Optimal Control of a Mechanical Two-Mass-Spring System Using Invariant Ellipsoids Technique

Boris T. Polyak, Pavel S. Shcherbakov, Mikhail V. Topunov

***Abstract*— A robust formulation is given for the problem of optimal rejection of persistent exogenous disturbances in dynamic systems subjected to matrix uncertainties. The solution technique based on the invariant ellipsoids concept is developed. The approach is exemplified through a well-known benchmark control problem for a mechanical two-mass-spring system.**

I. Introduction and a Motivating Example

Description and control of real-life physical systems suggests accounting for exogenous disturbances and uncertainties in the system coefficients. There exist various models for both; in this paper we adopt the unknown-but-bounded model [1], [2] due to its adequacy to many mechanical, electric and other systems encountered in practice and minimum a priory requirements imposed. Namely, no statistical properties, rates of variation, etc., are involved; the uncertainties are assumed to be arbitrary, and only bounds for their admissible values are known.

This viewpoint leads to the so-called guaranteed set-membership approach to various problems in control and system theory [1], [2] and the invariant sets ideology [3]. This ideology has got diverse applications in estimation, filtering, minimax control in the presence of uncertainty, etc., because it provides simple yet somewhat accurate outer approximation of reachable sets of dynamic systems.

In many cases, of the most adequate models of exogenous disturbances are the so-called persistent disturbances, which are the subject of l_1-optimization theory [4]. However, l_1-optimization technique often leads to high-dimensional controllers and is very hard to implement in the continuous-time case. Also, precise description of reachable sets for systems subjected to persistent disturbances is extremely cumbersome.

A natural way to overcome these difficulties is to appeal to the invariant sets ideology in order to reduce complexity and attain the control objectives. Among various possible "shapes" of invariant sets utilized in the research areas above, *ellipsoids* should be distinguished because of their simple structure and direct connection to the quadratic Lyapunov functions approach. On top of that, in the framework of the ellipsoidal description, a powerful apparatus of linear matrix inequalities (LMI) and semidefinite programming (SDP) [5] can be used as a technical solution tool. Among the first papers in this direction is [6], also see [7].

To further motivate the setup in this paper, consider the following simple mechanical system referred to as a double oscillator or two-mass-spring system, [8]. It consists of the two rigid bodies having masses m_1 and m_2 which are linked together by a spring with elasticity coefficient k and are allowed to slide without friction along a fixed horizontal rod as shown in Fig. 1. The bodies are subjected to exogenous disturbances w_1 and w_2, respectively,

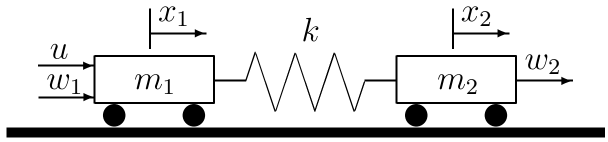

Fig. 1. The mechanical two-mass-spring system.

$$w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2,$$

for which the only available information is boundedness at any time instant: $w^{\mathrm{T}} w \leq 1$. The left body is governed by the control input $u \in \mathbb{R}$ aimed at compensating the effect of exogenous disturbances.

Letting x_1, x_2 and v_1, v_2 denote the position coordinates and the velocities of the bodies, the state vector of the system writes

$$x = \begin{pmatrix} x_1 & x_2 & v_1 & v_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^4.$$

Finally, let the output of the system be taken in the form

$$y = \begin{pmatrix} u & x_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2;$$

i.e., it is characterized by the control input and the coordinate of the right body, which is not directly affected by control.

With this description at hand, the laws of the classical mechanics lead to the following continuous-time model of disturbed oscillations of the system:

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} w,$$

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

Moreover, the uncertainty can be incorporated in the model description in the form of imprecise knowledge of the masses and/or the elasticity coefficient. The problem is then to design a static linear state feedback to *optimally* reject the effect of exogenous disturbances *robustly* against all admissible uncertainties.

Authors are with Institute of Control Science, Profsoyuznaya str., 65, 117997 Moscow, Russia; (boris,sherba,mtopunov)@ipu.ru

Precise formulations for the notions of persistent disturbance, invariant ellipsoids, optimal rejection, etc., will be given below.

This two-mass-spring system often serves as a benchmark for various control techniques (e.g., see [8]) due to its real-life nature, simple formulation and reasonable dimensions (four states, one control, two exogenous disturbances, one or two scalar uncertainties, and two outputs).

In this paper we propose an approach to such kind of problems, which is based on the method of invariant ellipsoids. The main contribution is extension of the results in [6], [7] to the presence of uncertainty in the model.

II. Invariant Ellipsoids. The Robust Analysis Problem

In this section, we give a precise general description of the uncertain dynamical system, formulate the analysis problem, and provide its solution using the invariant ellipsoids technique.

Consider the continuous-time dynamic system given by

$$\begin{aligned} \dot{x} &= \big(A+\Delta A(t)\big)x + \big(D+\Delta D(t)\big)w, \quad x(0)=0, \\ y &= Cx, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ are fixed known matrices, $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^l$ is the output, and $w(t) \in \mathbb{R}^m$ is the persistence exogenous *disturbance* satisfying the L_∞-norm constraint

$$w^{\mathrm{T}}(t)w(t) \le 1, \quad \forall\, t \ge 0. \tag{2}$$

Next, the model *uncertainty* is specified in the form

$$\Delta A(t) = F_A\Delta_A(t)H_A, \quad \Delta D(t) = F_D\Delta_D(t)H_D, \tag{3}$$

where F_A, F_D, H_A, H_D are known "frame" matrices of appropriate dimensions, and the matrix uncertainties $\Delta_A(t)$ and $\Delta_D(t)$ satisfy the condition

$$\|\Delta_A(t)\| \le 1, \qquad \|\Delta_D(t)\| \le 1 \quad \forall t \ge 0, \tag{4}$$

where $\|\cdot\|$ denotes the spectral or Frobenius matrix norm. Throughout the exposition, it is assumed that the nominal system (1) (i.e., the one without uncertainty) is stable (the matrix A is Hurwitz), the pair (A, D) is controllable, and C is a full-rank matrix.

Note that the system is subjected to both matrix uncertainty and exogenous disturbances. These two sources of uncertainty give rise to the *reachable set* of the system, which is by definition the set of all states of (1)–(4) attainable by the system at any time under any admissible uncertainty and disturbance. This set can be thought of as a characterization of the accumulated uncertainty in the system's state as time evolves.

We now introduce the notion of invariant ellipsoids. The ellipsoid

$$\mathcal{E}_x = \big\{x \in \mathbb{R}^n\colon \quad x^{\mathrm{T}}P^{-1}x \le 1\big\}, \quad P > 0, \tag{5}$$

centered at the origin and specified by the matrix P is said to be *invariant with respect to the variable* x (state-invariant) for the dynamic system (1)–(4), if the condition $x(0) \in \mathcal{E}_x$ implies $x(t) \in \mathcal{E}_x$ for all $t \ge 0$. In other words, starting at any point in $\mathcal{E}_x$, the state of the system is guaranteed to remain confined within $\mathcal{E}_x$ for all admissible disturbances (2) and uncertainties (3), (4).

It is important to note that every invariant ellipsoid contains the reachable set of the system, and our first goal in this section is to characterize invariant ellipsoids for system (1)–(4). The result is given below.

Theorem 1: Ellipsoid $\mathcal{E}_x$ (5) is state invariant for the dynamic system (1)–(4), if its matrix P satisfies the LMIs

$$\begin{pmatrix} \Omega & D & PH_A^{\mathrm{T}} & 0 \\ D^{\mathrm{T}} & -\alpha I & 0 & H_D^{\mathrm{T}} \\ H_AP & 0 & -\varepsilon_1 I & 0 \\ 0 & H_D & 0 & -\varepsilon_2 I \end{pmatrix} \le 0, \qquad P > 0,$$

for some $\alpha, \varepsilon_1, \varepsilon_2 > 0$, where $\Omega = AP + PA^{\mathrm{T}} + \alpha P + \varepsilon_1 F_AF_A^{\mathrm{T}} + \varepsilon_2 F_DF_D^{\mathrm{T}}$.

The first point to note is that the Hurwitz property and the controllability condition mentioned above are necessary for the the theorem to have a "nontrivial output," i.e., for the LMI to be feasible. Strictly speaking, these conditions should be satisfied *robustly* for all admissible uncertainties, which is not immediate to check in advance. However, if this is not the case, solving the LMI above will result in its infeasibility thus indicating the absence of invariant ellipsoids.

Next, it is noted that for $\Delta_A(t) = \Delta_D(t) \equiv 0$ we arrive at the uncertainty-free setup which was analyzed in [6], [7], [9] from the invariant ellipsoids viewpoint. Here, the robust version of the problem is addressed in a completely similar LMI style. The robust formulation above also extends to cover possible uncertainty in the initial state $x(0) = x_0$. Within the ellipsoidal framework, it is natural to specify this uncertainty in the form $x_0^{\mathrm{T}}P_0^{-1}x_0 \le 1$, where $P_0 > 0$ defines the ellipsoid $\mathcal{E}_0$ of initial uncertainty. Then the requirement $\mathcal{E}_0 \subset \mathcal{E}_x$ is formulated as $P \ge P_0$ and incorporated into the LMI constraints above.

Another important point is sufficiency of the conditions in Theorem 1. Without going deep into details, we note that matrix uncertainty of the form (3), (4) has been first introduced and studied in [10] (as applied to the disturbance-free LQR problem). To prove Theorem 1, we developed a generalization of the technical result in [10] to the case of multiple matrix uncertainties. Such a generalization is only possible in the form of sufficient condition thus leading to the sufficiency of the main result.

Consistent with the control objectives and physical motivation, our primary goal is to characterize the magnitude of the output y rather than the state x. In that respect, it is seen that associated with the state-invariant ellipsoid (5) is the *bounding* ellipsoid for the output variable y specified by

$$\mathcal{E}_y = \big\{y \in \mathbb{R}^m\colon \quad y^{\mathrm{T}}\big(CPC^{\mathrm{T}}\big)^{-1}y \le 1\big\}, \tag{6}$$

where P is the matrix of the state-invariant ellipsoid. Our goal is to distinguish the minimal bounding ellipsoid (6), where P satisfies the LMI in Theorem 1. There exist various meaningful criteria of minimality; here we adopt the following trace criterion:

$$f(P) = \mathrm{tr}\big[CPC^{\mathrm{T}}\big], \tag{7}$$

which characterizes the "size" (the sum of squared semiaxes) of the corresponding ellipsoid. An important thing to note is that for every fixed $\alpha > 0$, this trace criterion is linear in $P, \varepsilon_1, \varepsilon_2$; hence, for α fixed, the minimization of (7) under the LMI constraints above is a semidefinite program.

In other words, for system (1)–(4), the problem of finding the trace-optimal bounding ellipsoid (6) in the family specified by Theorem 1 reduces to solving an α-parametrized SDP with respect to one matrix and two scalar variables ($P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$) with subsequent one-dimensional optimization in α. Computationally, this is easily accomplished using any of the numerous appropriate toolboxes that are presently available, e.g., MATLAB-based packages SeDuMi and Yalmip.

III. Robust Optimal Design Problem

We now incorporate the control term into description and consider the system

$$\begin{aligned} &\dot{x} = \big(A+\Delta A(t)\big)x + \big(B_1+\Delta B_1(t)\big)u + \big(D+\Delta D(t)\big)w,\\ &y = Cx + B_2 u, \qquad\qquad x(0) = 0, \end{aligned} \tag{8}$$

where $u \in \mathbb{R}^p$ is control, $B_1 \in \mathbb{R}^{n\times p}$, the model uncertainty is specified in the same form as above:

$$\begin{aligned} \Delta A(t) &= F_A \Delta_A(t) H_A,\\ \Delta B_1(t) &= F_{B_1} \Delta_{B_1}(t) H_{B_1},\\ \Delta D(t) &= F_D \Delta_D(t) H_D, \end{aligned} \tag{9}$$

with F_A, F_{B_1}, F_D, H_A, H_{B_1}, H_D being fixed known matrices of compatible dimensions, and the matrix uncertainties $\Delta_A(t)$, $\Delta_{B_1}(t)$ and $\Delta_D(t)$ satisfy the norm-bound constraint (4). The rest of the quantities involved have the same meanings as in Section 2. The matrix A is not assumed to be Hurwitz, but the pair (A, B_1) is controllable and $B_2^{\mathrm{T}} C = 0$.

We are aimed at finding a gain matrix K for the linear static state feedback

$$u = Kx$$

which stabilizes the closed-loop system robustly against all matrix uncertainties and minimizes the trace of the bounding ellipsoid $\mathcal{E}_y$ defined above. It is this minimization that we refer to as the optimal rejection of exogenous disturbances $w(t)$.

We have the following result.

Theorem 2: Let $\widehat{P} > 0$ and $\widehat{Y}$ be solutions to the minimization problem

$$\mathrm{tr}\big[CPC^{\mathrm{T}} + B_2 Z B_2^{\mathrm{T}}\big] \longrightarrow \min \tag{10}$$

under constraints

$$\begin{pmatrix} \Omega & D & PH_A^{\mathrm{T}} & Y^{\mathrm{T}} H_{B_1}^{\mathrm{T}} & 0\\ D^{\mathrm{T}} & -\alpha I & 0 & 0 & H_D^{\mathrm{T}}\\ H_A P & 0 & -\varepsilon_1 I & 0 & 0\\ H_{B_1} Y & 0 & 0 & -\varepsilon_2 I & 0\\ 0 & H_D & 0 & 0 & -\varepsilon_3 I \end{pmatrix} \le 0, \tag{11}$$

$$\begin{aligned} \Omega = AP + PA^{\mathrm{T}} + B_1 Y + Y^{\mathrm{T}} B_1^{\mathrm{T}} + \alpha P +\\ + \varepsilon_1 F_A F_A^{\mathrm{T}} + \varepsilon_2 F_{B_1} F_{B_1}^{\mathrm{T}} + \varepsilon_3 F_D F_D^{\mathrm{T}}, \end{aligned} \tag{12}$$

$$\begin{pmatrix} Z & Y\\ Y^{\mathrm{T}} & P \end{pmatrix} \ge 0, \quad \alpha > 0, \tag{13}$$

with respect to the scalar variables $\alpha, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$, and matrix variables $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times n}$, $Z = Z^{\mathrm{T}} \in \mathbb{R}^{p\times p}$.

Then the state-feedback controller with matrix

$$\widehat{K} = \widehat{Y} \widehat{P}^{-1}$$

robustly stabilizes system (8), (2), (9), (4) and rejects the effect of disturbances $w(t)$, and the matrix $\widehat{P}$ defines the invariant ellipsoid for the closed-loop system.

Important remarks analogous to those following Theorem 1 are valid. Namely, due to the linearity of the trace criterion with respect to P, Z, for any fixed value of the parameter α, the problem above reduces to the minimization of the linear function (10) subject to the LMI constraints (11)–(13); i.e., to a well-defined semidefinite program. The subsequent scalar optimization over the parameter α leads to a (sub)optimal stabilizing controller, i.e., to the one that minimizes the trace criterion for the bounding ellipsoid of the closed-loop system. As far as the uncertainty in the initial state is considered, it can be specified and incorporated in the LMI constraints exactly in the same way as it was done in the analysis problem (Section 2).

Another comment relates to the issue of worst-case uncertainties and disturbances in the system. In proving Theorem 2, we build a quadratic Lyapunov function $V(x)$ for the closed-loop system having the property $\dot{V}(x) \le 0$ for $V(x) \ge 1$ and $w^{\mathrm{T}}(t) w(t) \le 1$. It is natural to determine exogenous disturbances $\widetilde{w}(t)$ and matrix uncertainties $\widetilde{\Delta}_A(t)$, $\widetilde{\Delta}_{B_1}(t)$, $\widetilde{\Delta}_D(t)$, which maximize $\dot{V}(x)$. These are referred to as *worst-case* ones. The explicit formulae for such worst-case uncertainties and disturbances are given by the lemma below.

Lemma 1: For system (8), (2), (9), (4), with Frobenius matrix norm in (4), the worst-case exogenous disturbance $\widetilde{w}(t)$ is given by

$$\widetilde{w}(t) = \frac{\big(D + F_D \Delta_D(t) H_D\big)^{\mathrm{T}} \widehat{P}^{-1} x(t)}{\big\| \big(D + F_D \Delta_D(t) H_D\big)^{\mathrm{T}} \widehat{P}^{-1} x(t) \big\|}.$$

The worst-case matrix uncertainties $\widetilde{\Delta}_A(t)$, $\widetilde{\Delta}_{B_1}(t)$ and $\widetilde{\Delta}_D(t)$ are defined by

$$\widetilde{\Delta}_A(t) = \frac{F_A^{\mathrm{T}}\widehat{P}^{-1}x(t)x^{\mathrm{T}}(t)H_A^{\mathrm{T}}}{\left\|F_A^{\mathrm{T}}\widehat{P}^{-1}x(t)x^{\mathrm{T}}(t)H_A^{\mathrm{T}}\right\|_F},$$

$$\widetilde{\Delta}_{B_1}(t) = \frac{F_{B_1}^{\mathrm{T}}\widehat{P}^{-1}x(t)x^{\mathrm{T}}(t)(H_{B_1}\widehat{K})^{\mathrm{T}}}{\left\|F_{B_1}^{\mathrm{T}}\widehat{P}^{-1}x(t)x^{\mathrm{T}}(t)(H_{B_1}\widehat{K})^{\mathrm{T}}\right\|_F},$$

$$\widetilde{\Delta}_D(t) = \frac{F_D^{\mathrm{T}}\widehat{P}^{-1}x(t)w^{\mathrm{T}}(t)H_D^{\mathrm{T}}}{\left\|F_D^{\mathrm{T}}\widehat{P}^{-1}x(t)w^{\mathrm{T}}(t)H_D^{\mathrm{T}}\right\|_F}.$$

Finally, we note that both Theorem 2 and Lemma 1 can be extended to the case of matrix uncertainties of a more general form (cf. (9)):

$$\Delta A(t) = \sum_{i=1}^{r} F_A^i \Delta_i(t) H_A^i$$

(and same for $\Delta B_1(t)$, $\Delta D(t)$), where $\Delta_i(t)$, $i = 1, \ldots, r$, satisfy constraints (4). The associated formulas are bulky and are not presented here.

IV. Application to the Two-Mass-Spring System

We turn back to the benchmark problem described in Section 1 and illustrate the theoretical results of Section 3.

For simplicity, we consider the case where the masses m_1, m_2 are assumed known and both equal to unity, and the uncertainty is concentrated in the elasticity coefficient, which is specified in the form

$$k = 1 + \delta\Delta(t), \quad \delta = \text{const.}$$

This leads to system (8), (9) with one scalar uncertainty $\Delta(t)$, $|\Delta(t)| \le 1$.

Application of Theorem 2 gives the optimal controller $\widehat{K}$ that minimizes the trace criterion for the two-dimensional bounding ellipse.

For the numerical solution of the SDP problem (10)–(13) we made use of the SeDuMi and Yalmip Toolboxes in MATLAB. For the specified value $\delta = 0.2$, the calculations yielded the gain matrix

$$\widehat{K} \approx \begin{pmatrix} -3.3443 & 1.6057 & -2.7810 & -2.1620 \end{pmatrix}$$

and the associated bounding ellipse.

Figure 2 depicts the minimal bounding ellipse for the system with controller $\widehat{K}$ in the feedback loop. The figure also shows the output trajectory $y(t)$ corresponding to a certain initial position inside this ellipse and the worst-case uncertainty $\widetilde{\Delta}(t)$ and exogenous disturbances $\widetilde{w}_1(t)$, $\widetilde{w}_2(t)$ calculated according to Lemma 1. These worst-case uncertainty and disturbances are depicted in Fig. 3 along with the optimal control $u(t)$.

¿From Fig. 2 it is seen that the sample output trajectory nearly touches the boundary of the calculated invariant ellipse; experiments show that this behavior is typical for the system. In other words, the proposed characterization of the reachable set by means of invariant ellipsoids is deemed to have low degree of conservatism.

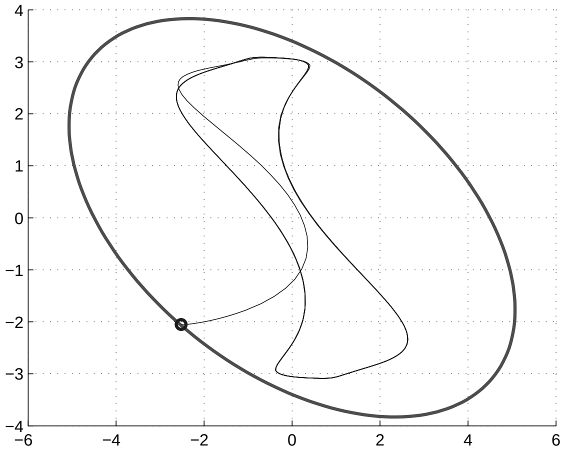

Fig. 2. The optimal bounding ellipse for the two-mass-spring system.

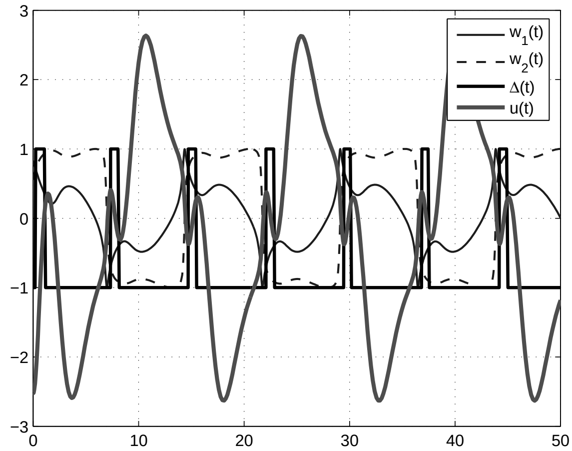

Fig. 3. The worst-case disturbances $\widetilde{w}_1(t)$, $\widetilde{w}_2(t)$ and uncertainty $\widetilde{\Delta}(t)$, and the optimal control $u(t)$.

The case where the masses contain uncertainty reduces to the setup mentioned at the end of Section 3 and can be completely analyzed in a similar way using the respective modifications of Theorem 2 and Lemma 1.

V. Conclusion

We have proposed a simple yet universal approach to rejection of unknown-but-bounded exogenous disturbances robustly against norm-bounded matrix uncertainties by means of linear static state feedback. This approach is based on the method of invariant ellipsoids, by which means the optimal control design problem reduces to finding the minimal invariant ellipsoid for the closed-loop system.

By using the invariant ellipsoids ideology, the original problem can be reformulated in terms of linear matrix inequalities, and the control design problem directly reduces to semidefinite programs and one-dimensional minimization, which is straightforward to implement numerically.

The efficacy of the approach is illustrated through application to a benchmark problem, which has a transparent physical motivation.

Another attractive property of the approach is that it is equally applicable to discrete-time systems. These results are not presented here and will be addressed in the journal version of the paper.

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