsov and Baklanov, 2015, Chentsov et al., 2016] *The set $\widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big)$ represents the universal AS as the asymptotic version of reachable sets*:

$$\begin{aligned} \widetilde{\Pi}^1\big(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})\big) = (\mathbf{as})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\mathfrak{Y}] =\\ = (\mathbf{as})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\widehat{\mathfrak{Y}}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\mathfrak{Y}] =\\ = (\mathbf{sas})[\mathbf{F};\mathbb{R}^n;\tau^{(n)}_{\mathbb{R}};\Pi;\widehat{\mathfrak{Y}}]. \end{aligned}$$

In connection with the representation of $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A}))$, we highlight two properties (see [Chentsov, 2011b]). First, if $t \in]a, b]$ and $g \in B(I, \mathcal{A})$, then g has a left-sided limit at t, and

$$\int_I g\, d\kappa(\mathcal{U}_t^{(-)}) = \lim_{\theta \uparrow t} g(\theta).$$

Secondly, if $t \in [a, b[$ and $h \in B(I, \mathcal{A})$, then h has a right-sided limit at t, and

$$\int_I h\, d\kappa(\mathcal{U}_t^{(+)}) = \lim_{\theta \downarrow t} h(\theta).$$

We use these properties to introduce the following definitions. For all $t \in]a, b]$,

$$\Big(\hat{\rho}_\uparrow(t) \triangleq \big(\lim_{\theta\uparrow t} \rho_i(\theta)\big)_{i \in J[N]}\Big) \,\&\, \Big(\overrightarrow{\pi}(t) \triangleq \big(\lim_{\theta\uparrow t} \pi_i(\theta)\big)_{i \in J[n]}\Big);$$

for all $t \in [a, b[$

$$\Big(\hat{\rho}_\downarrow(t) \triangleq \big(\lim_{\theta\downarrow t} \rho_i(\theta)\big)_{i \in J[N]}\Big) \,\&\, \Big(\overleftarrow{\pi}(t) \triangleq \big(\lim_{\theta\downarrow t} \pi_i(\theta)\big)_{i \in J[n]}\Big).$$

We put the following definitions

$$\Gamma \triangleq \{z \in]a, b[\times [0,1] \mid \mathrm{pr}_2(z)\hat{\rho}_\uparrow(\mathrm{pr}_1(z)) + (1 - \mathrm{pr}_2(z))\hat{\rho}_\downarrow(\mathrm{pr}_1(z)) \in \mathbf{Y}\},$$

$\Omega \triangleq \{\mathrm{pr}_2(z)\overrightarrow{\pi}(\mathrm{pr}_1(z)) + (1 - \mathrm{pr}_2(z))\overleftarrow{\pi}(\mathrm{pr}_1(z)) : z \in \Gamma\}$.

Theorem 3. [Chentsov and Baklanov, 2015, Chentsov et al., 2016] *The universal AS $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A}))$ has one of the following forms:*

1) if $\hat{\rho}_\downarrow(a) \notin \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \notin \mathbf{Y}$, then $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})) = \Omega$;

2) if $\hat{\rho}_\downarrow(a) \notin \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \in \mathbf{Y}$, then $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})) = \Omega \cup \{\overrightarrow{\pi}(b)\}$;

3) if $\hat{\rho}_\downarrow(a) \in \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \notin \mathbf{Y}$, then $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})) = \Omega \cup \{\overleftarrow{\pi}(a)\}$;

4) if $\hat{\rho}_\downarrow(a) \in \mathbf{Y}$ and $\hat{\rho}_\uparrow(b) \in \mathbf{Y}$, then $\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})) = \Omega \cup \{\overleftarrow{\pi}(a); \overrightarrow{\pi}(b)\}$.

For $H \in \mathcal{P}(\mathbb{R}^n)$ and $\varepsilon \in]0, \infty[$, we define

$$\mathbb{O}(H, \varepsilon) \triangleq \{(z_i)_{i \in J[n]} \in \mathbb{R}^n \mid \exists (h_i)_{i\in J[n]} \in H : |h_j - z_j| < \varepsilon \ \forall j \in J[n]\}.$$

From Theorem 2 and [Engelking, 1977, Proposition 3.5.1], it follows that $\forall \xi \in]0, \infty[\ \exists \varepsilon \in]0, \infty[\ \forall \delta \in]0, \varepsilon[$

$$\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})) \subset \mathrm{cl}(\Pi^1(\widehat{\mathbb{Y}}_\delta), \tau_{\mathbb{R}}^{(n)}) \subset \subset \mathrm{cl}(\Pi^1(\mathbb{Y}_\delta), \tau_{\mathbb{R}}^{(n)}) \subset \mathbb{O}\Big(\widetilde{\Pi}^1(\widetilde{\mathbb{P}}^0_\eta(\mathcal{A})), \xi\Big).$$

Furthermore, Theorem 2 delivers the following illustration for the property of asymptotic insensitivity w.r.t. the relaxation of a part of moment constraints: $\forall \xi \in]0, \infty[\ \exists \varepsilon \in]0, \infty[$:

$$\Pi^1(\widehat{\mathbb{Y}}_\delta) \subset \Pi^1(\mathbb{Y}_\delta) \subset \mathbb{O}(\Pi^1(\widehat{\mathbb{Y}}_\delta), \xi) \ \forall \delta \in]0, \varepsilon[.$$

5 The case of the double integrator

In this section we apply the developed theoretical framework to the case of the double integrator model and present some examples. Without loss of generality, we assume that the time interval is equal to $[0, 1]$; thus, $a = 0, b = 1$, and $I = [0, 1]$. We consider the following model of the double integrator:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \mathbf{b}(t) f(t) \end{cases}; \tag{3}$$

here $f \in \mathbf{F}$ is a control, and $\mathbf{b} \in B(I, \mathcal{A})$ is the coefficient at the control. The initial conditions in all examples are set as follows: $x_1(0) = x_2(0) = 0$. Let us define the functions $\pi_1 : I \to \mathbb{R}$ and $\pi_2 : I \to \mathbb{R}$ by the following rules: $\pi_1(t) \triangleq (1-t)\mathbf{b}(t)$, $\pi_2(t) \triangleq \mathbf{b}(t) \quad \forall t \in I$. Basically, given the initial conditions equal to zero, the vector function π generates the terminal position of the double integrator for an idealized short-pulse control applied at the time t. To employ Theorem 3, we have to specify the functions

$$\overrightarrow{\pi} :]0, 1] \to \mathbb{R}^2, \ \overleftarrow{\pi} : [0, 1[\to \mathbb{R}^2.$$

To this end, we introduce the vector function $\mathbf{p} : [0, 1] \to \mathbb{R}^2$ by the following rule: $\forall t \in [0, 1]$

$$(\mathbf{p}_1(t) \triangleq 1 - t) \,\&\, (\mathbf{p}_2(t) \triangleq 1).$$

It is easy to see that $\overrightarrow{\pi}(t) = \lim_{\theta\uparrow t} \mathbf{b}(\theta)\mathbf{p}(t) \ \forall t \in]0, 1]$ and $\overleftarrow{\pi}(t) = \lim_{\theta\downarrow t} \mathbf{b}(\theta)\mathbf{p}(t) \ \forall t \in [0, 1[$.

It is easy to see that the construction of the AS is essentially the matter of finding one-sided limits of $\mathbf{b}$. Note that $\mathbf{p}(1) = (0, 1)$ and $\mathbf{p}(0) = (1, 1)$. Let us introduce the vector function $\rho(t)$ (with the components $\rho_1(t)$ and $\rho_2(t)$) : $\forall t \in I$

$$\rho_1(t) = \mathbf{b}(t)(\mathbf{t} - t)\chi_{[0, \mathbf{t}[}(t), \quad \rho_2(t) = \mathbf{b}(t)\chi_{[0, \mathbf{t}[}(t).$$

This specification of ρ can be roughly understood as a requirement for the double integrator to reach the set $\mathbf{Y}$ given that control resources are available until $\mathbf{t}$. In the following examples, we set $\mathbf{t} = 0.9$.

To construct ASs, we apply Theorem 3. In the examples below, it easy to see that $\hat{\rho}_{\downarrow}(0) \notin \mathbf{Y}$, $\hat{\rho}_{\uparrow}(1) \notin \mathbf{Y}$. In this case, Theorem 3 (see the form 1) states that the ASs coincide with the set Ω :

$$\Omega = \{\alpha \overrightarrow{\pi}(t) + (1-\alpha)\overleftarrow{\pi}(t) : (t, \alpha) \in \Gamma\}.$$

We implemented a computer program for the numerical computation of Ω; the examples follow.

5.1 Example 1. One switching

This example models the control system (3) in the case of an instantaneous drop of mass (e.g., due to rocket staging). Assume that $\mathbf{b} = b_1\chi_{[0,t_0[} + b_2\chi_{[t_0,1]}$; here, $b_1 = 1, b_2 = 2$, and $t_0 = 0.6$. The set $\mathbf{Y}$ is specified as $\mathbf{Y} = [0.3,\ 0.5] \times [0.8,\ 2.1]$ and depicted in Fig. 1.

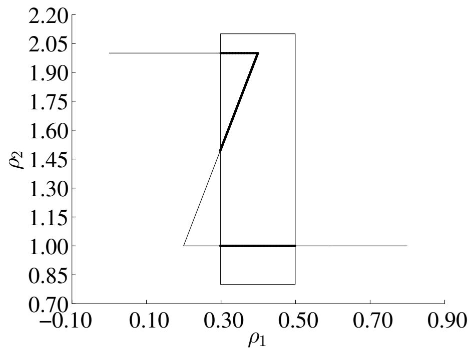

Figure 1. The constraint set $\mathbf{Y}$ and its elements representing all admissible generalized elements (bold line) in Example 1

Given this data, we depict the AS in Fig. 2.

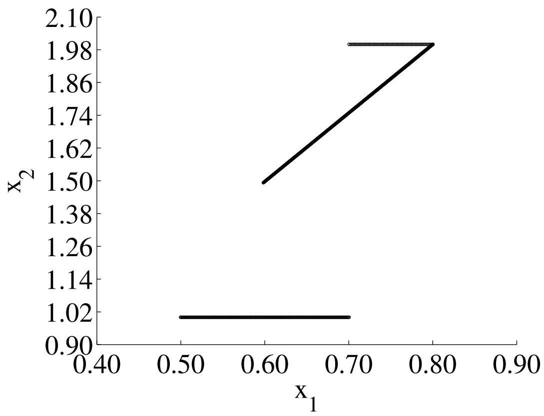

Figure 2. The attraction set in Example 1

5.2 Example 2. Several switchings

Let us consider a more sophisticated example with $m = 4$ points of discontinuity. We assume that $(t_i)_{i \in J[m]}$ is defined by $t_i = 0.1 * i \ \ \forall i \in J[m]$ and $b_i = i \ \forall i \in J[m+1]$. We now introduce $\mathbf{b}$ as follows:

$$\mathbf{b} = b_1\chi_{[0,t_1[} + \sum_{i \in \{2;3;4;5\}} b_i\chi_{[t_{i-1},t_i[} + b_5\chi_{[t_4,1]}, \quad (4)$$

We set $\mathbf{Y} = [0.3,\ 1.2] \times [1.4,\ 4.5]$ (see Fig. 3).

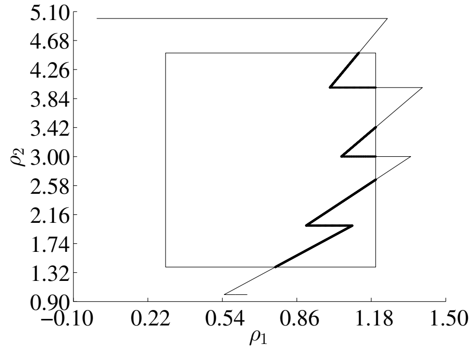

Figure 3. The constraint set $\mathbf{Y}$ and its elements representing all admissible generalized elements (bold line) in Example 2

We depict the corresponding AS on Fig. 4.

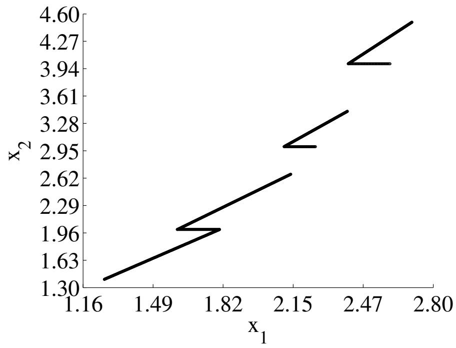

Figure 4. The attraction set in Example 2

5.3 Example 3. Bang-bang switching

In this example $\mathbf{b}$ has the ‘bang-bang’ type of switching. Assume that $m = 4, t_k = 0.2 * k \ \ \forall k \in J[m]$, and $b_k = (-1)^{(k+1)} \ \ \forall k \in J[m+1]$. We specify $\mathbf{b}$ by means of (4) and assume that $\mathbf{Y} = [-1.1,\ 1.1] \times [0,\ 1.1]$ (see Fig. 5).

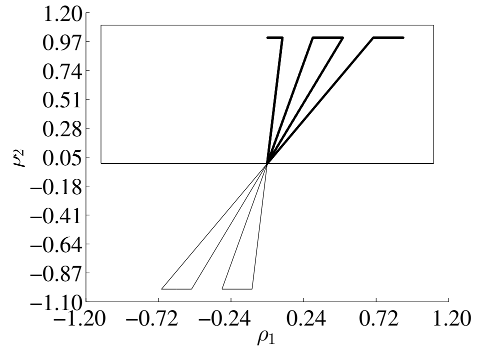

Figure 5. The constraint set $\mathbf{Y}$ and its elements representing all admissible generalized elements (bold line) in Example 3

The corresponding AS is shown on Fig.6.

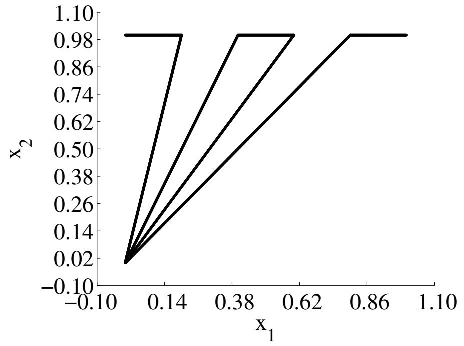

Figure 6. The attraction set in Example 3

6 Conclusion

In this paper we obtained a full characterization of ASs, which are asymptotic versions of reachable sets in the case of asymptotic constraints. The novelty is in the combination of constraints of asymptotic character corresponding to the short-time pulse control mode and the relaxation of the moment constraints. The developed extension scheme heavily relies on the results [Chentsov, 2011b] and uses finitely-additive measures as generalized elements (controls); see also [Chentsov, 1996, Chentsov, 1997, Chentsov and Morina, 2002, Chentsov, 2006]. More importantly, the abstract scheme was fully determined in terms of the solution of a finite-dimensional problem. For the case of the double integrator, the corresponding numerical procedure was developed and tested on the above examples.

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