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SYNCHRONIZATION OF DIFFUSIVELY COUPLED ELECTRONIC HINDMARSH-ROSE OSCILLATORS

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Abstract
In this paper we study the synchronization of diffusively coupled Hindmarsh-Rose (HR) electronic oscillators. These electronic oscillators are analog electrical circuits which integrate the differential equations of the HR model. An experimental setup consisting of four chaotic HR oscillators, is used to evaluate the existence and stability of partially or fully synchronized states.

Key words
Synchronization, Hindmarsh-Rose, Electrical circuits

1 Introduction

Synchronous behavior of systems is witnessed in a vast number of research area's. Beautiful examples are, for instance, the simultaneous flashing of male fireflies on banks along rivers in Malaysia, Thailand and New Guinea (Strogatz and Stewart, 1993), and the synchronous release of action potentials in parts of the mammalian brain (Gray, 1994). A large number of examples of synchronization in nature can be found in (Pikovsky *et al.*, 2003). Synchronization can also be found in robotics, usually referred to as coordination (Rodriguez-Angeles and Nijmeijer, 2001), and it is potentially of interest in the field of (secure) communication, see for instance (Pecora and Carroll, 1990; Huijberts *et al.*, 1998).

Most of the studies of synchronization in networks of coupled systems deal with analysis supported by simulations. Significantly less attention is given to validate results in an experiment setup. In this paper we present the synchronization of coupled HR electronic oscillators. The HR model (Hindmarsh and Rose, 1984) is a well-known model in the field of neuroscience that provides a description of the action potential generation in neuronal cells. This model consists of three coupled nonlinear differential equations and is capable, as function of specific parameters, of producing both simple and complex oscillatory motion. Here, an experimental setup consisting of four circuits, operating in a chaotic regime, is used to investigate the existence and stability of synchronized states.

The HR oscillators in the experimental setup are diffusively coupled; that is the systems are mutually coupled using (linear) functions of the outputs of the systems. Using a semipassivity based approach (Pogromsky *et al.*, 2002) we derive conditions that guarantee the existence of synchronized regimes. These regimes might correspond to the fully synchronized state, i.e. all systems perform an identical motion, as well as to partial synchronization where only some systems do synchronize.

This paper is organized as follows. In section 2 the mathematical notations are being introduced and we present the notions of *semipassivity* and *convergent systems*. In section 3 we present a theoretical passivity-based framework, introduced in (Pogromsky, 1998), that provides conditions under which the coupled oscillators synchronize. In addition, we show that the HR systems satisfy the assumptions of this framework. Next, in section 4 the experimental setup is discussed and in section 5 we show that the coupled electronic HR systems synchronize. Finally, in section 6 conclusions are drawn.

2 Preliminaries

Throughout this paper we use the following notations. The Euclidian norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$, $\|x\|^2 = x^\top x$ where the symbol $^\top$ stand for transposition. The symbol I_n defines the $n \times n$ identity matrix and the notation $\operatorname{col}(x_1, \ldots, x_n)$ stands for the column vector containing the elements $x_1, \ldots, x_n$. A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called positive definite if $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. It is radially unbounded if $V(x) \to \infty$ if $\|x\| \to \infty$. If the quadratic form $x^\top Px$ with a symmetric matrix $P = P^\top$ is positive definite, then the matrix P is positive definite, denoted as $P > 0$. The notation $A \otimes B$ stands for the Kronecker product of the matrices A and B.

Let us, in addition, present the notions of *semipassivity* and *convergent systems*.

Definition 2.1 (semipassivity). *(Pogromsky and Nijmeijer, 2001) Consider the following system:*

$$\begin{aligned} \dot{x} &= f(x) + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$, vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and matrices B and C of appropriate dimensions. Let $V : \mathbb{R}^n \to \mathbb{R}_+$, $V(0) = 0$ be a differentiable nonnegative (storage) function, then the system (1) *is called semipassive if the following inequality is satisfied:*

$$\dot{V} \leq y^\top u - H(x) \tag{2}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is nonnegative outside some ball

$$\exists \rho > 0, \quad \forall \|x\| \geq \rho \Rightarrow H(x) \geq \varrho(\|x\|)$$

for some continuous nonnegative function $\varrho(\cdot)$ defined for $\|x\| \geq \rho$. If the inequality (2) *is strict, the system* (1) *is called strictly semipassive.*

The most useful property of semipassive systems is that being linearly interconnected, the solutions of all systems in the network exist for all $t \geq 0$ and are *ultimately* bounded (Pogromsky, 1998).

Consider the following system:

$$\dot{z} = q(z, w) \tag{3}$$

where $z(t) \in \mathbb{R}^s$, $w(t) \in \mathcal{D}$, $\mathcal{D}$ is some compact subset of $\mathbb{R}^p$, continuous function $w : \mathbb{R}_+ \to \mathcal{D}$ and the vector field $q : \mathbb{R}^s \times \mathcal{D} \to \mathbb{R}^s$.

Definition 2.2 (convergent systems). *(Demidovich, 1967; Pavlov* et al.*, 2006) The system* (3) *is said to be convergent if for any $w(\cdot)$:*

1. *all solutions $z(t)$ are well-defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^s$;*
2. *there exists a unique globally asymptotically stable solution $z_w(t)$ defined and bounded for all $t \in (-\infty, +\infty)$, i.e. for any solution $z(t)$ it follows that*

$$\lim_{t \to \infty} \|z(t) - z_w(t)\| = 0.$$

According to Demidovich (Demidovich, 1967), there exists a simple sufficient condition to determine whether the system (3) is convergent:

Lemma 2.1. *(Demidovich, 1967; Pavlov* et al.*, 2006) If there exists a matrix $P = P^\top > 0$ such that the eigenvalues $\lambda_i(Q)$ of the symmetric matrix*

$$Q(z, w) = \left(P \left(\frac{\partial q}{\partial z}(z, w) \right) + \left(\frac{\partial q}{\partial z}(z, w) \right)^\top P \right) \tag{4}$$

are negative and separated away from the imaginary axis for all $z \in \mathbb{R}^s$, $w \in \mathcal{D}$, then the system (3) *is convergent.*

3 Synchronization of Diffusively Coupled HR Oscillators

In this section conditions are posed that guarantee (partial) synchronization in a network of coupled HR oscillators. First the semipassivity based framework as described in (Pogromsky *et al.*, 2002) is presented, and next we show that the HR oscillators satisfy the assumptions of this framework.

Consider the k systems of the following form

$$\begin{aligned} \dot{x}_i &= f(x_i) + Bu_i \\ y_i &= Cx_i \end{aligned} \tag{5}$$

where $i = 1, \dots, k$ denotes the number of each system in the network, $x_i \in \mathbb{R}^n$ the state, $u_i \in \mathbb{R}^m$ the input and $y_i \in \mathbb{R}^m$ the output of the i^{th} system, smooth vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and matrices B and C of appropriate dimensions. Let, in addition, the matrix CB be positive definite and nonsingular.

The k dynamical systems (5) are coupled via *diffusive* coupling, i.e. mutual interconnection through linear output coupling of the form

$$\begin{aligned} u_i = &- \gamma_{i1}(y_i - y_1) - \gamma_{i2}(y_i - y_2) \\ &- \dots - \gamma_{ik}(y_k - y_1) \end{aligned} \tag{6}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ denotes the strength of the interconnection between the systems i and j.

Defining the $k \times k$ *coupling matrix* as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix}$$

the diffuse coupling functions (6) can be written as

$$\underline{u} = -\Gamma \underline{y} \tag{7}$$

where $\underline{u} = \operatorname{col}(u_1, \dots, u_k)$, $\underline{y} = \operatorname{col}(y_1, \dots, y_k)$. Since $\Gamma = \Gamma^\top$ all its eigenvalues are real. Moreover, applying Gerschgorin's theorem about the localization of the eigenvalues, it is easy to verify that Γ is positive semidefinite.

A network might possess certain symmetries. In particular, the network may contain repeating patterns. Hence, a permutation of some elements in the network, with respect to the interconnections, will leave the network unchanged. The mathematical representation of the permutation of the elements is a permutation matrix $\Pi \in \mathbb{R}^{k\times k}$. The matrix Π defines a symmetry for the network if Γ and Π commute, i.e. $\Pi\Gamma = \Gamma\Pi$. Moreover, given a permutation matrix Π that commutes with Γ, the set $\ker\left(I_{kn} - \Pi \otimes I_n\right)$ defines a *linear invariant manifold* for the closed loop systems (5) and (7). To be precise, the set $\ker\left(I_{kn} - \Pi \otimes I_n\right)$ describes a set of linear equations of the form $x_i - x_j = 0$ for some i and j. Hence, we want to guarantee asymptotic stability of such a set. Therefore, introduce a linear change of coordinates $x_i \mapsto (z_j, y_j)$. Under the assumption that CB is nonsingular, the systems (5) can be written after the coordinate transformation in the normal form:

$$\begin{cases} \dot{z}_i = q(z_i, y_i) \\ \dot{y}_i = a(z_i, y_i) + CBu_i \end{cases} \tag{8}$$

where $z_i \in \mathbb{R}^{n-m}$ and smooth vector fields $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$. A sufficient condition for asymptotic stability of the set $\ker\left(I_{kn} - \Pi \otimes I_n\right)$ is given in the following theorem:

Theorem 3.1. *(Pogromsky* et al.*, 2002) Let λ' be the smallest nonzero eigenvalue of Γ under the restriction that the eigenvectors are taken from the set* $\text{range}\left(I_k - \Pi\right)$. *Assume that:*

A1. the free system (8) *is strictly semipassive with respect to input u_i and output y_i with a radially unbounded storage function;*
A2. there exists a matrix $P = P^\top > 0$ such that the conditions of Lemma 2.1 are satisfied for q as defined in (8).

Then for all positive semidefinite matrices Γ all solutions of the diffusive network (8) *and* (7) *are ultimately bounded and there exists a positive number $\bar{\lambda}$ such that if $\lambda' \geq \bar{\lambda}$ the set* $\ker\left(I_{kn} - \Pi \otimes I_n\right)$ *contains a globally asymptotically stable subset.*

A network of HR oscillators is given by the following set of equations:

$$\begin{cases} \frac{1}{\tau_s}\dot{y}_i = -ay_i^3 + \varphi_1 y_i + \varphi_2 + g_{z_1} z_{1,i} - g_{z_2} z_{2,i} \\ \qquad\qquad +\mathcal{I} + u_i \\ \frac{1}{\tau_s}\dot{z}_{1,i} = -c - dy_i^2 - \varphi_3 y_i - \beta z_{1,i} \\ \frac{1}{\tau_s}\dot{z}_{2,i} = r\left(s\left(y_i + y_0\right) - z_{2,i}\right) \end{cases} \tag{9}$$

where $i = 1, \ldots, k$ denotes the number of each oscillator in the network, y_i represents the membrane potential, which can be regarded as the natural output of a neuron, $z_{1,i}$ is an internal recovery variable and $z_{2,i}$ is a slow internal recovery variable and input u_i. The constant $\mathcal{I}$ represents an external applied stimulus. Parameters $a, \varphi_1, \varphi_2, g_{z_1}, g_{z_2}, c, d, \varphi_3, \beta, r, s, x_0$ are all positive constants and $\tau_s > 0$ denotes a time scaling factor.

Proposition 3.1. *Each free HR system is strictly semipassive with respect to the input u_i and the output y_i with a radially unbounded storage function*

Proof. Following (Oud and Tyukin, 2004), consider the following storage function $V : \mathbb{R}^3 \to \mathbb{R}_+$:

$$V(y_i, z_{1,i}, z_{2,i}) = \frac{1}{2\tau_s}\left(c_1 y_i^2 + c_2 z_{1,i}^2 + c_3 z_{2,i}^2\right) \tag{10}$$

where c_1, c_2, c_3 are positive constants. Let $c_1 = 1$, $c_2 = \frac{g_{z1}}{\varphi_3}$ and $c_3 = \frac{g_{z2}}{rs}$, then one can easily verify that (2) is satisfied when

$$H(y_i, z_{1,i}, z_{2,i}) = ay_i^4 - \varphi_1 y_i^2 - (\varphi_2 + \mathcal{I})y_i$$
$$+ \frac{g_{z_1}}{\varphi_3} z_{1,i}\left(c + dy_i^2 + \beta z_{1,i}\right) + g_{z_2} z_{2,i}\left(\frac{1}{s} z_{2,i} - y_0\right).$$

Proposition 3.2. *The system*

$$\begin{cases} \frac{1}{\tau_s}\dot{z}_{1,i} = -c - dy_i^2 - \varphi_3 y_i - \beta z_{1,i} \\ \frac{1}{\tau_s}\dot{z}_{2,i} = r\left(s\left(y_i + y_0\right) - z_{2,i}\right) \end{cases} \tag{11}$$

is convergent.

Proof. Set $P = I_2$, then the matrix Q as defined in (4) is given by

$$Q = \tau_s \begin{pmatrix} -\beta & 0 \\ 0 & -r \end{pmatrix}.$$

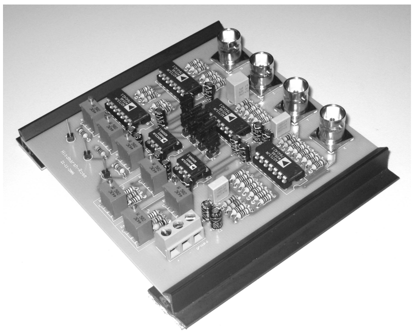
Since $\beta, r, \tau_s > 0$, it follows directly that the condition of Lemma 2.1 is satisfied, i.e. the system (11) is convergent.

Propositions 3.1 and 3.2 show that assumptions A1 and A2 of Theorem 3.1 hold and therefore we ensure that for sufficiently strong coupling the HR systems (9) in the network will (partially) synchronize.

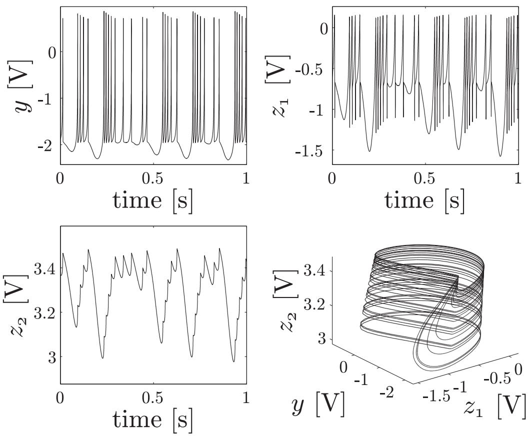
4 Experimental Setup

We realized four analog electronic equivalents of the HR equations (9), partially based on the implementation as presented in (Lee *et al.*, 2004), with the following set of nominal parameters:

$$\begin{array}{lllll} a = 1, & \varphi_1 = 3,\ \varphi_2 = 2, & & g_{z_1} = 5, & g_{z_2} = 1, \\ \mathcal{I} = 3.25, & c = 0.8,\ d = 1, & & \varphi_3 = 2, & \beta = 1, \\ r = 0.005, & s = 4, & y_0 = 2.618, & \tau_s = 1000. & \end{array}$$

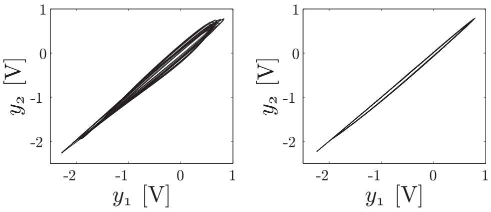


(a) The HR circuit.



(b) Signals of the electronic circuit.

Figure 1. The HR electronic oscillator.



(a) $K = 0.6$. (b) $K = 1.2$.

Figure 2. Synchronization of the two coupled systems.

Each electronic HR system consists of three integrator circuits, which integrate the HR equations, and a multiplier circuit, build using *AD633j* multipliers, that generates the squared and cubic terms in the HR equations. Figure 1(a) shows a single electronic HR circuit, and Figure 1(b) shows measured chaotic signals from such an experimental oscillator. In particular, in this figure the time series of the states y, z_1, z_2 are depicted and (a part of) the chaotic attractor in the phase-space is shown.

There are slight differences between the measured signals and the signals that can obtained through numerical integration of the equations (9). This mismatch is due to tolerances of the used components, i.e. the parameters of each circuit differ a little from the nominal ones. This implies that there are small differences between the individual circuits as well. Hence, synchronization in the sense that $x_i = x_j$, where $x_i = [y_i, z_{1,i}, z_{2,i}]^\top$, is not possible. Therefore, we introduce a weakened form, referred to as *practical synchronization*, defined as $\sup \|x_i - x_j\| \leq \delta$, with a fixed, sufficiently small $\delta > 0$. Here, $\delta = 0.5$ [V] will be used.

Remark 4.1. *Although the value $\delta = 0.5$ [V] seems rather high, one has to realize that due to the spiking behavior of the signals (see Figure 1(b)), a small mismatch induces a relatively large error.*

In order to define the connections between the HR electronic systems, a synchronization interface is developed that makes use of a microcontroller in which the coupling functions (7) can be programmed. The use of a microcontroller to define the coupling functions allows convenient experimenting with different network topologies and changes in coupling strength.

5 Synchronization Experiments

5.1 Two systems

Before synchronization in a network with all four systems is considered, the most obvious case is investigated first, i.e. the synchronization of two diffusively coupled systems being interconnected with coupling strength K. The two systems are connected by feedback (7) with Γ defined as:

$$\Gamma = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}.$$

It turns out that the two electronic HR oscillators practically synchronize when $K \geq 0.6$. This experimentally obtained coupling strength is pretty close to the value $K \geq 0.5$ that is found in simulations. Figure 2(a) shows the practical synchronization of the two systems for $K = 0.6$. In the left pane the y-states of the two systems are shown as function of time. The synchronization phase portrait is depicted on the right pane. The same is shown in Figure 2(b) in case that the coupling between the two systems is twice that large. One can see that the error between both signals decreases when K increases, i.e. δ becomes smaller.

5.2 Three systems

Next, three electronic HR oscillators coupled in a ring are considered. The corresponding coupling matrix is given by:

$$\Gamma' = \begin{pmatrix} 2K & -K & -K \\ -K & 2K & -K \\ -K & -K & 2K \end{pmatrix}.$$

Using a conjecture stated by Wu and Chua (Wu and Chua, 1996), the coupling strength required to synchronize the three systems can be determined from the coupling that is required to synchronize two systems. The conjecture is formulated as follows: given two networks of diffusively coupled systems, if the systems in the network with coupling matrix Γ synchronize, then the systems in the network with coupling matrix Γ' synchronize if and only if $\gamma = \gamma'$, where γ and γ' denote the smallest nonzero eigenvalues of Γ and Γ', respectively. Applying the Wu-Chua conjecture we expect the systems to synchronize when $K \geq 0.4$ in the experimental setup and for $K \geq 0.34$ in the simulation. Indeed, the three connected HR oscillators show synchronized behavior for $K = 0.34$ in a simulation study, and the three experimental systems in the network practically synchronize when $K \geq 0.4$.

Remark 5.1. *Although it can be shown that the Wu-Chua conjecture is not valid in general, cf. (Pecora, 1998), it is stated in (Pogromsky and Nijmeijer, 2001) that the conjecture is true for systems satisfying assumption A2 of Theorem 3.1.*

5.3 Four systems

The four systems are coupled in a ring as shown schematically in Figure 3. The corresponding coupling matrix for this setup is given by

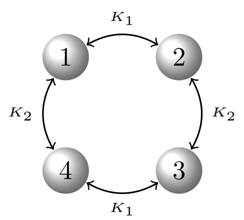


Figure 3. Setup with four systems

$$\Gamma'' = \begin{pmatrix} K_1+K_2 & -K_1 & 0 & -K_2 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_1+K_2 & -K_1 \\ -K_2 & 0 & -K_1 & K_1+K_2 \end{pmatrix}.$$

The network described by the matrix Γ'' does possess some symmetries. The following matrices define a permutation of the network:

$$\Pi_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix},$$

and $\Pi_4 = I_4$, where O denotes the 2×2 matrix with all its elements equal to zero and

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The action of Π_1 is to switch the systems 1 and 2, and simultaneously switching of the systems 3 and 4. It follows immediately from Figure 3 that the network is left invariant with respect to its interconnections. The matrices Π_2 and Π_3 define similar actions, while Π_4 leaves everything unchanged.

The matrices Π_1, Π_2 and Π_3 define, respectively, the following linear invariant manifolds:

$$\mathcal{A}_1 = \left\{ x \in \mathbb{R}^{12} | x_1 = x_2, x_3 = x_4 \right\},$$
$$\mathcal{A}_2 = \left\{ x \in \mathbb{R}^{12} | x_1 = x_4, x_2 = x_3 \right\},$$
$$\mathcal{A}_3 = \left\{ x \in \mathbb{R}^{12} | x_1 = x_3, x_2 = x_4 \right\}.$$

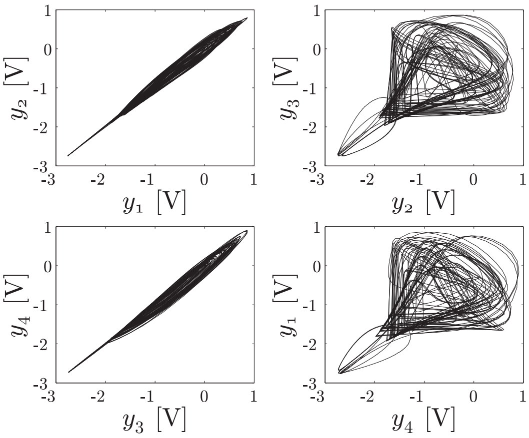
Applying Theorem 3.1, we have $\lambda' = 2K_1$ for Π_1, $\lambda' = 2K_2$ for Π_2, and $\lambda' = \min(2K_1, 2K_2)$ for Π_3. This means that for large enough K_1 we can expect asymptotic stability of a subset of $\mathcal{A}_1$ and for large enough K_2 a subset of $\mathcal{A}_2$ is asymptotically stable. A subset of $\mathcal{A}_3$ can only be stable as the stable intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$, which describes the fully synchronized state. In our experimental setup practical partial synchronization with respect to the manifold $\mathcal{A}_1$ is found for $K_1 \geq 0.6 > K_2$, while practical partial synchronization with respect to $\mathcal{A}_2$ follows, obviously, when $K_2 \geq 0.6 > K_1$. The phase portraits corresponding to these synchronization regimes are depicted in Figure 4. Depending on the values of K_1 and K_2, in this ring setup there are two possible routes from no synchronization to full synchronization:

$$\text{no synchrony} \to \mathcal{A}_1 \to \mathcal{A}_1 \cap \mathcal{A}_2 \text{ (full synchrony)},$$
$$\text{no synchrony} \to \mathcal{A}_2 \to \mathcal{A}_1 \cap \mathcal{A}_2 \text{ (full synchrony)}.$$

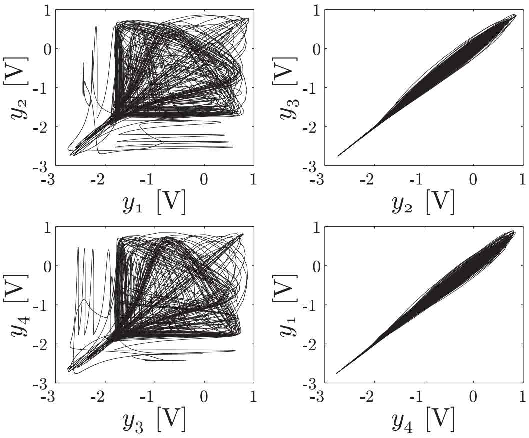
The systems in the experimental setup will indeed practically synchronize when $K_1 = K_2 \geq 0.6$. Figure 5 shows the synchronization phase portraits for the four systems in case that $K = 0.6$.

6 Conclusions

We have presented our experimental finding of synchronous behavior in an experimental setup with up to four coupled electronic, chaotic HR oscillators. At first, it is shown that each free HR system is semipassive and the internal dynamics are convergent. Therefore, under the condition that the coupling between the systems is large enough, the systems in the network should show (partially) synchronized behavior. Indeed, in the experimental setup synchronization (and partial synchronization) in networks consisting of two, three or four oscillators is witnessed. We remark that because of small differences in the behavior of the individual circuits, we are not able to achieve a perfect zero synchronization error, but practical (partial) synchronization is achieved.



(a) $K_1 = 0.6,\ K_2 = 0.3.$



(b) $K_1 = 0.3,\ K_2 = 0.6.$

Figure 4. Partial practical synchronization of four systems with respect to the linear invariant manifolds: (a) $\mathcal{A}_1$, (b) $\mathcal{A}_2$.

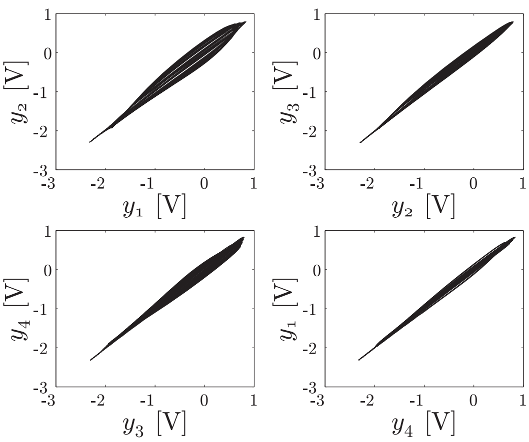


Figure 5. Synchronization phase portraits for four system connected in a ring with the couplings $K_1 = K_2 = 0.6$.

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