Necessary and sufficient conditions for stability of MRAC systems

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Abstract

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1 Introduction

In many interesting nonlinear control problems, the closed-loop control system can be modeled by the non-auotnomous differential equation

$$\dot{x} = F(t, x) \tag{1}$$

where $F(\cdot, x)$ is not necessarily periodic.

We will consider general nonlinear time-varying systems of the form (1) where $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that solutions of (1) exist over finite intervals.

We consider a particular class of nonlinear systems (1) with $x := \mathrm{col}[x_1, x_2]$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ which may be viewed as a generalization of the classical strictly positive real system well studied in the context of linear systems. Hence, the structure of systems we consider in this section is important because, when inputs and outputs are considered[1], they naturally yield *passive* systems. Consequently, the type of results that we will present here may be used in the analysis of passivity-based (adaptive) control systems. This will be illustrated further below, with the application to adaptive control of robot manipulators.

A typical example of systems with matching nonlinearities is what we may call *of the "MRAC-type"* where MRAC stands for Model Reference Adaptive Control. These systems appear as closed loop equations in MRAC of *linear* plants (cf. [15]). In the purely nonlinear context, they have also been for instance, in [25, 28, 8, 5].

Our results rely on the key condition, well-known in adaptie control literature, of persistency od excitation; particularly, a PE property tailored for nonlinear systems which we call δ-PE. To put our contriutions in perspective let us recall that for linear systems

$$\dot{x} = -P(t)x, \qquad P(t) = P(t)^T \geq 0, \ \forall\, t \geq 0$$

it was shown in [19] that it is necessary and sufficient for uniform asymptotic stability that $P(t)$ be *persistently exciting* (PE), namely that there exist $a > 0$ and $b \in \mathbb{R}$ such that for *all* unitary vectors $x \in \mathbb{R}^n$,

$$\int_{t_\circ}^{t} |P(s)x|\, ds \geq a(t - t_\circ) + b \qquad \forall\, t \geq t_\circ \geq 0\,. \tag{2}$$

[1] E.g. in applications of mechanical systems these may be external generalized torques and generalized velocities respectively.

Equivalently, this system is uniformly (in $t_\circ$) completely (i.e. for all initial states) observable (see e.g. [1]) from the output $y(t) := P(t)x(t)$ if and only if $P(\cdot)$ is PE. Notice that $P(t)$ does not need to be full rank for any fixed t.

That PE is a necessary condition for uniform asymptotic (exponential) stability for linear systems has been well-known for many years now. In the case of general nonlinear systems, this was established for a generalized notion of persistency of excitation by Artstein in [2, Theorem 6.2] using limiting equations. Our method is different: our definition is not trajectory-dependent and, sufficiency relies on a recent extension of Matrosov's theorem.

2 Mathematical preliminaries

2.1 Notation and definitions

Notation. For two constants $\Delta \geq \delta \geq 0$ we define $\mathcal{H}(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta \leq |x| \leq \Delta\}$. We also will use $\mathcal{B}(r) := \mathcal{H}(0, r)$. A continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{N}$ if it is non decreasing. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\gamma \in \mathcal{K}$), if it is strictly increasing and $\gamma(0) = 0$; $\gamma \in \mathcal{K}_\infty$ if in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_{\geq 0}$ and $\beta(s, t) \to 0$ as $t \to +\infty$ for each $s \geq 0$. We denote by $x(\cdot, t_\circ, x_\circ)$, the solutions of the differential equation $\dot{x} = F(t, x)$ with initial conditions $(t_\circ, x_\circ)$.

We recall that a function $F(\cdot, \cdot)$ is locally Lipschitz in x uniformly in t if for each x_0 there exists L such that

$$|F(t, x) - F(t, y)| \leq L\,|x - y|$$

for all x and y in a neighbourhood of x_0 and for all $t \in \mathbb{R}$. For a locally Lipschitz function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we define its total time derivative along the trajectories of $\dot{x} = F(t, x)$ as, $\dot{V}(t, x) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} F(t, x)$. In general, this quantity is defined almost everywhere.

As it has been motivated e.g. in [28, 18], for these systems the most desirable forms of stability are those which are uniform in the initial time:

Definition 1 (Uniform global stability) The origin of the system (1) is said to be uniformly globally stable (UGS) if there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R} \times \mathbb{R}^n$ each solution $x(\cdot, t_\circ, x_\circ)$ satisfies

$$|x(t, t_\circ, x_\circ)| \leq \gamma(|x_\circ|) \qquad \forall\, t \geq t_\circ\,. \tag{3}$$

Definition 2 (Uniform global attractivity) The origin of the system (1) is said to be uniformly globally attractive if for each r, $\sigma > 0$ there exists $T > 0$ such that

$$|x_\circ| \leq r \implies \|x(t, t_\circ, x_\circ)\| \leq \sigma \qquad \forall\, t \geq t_\circ + T\;. \tag{4}$$

Furthermore, we say that the (origin of the) system is uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive.

We will also make use of the following.

Definition 3 (Uniform exponential stability) The origin of the system (1) is said to be uniformly (locally) exponentially stable (ULES) if there exist constants γ_1, γ_2 and $r > 0$ such that for all $(t_\circ, x_\circ) \in \mathbb{R} \times B_r$ and all corresponding solutions

$$\|x(t, t_\circ, x_\circ)\| \leq \gamma_1 \|x_\circ\| e^{-\gamma_2(t - t_\circ)} \qquad \forall t \geq t_\circ. \tag{5}$$

The system (1) is uniformly globally exponentially stable (UGES) if there exist $\gamma_1,\ \gamma_2 > 0$ such that (5) holds for all $(t_\circ, x_\circ) \in \mathbb{R} \times \mathbb{R}^n$. □

We recall the definition of "uδ-PE", a property originally introduced in [14, 28].

Let $x \in \mathbb{R}^n$ be partitioned as $x := \text{col}[x_1, x_2]$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Define the *column vector* function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and the set $\mathcal{D}_1 := (\mathbb{R}^{n_1} \setminus \{0\}) \times \mathbb{R}^{n_2}$.

Definition 4 A function $\phi(\cdot, \cdot)$ where $t \mapsto \phi(t, x)$ is locally integrable, is said to be uniformly δ-persistently exciting (uδ-PE) with respect to x_1 if for each $x \in \mathcal{D}_1$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ s.t. $\forall t \in \mathbb{R}$,

$$|z - x| \leq \delta \quad \Longrightarrow \quad \int_t^{t+T} |\phi(\tau, z)| \, d\tau \geq \mu \,. \tag{6}$$

□

If $\phi(\cdot, \cdot)$ is uδ-PE with respect to the whole state x then we will simply say that "ϕ is uδ-PE". This notation will allow us to establish some results for nonlinear systems with state x by imposing, on a certain function, the condition of uδ-PE w.r.t. only *part* of the state.

Furthremore, the following characterizations are useful to establish certain proofs.

Our first "characterization" of uδ-PE is actually a relaxed property. It states that when dealing with the particular (but fairly wide) class of uniformly continuous functions, it is sufficient to verify the integral in (6) only for each fixed x such that $x_1 \neq 0$ (i.e., for "large" states).

Lemma 1 If $x \mapsto \phi(t, x)$ is continuous uniformly in t then $\phi(\cdot, \cdot)$ is uδ-PE with respect to x_1 if and only if

(**A**) *for each $x \in \mathcal{D}_1$ there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$,*

$$\int_t^{t+T} |\phi(\tau, x)| d\tau \geq \mu \ . \tag{7}$$

□

The following Lemma helps us to see that Definition 4 state in words that "a function $\phi(t, x)$ is uδ-PE with respect to x_1 if $t \mapsto \phi(t, x)$ is PE in the usual sense[2] whenever the states x_1 (or similarly, the trajectories $x_1(t)$) are large". This is important since it is the central idea to keep in mind when establishing sufficiency results based on the uδ-PE property. This idea also establishes a relation with the original but also technically different definition given in [14].

Lemma 2 The function $\phi(\cdot, \cdot)$ is uδ-PE w.r.t. x_1 if and only if

(**B**) *for each $\delta > 0$ and $\Delta \geq \delta$ there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$,*

$$|x_1| \in [\delta, \Delta] \, , \ |x_2| \in [0, \Delta] \qquad \Longrightarrow \qquad \int_t^{t+T} |\phi(\tau, x)| \, d\tau \geq \mu \,. \tag{8}$$

□

The last characterization is useful as a technical tool in the proof of convergence results.

Lemma 3 The function $\phi(\cdot, \cdot)$ is uδ-PE w.r.t. x_1 if and only if

(**C**) *for each $\Delta > 0$ there exist $\gamma_\Delta \in \mathcal{K}$ and $\theta_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ continuous strictly decreasing such that, for all $t \in \mathbb{R}$,*

$$\{ \, |x_1| \, , \ |x_2| \in [0, \, \Delta] \backslash \{x_1 = 0\} \, \} \qquad \Longrightarrow \qquad \int_t^{t+\theta_\Delta(|x_1|)} |\phi(\tau, x)| \, d\tau \geq \gamma_\Delta(|x_1|) \,. \tag{9}$$

□

[2] That is, as defined for functions which depend only on time: that the function $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m \times n}$, $m \leq n$ is PE if there exist $T > 0$ and $\mu > 0$ such that for all unitary vectors $z \in \mathbb{R}^n$ we have that $\int_t^{t+T} z^\top A(s)^\top A(s) z \, ds \, \geq \, \mu$.

2.3 uδ-PE is necessary and sufficient for UGAS

We present in this section our main results. We will show that for a fairly general class of nonlinear time-varying systems the property of uδ-PE is necessary and sufficient for uniform attractivity of the origin. In other words, we establish that for UGS systems uδ-PE is necessary and sufficient for UGAS.

The following result, contained in [2, Theorem 6.2], gives conditions under which uδ-PE of the right-hand side of a differential equation is necessary for uniform asymptotic stability. The technical conditions that we use permit a relatively straightforward proof, based on Gronwall's lemma, without recourse to the notion of limiting equations which are used in [2, Theorem 6.2].

Theorem 1 (UGAS $\Rightarrow$ uδ-PE) Assume that $F(\cdot,\cdot)$ in (1) is Lipschitz in x uniformly in t. If (1) is UGAS, then $F(\cdot,\cdot)$ is uδ-PE with respect to $x \in \mathbb{R}^n$. $\square$

Our main results on sufficiency of uδ-PE for UGAS of MRAC systems derive from the sufficient conditions for UGAS that we have recently established in [13].

Theorem 2 Under Assumptions 1-6 below, the origin of (1) is UGAS $\square$

Assumption 1 The origin is UGS.

Assumption 2 There exist integers j, $m > 0$ and for each $\Delta > 0$ there exist

- a number $\mu > 0$
- locally Lipschitz continuous functions $V_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, j\}$
- a (continuous) function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$,
- continuous functions $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \{1, \ldots, j\}$

such that, for almost all $(t, x) \in \mathbb{R} \times \mathcal{B}(\Delta)$,

$$\max\{|V_i(t,x)|\,,\ |\phi(t,x)|\} \le \mu, \tag{10}$$

$$\dot{V}_i(t,x) \le Y_i(x, \phi(t,x))\,. \tag{11}$$

Assumption 3 For each $k \in \{1, \cdots, j\}$ we have that[3]

(A): $\{ \quad (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ ,\ Y_i(z,\psi) = 0\ \ \forall i \in \{1, \ldots, k-1\} \quad \}$

implies

(B): $\{\ Y_k(z,\psi) \le 0\ \}$.

Assumption 4 We have that

(A): $\{\ (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ , \quad Y_i(z,\psi) = 0\ \ \forall i \in \{1, \ldots, j\}\ \}$

implies

(B): $\{\ z_1 = 0, \psi_1 = 0\ \}$.

Remark 1 In Assumption 4, there is no requirement that the size of z_1 matches the size of ψ_1. $\square$

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[3]For the case $k = 1$ one should read Assumption 3 as $Y_1(x, \phi(t,x)) \le 0$ for all $(z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)$.

Assumption 5 The first component of $\phi(t,x)$ i.e., ϕ_1, is independent of x_1, locally Lipschitz in x_2 uniformly in t, $u\delta$-PE w.r.t. x_2 and zero at the origin.

Assumption 6 For all $(t,x) \in \mathbb{R} \times \mathcal{B}(\Delta)$, we have $|F_2(t,x)| \leq \rho_\Delta(x_1, \phi_1(t,x))$ where ρ_Δ is continuous and zero at zero.

The following corollary covers [2, Theorem 6.3].

Corollary 1 If the origin of (1) is UGS and the following assumptions hold then, the origin of (1) is also UGAS. □

Assumption 7 For each $\Delta > 0$ there exist

- a number $\mu > 0$
- a locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$,
- a continuous function $Y : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

such that, for almost all $(t,x) \in \mathbb{R} \times \mathcal{B}(\Delta)$,

$$\max\{|V(t,x)|\,,\ |F(t,x)|\} \leq \mu, \tag{12}$$

$$\dot{V}(t,x) \leq Y(x, F(t,x))\,. \tag{13}$$

Assumption 8 We have that

(A): $\{\quad (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\}$

implies

(B): $\quad \{\ Y(z,\psi) \leq 0\ \}\ .$

Assumption 9 We have that

(A): $\quad \{\ (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ ,\qquad Y(z,\psi) = 0\}$

implies

(B): $\quad \{\psi = 0\ \}\ .$

Assumption 10 $(t,x) \mapsto F(t,x)$ is locally Lipschitz in x uniformly in t and $u\delta$-PE with respect to x.

3 Main results

We consider MRAC-type systems or, systems with matching nonlinearities, of the general form (1) where

$$F(t,x) := \begin{bmatrix} A(t,x) + B(t,x) \\ C(t,x) + D(t,x) \end{bmatrix} \tag{14}$$

for which it is assumed that all functions are zero at $x = 0$. Moreover, we will make the standing hypothesis that the system is UGS.

The reason why we decompose $F(t,x)$ in 4 terms is that we will impose different conditions on each of them. Roughly speaking, we will require that $\dot{x}_1 = A(t,x)$ is UGAS with respect to[4] x_1, A and C vanish at $x_1 = 0$, D and B vanish at $x_2 = 0$ and moreover that the remainder of $B(t,x)$, i.e. when $x_1 = 0$, is uδ-PE.

The UGS assumption implies by converse Lyapunov theorems that there exists $V(t,x)$ with a negative semidefinite derivative. Roughly speaking, this requires that the nonlinearities from the $\dot{x}_1$-equation "match" with those in the $\dot{x}_2$-equation. This motivates the title of the subsection. To illustrate further this idea we stress that though restrictive in general, the hypothesis on UGS holds for a large class of systems, including systems with the following interesting structure:

$$\dot{x}_1 = \tilde{A}(t,x_1)x_1 + G(t,x)x_2 \tag{15a}$$

$$\dot{x}_2 = -P^{-1}G(t,x)^\top \left(\frac{\partial W(t,x_1)}{\partial x_1}\right)^\top, \qquad P = P^\top > 0 \tag{15b}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and $W : \mathbb{R}^{n_1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^1$ positive definite radially unbounded function such that

$$\frac{\partial W(t,x_1)}{\partial x_1}\tilde{A}(t,x_1)x_1 \leq 0\,. \tag{16}$$

Indeed, it is sufficient to take $V(t,x) := W(t,x_1) + 0.5x_2^\top P x_2$ to see that the system is UGS since $\dot{V}(t,x) \leq 0$. Notice that for this inequality to hold it is instrumental that the nonlinearities in the x_2-equation match with the second term in the x_1-equation. This motivates the title of the subsection.

It may be also apparent that the structure (15) is roughly, a direct generalization of linear positive real systems and strictly positive real systems in the case when (16) holds with a bound of the form $-\alpha(|x_1|)$, with $\alpha \in \mathcal{K}$. To better see this, let us restrict $W(t,x_1)$ to be quadratic then, we can view (15) as two passive systems interconnected through the nonlinearity $G(t,x)$, i.e., the x_1 equation defines a passive map $x_2 \mapsto x_1$ (output feedback passive[5] if (16) holds with $-\alpha(|x_1|)$, with $\alpha \in \mathcal{K}$) and the x_2 equation is an integrator (hence passive). See [6] for a real-world example and [15, 5] for further discussions.

We address two cases: when $D(t,x_2) \equiv 0$ and when $D(t,x_2) \not\equiv 0$. With reference to the observations above, if we regard x_2 as an input, we restrict $G(t,x)$ to depend only on x_1, and regard $\dot{x}_2$ as an output, these two cases actually correspond to those of relative degrees 1 and 0 respectively. This may be more clear if we restrict further our attention to linear time-varying systems and define: $W(t,x_1) := \frac{1}{2}|x_1|^2$, $z := x_1$, $\dot{z} = A(t)z + B(t)u$ with $B(t) = G(t)$, $u := x_2$, and output $y := C(t)z + \tilde{D}(t)u$ with $C(t) := P^{-1}B(t)$ and $\tilde{D}(t)u =: D(t,u)$.

We also present some concrete examples in the following subsections: we first revisit some stabilization results for feedforward systems *a la* [16] and then, we see how our results apply to closed-loop identification (or adaptive control) of mechanical systems.

We now present the main results of this section. To that end, let us define

$$B_\circ(t,x_2) := B(t,x)\big|_{x_1=0} \tag{17}$$

and notice that necessarily, $B_\circ(\cdot,0) \equiv 0$.

We also introduce the following hypothesis which together with (16), roughly speaking, is related to the attractivity of the set $\{x_1 = 0\}$ or in other words, to the inherent stability of $\dot{x}_1 = A(t,x)$ with respect to x_1 which, in particular implies that $x_1(t) \to 0$ as $t \to \infty$. Notice also that in the case that A in (15) is linear time-independent and under (16), the following assumption is equivalent to requiring that $\tilde{A}$ is stable.

[4]We recall that a system is stable with respect to part of the state if, roughly speaking, the classical Lyapunov stability properties hold for that part of the state. See [38] for details.

[5]See [33] for a precise definition

Assumption 11 For the system defined by (14) Assumption 7 holds and for the function $Y(\cdot,\cdot)$ in (13) we have that

(A): $\{\ (z,\psi) \in \mathcal{B}(\Delta) \times \mathcal{B}(\mu)\ , \quad Y(z,\psi) = 0\}$

implies

(B): $\{x_1 = 0\ \}$.

The next hypothesis imposes a regularity condition on $B(\cdot,\cdot)$ and that A and C vanish when $x_1 = 0$. Within the framework of linear adaptive control systems, we may say that the first part is a generalization of the global Lipschitz assumption on the regressor function. See [28] for further discussions.

Assumption 12 The functions A, B and C are locally Lipschitz in x uniformly in t. Moreover, for each $\Delta \geq 0$ there exist $b_M > 0$ and continuous nondecreasing functions $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\rho_i(0) = 0$ and for almost all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$

$$\max\left\{ |B_\circ(t,x_2)|\,, \left|\frac{\partial B_\circ}{\partial t}\right|, \left|\frac{\partial B_\circ}{\partial x_2}\right| \right\} \leq \rho_1(|x_2|)\,, \tag{18}$$

and for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $|x_2| \leq \Delta$,

$$|B(t,x) - B_\circ(t,x_2)| \leq \rho_2(|x_1|) \tag{19}$$

$$\max_{|x_2|\leq\Delta} \{|A(t,x)|\,, |C(t,x)|\} \leq \rho_3(|x_1|) \tag{20}$$

The following theorem generalizes related results in the previously cited papers, including [25, 8, 28]. See the last reference for a detailed but non exhaustive literature review.

Theorem 3 Consider the system (1), (14) under Assumptions 1, 7, 8, 11 and 12. Suppose also that

(Assumption 13) there exists a continuous non decreasing function $\rho_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\rho_4(0) = 0$,

$$|D(t,x)| \leq \rho_4(|x_2|) \tag{21}$$

and Statement **C** holds with θ_Δ and γ_Δ such that for all $x_2 \neq 0$

$$\mathrm{e}^{-\theta_\Delta(|x_2|)}\gamma_\Delta(|x_2|) \geq 3\rho_1(\Delta)\rho_4(|x_2|)\,. \tag{22}$$

Then, the origin is UGAS. □

Remark 2 Notice that for the common case when $D \equiv 0$, Assumption 13 reduces to requiring that $B_\circ(\cdot,\cdot)$ is uδ-PE with respect to x_2. Also, in this case the necessity of the latter follows directly from Theorem 1 by observing that uδ-PE of $F(\cdot,\cdot)$ as defined in (14) implies by virtue of (20), that $B_\circ(\cdot,\cdot)$ is uδ-PE with respect to x_2. □

Proof of Theorem 3. We appeal to Theorem 2. Assumption 1 is our standing hypothesis. To verify the rest of the assumptions of Theorem 2 we introduce $V_1(t,x) := V(t,x)$ where $V(t,x)$ comes from Assumption 7 and the locally Lipschitz (due to Assumption 12) function

$$V_2(t,x) = -x_1^\top B_\circ(t,x_2) - \int_t^\infty \mathrm{e}^{(t-\tau)}\,|B_\circ(\tau,x_2)|^2\,d\tau \tag{23}$$

hence, $j = m = 2$ in Assumption 2. We also introduce

$$\phi(t, x) := \begin{bmatrix} x_2 \\ F(t, x) \end{bmatrix} . \tag{24}$$

With these definitions, it is clear that (10) holds since all the functions are locally Lipschitz in x uniformly in t and $V_i(t, 0) \equiv \phi(t, 0) \equiv 0$. To see that (11) also holds, we observe first that it is satisfied for $V_1(t, x)$ in view of (13) hence, we only need to check that there exists a continuous function $Y_2(\cdot, \cdot)$ satisfying the required conditions. To that end, we evaluate the total time derivative of $V_2(t, x)$ along the solutions of (1), (14) to obtain

$$\begin{aligned} \dot{V}_2(t, x) \;\; = \;\; & -x_1^\top \left(\frac{\partial B_\circ}{\partial t} + \frac{\partial B_\circ}{\partial x_2} [C(t, x) + D(t, x)] \right) + V_2(t, x) + x_1^\top B_\circ(t, x_2) \\ & -[A(t, x) + B(t, x) - B_\circ(t, x_2)]^\top B_\circ(t, x_2) \\ & - \left(\int_t^\infty \mathrm{e}^{(t-\tau)} B_\circ(\tau, x_2)^\top \frac{\partial B_\circ(\tau, x_2)}{\partial x_2} \right) [C(t, x) + D(t, x)] \quad \text{a.e.} \end{aligned} \tag{25}$$

Notice also that by the uδ-PE condition on $B_\circ(\cdot, \cdot)$ and in view of Assumption 12 we have that

$$V_2(t, x) \leq |x_1| \, \rho_1(|x_2|) - \mathrm{e}^{-\theta_\Delta(|x_2|)} \gamma_\Delta(|x_2|) \, . \tag{26}$$

Furthermore, using this and Assumption 12 again, we can over-bound several terms on the right hand side of (25) as follows. Define $b_M := \rho_1(\Delta)$ and $\bar{\rho}(r, s) := b_M[\, 3r + (r + 3)\rho_3(r) + r\rho_4(s) + \rho_2(r) \,]$ then, for almost all $(t, x) \in \mathbb{R} \times \mathcal{B}(\Delta)$,

$$\dot{V}_2(t, x) \leq \bar{\rho}(|x_1| \, , |x_2|) - \mathrm{e}^{-\theta_\Delta(|x_2|)} \gamma_\Delta(|x_2|) + 2b_M \rho_4(|x_2|) =: \overline{Y}_2(x, \phi(t, x)) \, . \tag{27}$$

Define $Y_2(x, \phi(t, x)) := \max\{- |x_2| \, , \overline{Y}_2(x, \phi(t, x))\}$. In view of (22) and since $\bar{\rho}(0, s) \equiv 0$ we have that $Y_2(x, \phi(t, x)) \leq \max\{- |x_2| \, , -\frac{1}{3}\mathrm{e}^{-\theta_\Delta(|x_2|)} \gamma_\Delta(|x_2|)\} \leq 0$ when $x_1 = 0$. Thus, Assumption 3 holds for $k = 1$ due to Assumption 8 (we recall that here, $Y_1 = Y$) and for $k = 2$, because $Y_2(x, \phi(t, x)) \leq 0$ when $x_1 = 0$.

Assumption 4 holds due to the following. Let $Y_1(x, \phi(t, x)) = Y_2(x, \phi(t, x)) = 0$. Then, by Assumption 11 we have that $x_1 = 0$ while by definition, $Y_2(x, \phi(t, x)) = 0$ implies that $x_2 = 0$.

Assumption 5 trivially holds and Assumption 6 holds with $\rho_\Delta(x_1, x_2) := \rho_4(|x_2|) + \rho_3(|x_1|)$. ■

In support to the discussion at the beginning of the section it is worth remarking that Assumptions 1, 7, 8 and 11 hold under the following more restrictive but commonly satisfied hypothesis, at least for a large class of *passive* systems. Below, we present a concrete example concerning the adaptive tracking control of mechanical systems.

Assumption 14 There exists a locally Lipschitz function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, class-$\mathcal{K}_\infty$ functions α_1, α_2 and a continuous, positive definite function α_3 such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \tag{28}$$

and, almost everywhere,

$$\dot{V}(t, x) \leq -\alpha_3(x_1) \; . \tag{29}$$

It may be more clear from this assumption that part of the conditions in Theorem 3 are in the spirit of imposing that the system be asymptotically stable with respect to the x_1-part of the state. Then, for the convergence of the x_2-part of the trajectories we impose the uδ-PE exciteness condition. The following result which covers a class of systems similar to those covered by the main results in [15, Appendix B.2], [25, 28] is a direct corollary of Theorem 3.

Proposition 1 The system (1), (14) with $D \equiv 0$ and under Assumptions 12, 14 is UGAS if and only if $B_\circ(\cdot,\cdot)$ is uδ-PE w.r.t. x_2. $\square$

For clarity of exposition we present separately the following result, for systems when the "feedthrough" term $D(t,x)$ is present.

Proposition 2 Consider the system (1), (14) under Assumptions 12, 14 and 13. Then, the origin is UGAS. $\square$

We close the section with the counterparts of Propositions 1 and 2 which establish sufficient conditions for ULES.

Proposition 3 Consider the system (1), (14) under Assumptions 12, 13, and 14 and assume further that $\alpha_i(s) := \alpha_i\, s^2$, $i = 1\ldots3$, $\rho_j(s) = \rho_j\, s$, $j = 1\ldots4$ for small s and assume that the function $B_\circ(t,x_2)$ is uδ-PE in the sense that Statement **C** holds with functions $\gamma_\Delta(s)$ and $\theta_\Delta(s)$ such that $\mathrm{e}^{-\theta_\Delta(s)}\gamma_\Delta(s) \geq \mu\, s^2$ for small s and $\mu \geq 3b_M\rho_4$. Then, the origin is ULES. $\square$

Proposition 4 Consider the system (1), (14) with $D(t,x) \equiv 0$ and let Assumptions 12, 13 and 14 hold with $\alpha_i(s) := \alpha_i\, s^2$, $i = 1\ldots3$, $\rho_j(s) = \rho_j\, s$, $j = 1\ldots3$ for small s and assume that the function $B_\circ(t,x_2)$ is uδ-PE. Then, the origin is ULES. $\square$

Proof of Propositions 3 and 4. We provide a combined proof for both propositions, based on standard Lyapunov theory.

Let Δ be generated by the uδ-PE assumption on $B_\circ(t,x_2)$. Let $0 < R \leq \Delta$ be such that $\rho_i(s) = \rho_i\, s$, $\alpha_i(s) := \alpha_i s^2$ and $\mathrm{e}^{-\theta_\Delta(s)}\gamma_\Delta(s) \geq \mu\, s^2$ for all $s \leq R$. Let $r := R\sqrt{\frac{\alpha_1}{\alpha_2}}$ then, from (28) and (29) we obtain that $|x_\circ| \leq r$ implies that $|x(t)| \leq R$. In the sequel, we will restrict the initial conditions to $x_\circ \in \mathcal{B}(r)$.

Consider the Lyapunov function candidate $\mathcal{V}(t,x) := V(t,x) + \varepsilon V_2(t,x)$ where $V_2(t,x)$ is defined in (23) and ε is a small positive number to be chosen. Notice that $V_2(t,x)$ satisfies on $\mathbb{R}\times\mathcal{B}(R)$,

$$-\varepsilon\rho_1\,|x_2|^2 - \varepsilon\rho_1\,|x_1|\,|x_2| \leq \varepsilon V_2(t,x) \leq \varepsilon\rho_1\,|x_1|\,|x_2| - \varepsilon\mu\,|x_2|^2\ . \tag{30}$$

So we have that for sufficiently small ε there exist $\alpha_1' > 0$ and $\alpha_2' > 0$ such that for all $(t,x) \in \mathbb{R}\times\mathcal{B}(R)$,

$$\alpha_1'\,|x|^2 \leq \mathcal{V}(t,x) \leq \alpha_2'\,|x|^2\ . \tag{31}$$

The total time derivative of $\mathcal{V}(t,x)$ on the points of existence and along the systems solutions yields, using (29) and (27),

$$\dot{\mathcal{V}}(t,x) \leq -(\alpha_3 - \varepsilon\nu_R)\,|x_1|^2 + \varepsilon\nu_R\,|x_1|\,|x_2| - \varepsilon\mu\,|x_2|^2 + 2\varepsilon b_M\rho_4\,|x_2|^2 \quad \text{a.e.} \tag{32}$$

where $\nu_R > 0$. Next, using the condition imposed on μ we obtain that

$$\dot{\mathcal{V}}(t,x) \leq -(\alpha_3 - \varepsilon\nu_R)\,|x_1|^2 + \varepsilon\nu_R\,|x_1|\,|x_2| - \varepsilon b_M\rho_4\,|x_2|^2 \quad \text{a.e.} \tag{33}$$

In the case that $D \equiv 0$ we have that $\rho_4 = 0$ and completing squares in (32) we obtain that for sufficiently small ε and sufficiently large α_3 there exists $c > 0$ such that $\dot{\mathcal{V}}(t,x) \leq -c\,|x|^2$. Otherwise, the latter holds from (33) under similar arguments. ULES follows invoking standard Lyapunov theorems.

■

4 Conclusions

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